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THE INVERSE PROBLEM OF CENTROSYMMETRIC MATRICES WITH A SUBMATRIX CONSTRAINT *1)

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Abstract

By using Moore-Penrose generalized inverse and the general singular value decomposition of matrices, this paper establishes the necessary and sufficient conditions for the existence of and the expressions for the centrosymmetric solutions with a submatrix constraint of matrix inverse problem $AX = B$. In addition, in the solution set of corresponding problem, the expression of the optimal approximation solution to a given matrix is derived.

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1. Introduction

Inverse eigenvalue problem has widely used in control theory [1, 2], vibration theory [3, 4], structural design [5], molecular spectroscopy [6]. In recent years, many authors have been devoted to the study of this kind of problem and a serial of good results have been made [7, 8, 9]. Centrosymmetric matrices have practical application in information theory, linear system theory and numerical analysis theory. However, inverse problems of centrosymmetric matrices, specifically centrosymmetric matrices with a submartix constraint, have not be concerned with. In this paper, we will discuss this problem.

Denote the set of all n-by-m real matrices by $R^{n\times m}$ and the set of all n-by-n orthogonal matrices in $R^{n\times n}$ by $OR^{n\times n}$. Denote the column space, the null space, the Moore-Penrose generalized inverse and the Frobenius norm of a matrix A by $R(A), N(A), A^+$ and $\|A\|$, respectively. I_n denotes the $n\times n$ unit matrix and S_n denotes the $n\times n$ reverse unit matrix. We define the inner product in space $R^{n\times m}$ by

$$\langle A, B\rangle = trace(B^H A),\ \forall A, B \in R^{n\times m}.$$

Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by this inner product space is the Frobenius norm. For $A = (a_{ij}), B = (b_{ij}) \in R^{n\times m}$, we using the notation $A * B = (a_{ij}b_{ij}) \in R^{n\times n}$ denotes the Hadmard product of matrices A and B.

Definition 1 [10,15]**.** *$A = (a_{ij}) \in R^{n\times n}$ is termed a Centrosymmetric matrix if*

$$a_{ij} = a_{n+1-j,n+1-i} \quad i, j = 1, 2, \ldots, n.$$

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The set of $n \times n$ centrosymmetric matrices denoted by $CSR^{n\times n}$.

In this paper, we consider the following two problems:

Problem 1. Given $X, B \in R^{n\times m}, A_0 \in R^{r\times r}$, find $A \in CSR^{n\times n}$ such that

$$AX = B, \quad A_0 = A([1,r]),$$

where $A([1,r])$ is a leading $r \times r$ principal submatrix of matrix A.

Problem 2. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\| = \min_{A\in S_E} \|A^* - A\|,$$

where S_E is the solution set of Problem 1.

In Section 2, we first discuss the structure of the set $CSR^{n\times n}$, and then present the solvability conditions and provide the general solution formula for Problem 1. In Section 3, we first show the existence and uniqueness of the solution for Problem 2, and then derive an expression of the solution when the solution set S_E is nonempty. Finally, in section 4, we first give an algorithm to compute the solution to Problem 2, and then give a numerical example to illustrate the results obtained in this paper are correction.

2. Solving Problem 1

We first characterize the set of all centrosymmetric matrices. For all positive integers k, let

$$D_{2k} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}, D_{2k+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}. \tag{1}$$

Clearly, D_n is orthogonal for all n.

Lemma 1 [10]**.** *$A \in CSR^{n\times n}$ if and only if $A = S_n A S_n$.*

Lemma 2. *$A \in CSR^{n\times n}$ if and only if A can be expressed as*

$$A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T, \tag{2}$$

where $A_1 \in R^{(n-k)\times(n-k)}, A_2 \in R^{k\times k}$.

Proof. We only prove the case for $n = 2k$, the case for $n = 2k+1$ can be discussed similarly. Partition the matrix A into the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11}, A_{22} \in R^{k\times k}.$$

If $A \in CSR^{2k\times 2k}$, then we have from Lemma 1 that

$$\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

which is equivalent to

$$A_{22} = S_k A_{11} S_k, \quad A_{12} = S_k A_{21} S_k.$$

Hence

$$\begin{aligned} D_n^T A D_n &= \frac{1}{2}\begin{pmatrix} I_k & S_k \\ I_k & -S_k \end{pmatrix}\begin{pmatrix} A_{11} & S_k A_{21} S_k \\ A_{21} & S_k A_{11} S_k \end{pmatrix}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix} \\ &= \begin{pmatrix} A_{11} + S_k A_{21} & 0 \\ 0 & A_{11} - S_k A_{21} \end{pmatrix}. \end{aligned}$$

Let

$$A_1 = A_{11} + S_k A_{21}, A_2 = A_{11} - S_k A_{21},$$

and note that D_n is a orthogonal matrix, we have (2).

Conversely, for every $A_1, A_2 \in R^{k\times k}$, we have

$$\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T.$$

It follows from Lemma 1 that $A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T \in CSR^{2k\times 2k}$.

Next, we give some lemmas. According to [7], the following lemma is easy to verify.

Lemma 3. *Given $Z \in R^{m\times s}, Y \in R^{n\times s}$, then the matrix equation $AZ = Y$ has a solution $A \in R^{n\times m}$ if and only if $YZ^+Z = Y$. In that case the general solution can be expressed as $A = YZ^+ + GP^T$, where $P \in R^{n\times(n-t)}$ is an unit column-orthogonal matrix, $R(P) = N(Z^T)$ and $t = rank(Z)$.*

Let

$$\Gamma = \{A \in CSR^{n\times n} | AX = B, X, B \in R^{n\times m}\}. \tag{3}$$

Partitioning $D_n^T X$ and $D_n^T B$ into to the following form

$$D_n^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, D_n^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{4}$$

where $X_1, B_1 \in R^{(n-k)\times m}, X_2, B_2 \in R^{k\times m}$. We have the following lemma.

Lemma 4. *Γ is nonempty if and only if $B_1 X_1^+ X_1 = B_1$ and $B_2 X_2^+ X_2 = B_2$. Furthermore, any matrix $A \in \Gamma$ can be expressed as*

$$A = D_n \begin{pmatrix} B_1 X_1^+ + G_1 U_2^T & 0 \\ 0 & B_2 X_2^+ + G_2 P_2^T \end{pmatrix} D_n^T, \tag{5}$$

where $G_1 \in R^{(n-k)\times(n-k-r_1)}, G_2 \in R^{k\times(k-r_2)}$ are arbitrary matrices, $r_1 = rank(X_1), r_2 = rank(X_2)$, $R(U_2) = N(X_1^T)$, $R(P_2) = N(X_2^T)$.

Proof. If Γ is nonempty, we have from Lemma 2 and D_n being an orthogonal matrix that

$$A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T, \tag{6}$$

with $A_1 \in R^{(n-k)\times(n-k)}, A_2 \in R^{k\times k}$ satisfy

$$A_1X_1 = B_1, \quad A_2X_2 = B_2. \tag{7}$$

It follows from Lemma 3 that

$$B_1X_1^+X_1 = B_1, B_2X_2^+X_2 = B_2 \tag{8}$$

and

$$A_1 = B_1X_1^+ + G_1U_2^T, A_2 = B_2X_2^+ + G_2P_2^T, \tag{9}$$

where $G_1 \in R^{(n-k)\times(n-k-r_1)}$, $G_2 \in R^{k\times(k-r_2)}$ are arbitrary matrix, r_1=rank(X_1) , r_2=rank(X_2), $R(U_2) = N(X_1^T)$, $R(P_2) = N(X_2^T)$. Substituting (9) into (6), we have (5).

Conversely, if $B_1X_1^+X_1 = B_1$ and $B_2X_2^+X_2 = B_2$, then we have from Lemma 3 that there exist $A_1 \in R^{(n-k)\times(n-k)}$ and $A_2 \in R^{k\times k}$ such that

$$A_1X_1 = B_1, \quad A_2X_2 = B_2,$$

which is equivalent to

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

It in turn is equivalent to

$$D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T X = B.$$

So the matrix $A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T \in \Gamma$. And this illustrate Γ is nonempty.

Now, we investigate the consistency of Problem 1 with a centrosymmetric condition on the solution.

Obviously, solving Problem 1 is equivalent to find $A \in \Gamma$ such that A_0 is a leading principal submatrix of A, i.e., find $A \in \Gamma$ such that

$$(I_r, 0)A(I_r, 0)^T = A_0, \tag{10}$$

where $(I_r, 0) \in R^{r\times n}$. Note that any matrix $A \in \Gamma$ can be expressed as (5), we partition the matrix $(I_r, 0)D_n$ into the following form

$$(I_r, 0)D_n = (W_1, W_2), W_1 \in R^{r\times(n-k)}, W_2 \in R^{r\times k}. \tag{11}$$

Let

$$\tilde{E} = A_0 - W_1B_1X_1^+W_1^T - W_2B_2X_2^+W_2^T. \tag{12}$$

Then equation (10) is equivalent to find G_1 and G_2 as in (5) such that

$$W_1G_1N_1 + W_2G_2N_2 = \tilde{E}, \tag{13}$$

where $N_1 = U_2^TW_1^T, N_2 = P_2^TW_2^T$. Decomposing the matrix pairs (W_1^T, W_2^T) and (N_1, N_2) using the General Singular-Value Decomposition ($GSVD$) (see[12, 13, 14]):

$$W_1^T = \tilde{U}\Sigma_1M_1, \quad W_2^T = \tilde{V}\Sigma_2M_1, \quad N_1 = \tilde{P}\Sigma_3M_2, \quad N_2 = \tilde{Q}\Sigma_4M_2, \tag{14}$$

where $\tilde{U}, \tilde{V}, \tilde{P}$ and $\tilde{Q}$ are orthogonal matrices, and M_1 and M_2 are nonsingular matrices of order r, $\Sigma_1 \in R^{(n-k)\times r}$, $\Sigma_2 \in R^{k\times r}$, $\Sigma_3 \in R^{(n-k-r_1)\times r}$, $\Sigma_4 \in R^{(k-r_2)\times r}$, with

$$\Sigma_1 = \begin{pmatrix} I_1 & & & \vdots & \\ & S_1 & & \vdots & 0 \\ & & 0_1 & \vdots & \end{pmatrix}_{\; r_3 \;\; r_4 \;\; p_1-r_3-r_4 \;\; r-p_1}, \quad \Sigma_2 = \begin{pmatrix} 0_2 & & & \vdots & \\ & S_2 & & \vdots & 0 \\ & & I_2 & \vdots & \end{pmatrix}_{\; r_3 \;\; r_4 \;\; p_1-r_3-r_4 \;\; r-p_1}, \tag{15}$$

$$\Sigma_3 = \begin{pmatrix} I_3 & & & \vdots & \\ & S_3 & & \vdots & 0 \\ & & 0_3 & \vdots & \end{pmatrix}_{\; r_5 \;\; r_6 \;\; p_2-r_5-r_6 \;\; r-p_2}, \quad \Sigma_4 = \begin{pmatrix} 0_4 & & & \vdots & \\ & S_4 & & \vdots & 0 \\ & & I_4 & \vdots & \end{pmatrix}_{\; r_5 \;\; r_6 \;\; p_2-r_5-r_6 \;\; r-p_2}, \tag{16}$$

here I_1, I_2, I_3 and I_4 are identity matrices, $0_1, 0_2, 0_3$ and 0_4 are zero matrices, p_1=rank(C_1) =rank(W_1, W_2), p_2=rank(C_2)=rank(N_1, N_2), and

$$S_1 = diag(\alpha_1, \ldots, \alpha_{r_4}), S_2 = diag(\beta_1, \ldots, \beta_{r_4}), \tag{17}$$

$$S_3 = diag(\sigma_1, \ldots, \sigma_{r_6}), S_4 = diag(\delta_1, \ldots, \delta_{r_6}), \tag{18}$$

with $1 > \alpha_1 \geq \ldots \geq \alpha_{r_4} > 0, 0 < \beta_1 \leq \ldots \leq \beta_{r_4} < 1$, $1 > \sigma_1 \geq \ldots \geq \sigma_{r_6} > 0, 0 < \delta_1 \leq \ldots \leq \delta_{r_6} < 1$, and $\alpha_i^2 + \beta_i^2 = 1, (i = 1, \ldots, r_4)$, $\sigma_i^2 + \delta_i^2 = 1, (i = 1, \ldots, r_6)$. Some submatrices in equations (15) and (16) may disappear, depending on the structure of the matrices W_1, W_2, N_1 and N_2. Define $Y = \tilde{U}^T G_1 \tilde{P}, Z = \tilde{V}^T G_2 \tilde{Q}$ and $E = M_1^{-T} \tilde{E} M_2^{-1}$. Equation (13) now reads

$$\Sigma_1^T Y \Sigma_3 + \Sigma_2^T Z \Sigma_4 = E. \tag{19}$$

Note that transforming equation (13) to (19) does not change the equation's consistency. Partitioning the matrices Y, Z and E according to the Σ's, equation (19) is equivalent to

$$\begin{pmatrix} Y_{11} & Y_{12}S_3 & 0 & 0 \\ S_1Y_{21} & S_1Y_{22}S_3 + S_2Z_{22}S_4 & S_2\tilde{Y}_{23} & 0 \\ 0 & Z_{32}S_4 & Z_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix}. \tag{20}$$

From the above discussion, we can prove the following theorem:

Theorem 1. *Problem 1 is solvable if and only if*

(a) $B_i X_i^+ X_i = B_i \quad (i = 1, 2)$.

(b) $E_{13} = 0, E_{31} = 0, E_{14} = 0, E_{24} = 0, E_{34} = 0, E_{41} = 0, E_{42} = 0, E_{43} = 0, E_{44} = 0.$

When the conditions (a) and (b) are satisfied, the general solution can be expressed as

$$A = D_n \begin{pmatrix} B_1X_1^+ + G_1U_2^T & 0 \\ 0 & B_2X_2^+ + G_2P_2^T \end{pmatrix} D_n^T, \tag{21}$$

where

$$G_1 = \tilde{U}\begin{pmatrix} E_{11} & E_{12}S_3^{-1} & Y_{13} \\ S_1^{-1}E_{21} & S_1^{-1}(E_{22}-S_2Z_{22}S_4)S_3^{-1} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix}\tilde{P}^T, \tag{22}$$

$$G_2 = \tilde{V}\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & S_2^{-1}E_{23} \\ Z_{31} & E_{32}S_4^{-1} & E_{33} \end{pmatrix}\tilde{Q}^T, \tag{23}$$

with $Y_{13}, Y_{23}, Y_{31}, Y_{32}, Y_{33}, Z_{11}, Z_{12}, Z_{13}, Z_{21}, Z_{22}, Z_{31}$ *are arbitrary matrices.*

Proof. Problem 1 having a solution $A \in CSR^{n\times n}$ is equivalent to there exists $A \in \Gamma$ such that (10) holds. Condition (a) is the necessary and sufficient conditions for Γ is nonempty. Condition (b) is the necessary and sufficient conditions for equation (10) has centrosymmetric solution with the given matrix as its submatrix. The general solution can be obtained by the definition of Y and Z and the Equations (5) and (20).

3. Solving Problem 2

Let us first introduce a lemma.

Lemma 5 [11]**.** *Given* $E, F \in R^{n\times n}$*,* $\Omega_1 = diag(a_1, \ldots, a_n) > 0$*,* $\Omega_2 = diag(b_1, \ldots, b_n) > 0$*,* $\Phi = (\frac{1}{1+a_i^2b_j^2}) \in R^{n\times n}$*, then there exists an unique matrix* $\hat{S} \in R^{n\times n}$ *such that* $\|S-E\|^2 + \|\Omega_1 S\Omega_2 - F\|^2 = min$*, and* $\hat{S}$ *can be expressed as* $\hat{S} = \Phi * (E + \Omega_1 F\Omega_2)$*.*

Partition the matrix $D_n^T A^* D_n$ into a 2×2 block matrix $(A_{ij}^*)_{2\times 2}$ with $A_{11}^* \in R^{(n-k)\times(n-k)}$, $A_{22}^* \in R^{k\times k}$. Partition matrices $\tilde{U}^T A_{11}^* U_2 \tilde{P}$ and $\tilde{V}^T A_{22}^* P_2 \tilde{Q}$ into 3×3 block matrices $(\tilde{Y}_{ij})_{3\times 3}$ with $\tilde{Y}_{11} \in R^{r_3\times r_5}, \tilde{Y}_{22} \in R^{r_4\times r_6}, \tilde{Y}_{33} \in R^{(n-k-r_3-r_4)\times(n-k-r_1-r_5-r_6)}$, and $(\tilde{Z}_{ij})_{3\times 3}$ with $\tilde{Z}_{11} \in R^{(k+r_3-p_1)\times(k-r_2+r_3-p_2)}, \tilde{Z}_{22} \in R^{r_4\times r_6}, \tilde{Z}_{33} \in R^{(p_1-r_3-r_4)\times(p_2-r_5-r_6)}$, respectively. Then we have the following theorem.

Theorem 2. *Given* $X, B \in R^{n\times m}, A_0 \in R^{r\times r}$ *and* $A^* \in R^{n\times n}$*.* X, B, A_0 *satisfy conditions of Theorem 1. Then Problem 2 has an unique optimal approximate solution which can be expressed as*

$$\hat{A} = D_n\begin{pmatrix} B_1X_1^+ + \hat{G}_1U_2^T & 0 \\ 0 & B_2X_2^+ + \hat{G}_2P_2^T \end{pmatrix}D_n^T \tag{24}$$

where

$$\hat{G}_1 = \tilde{U}\begin{pmatrix} E_{11} & E_{12}S_3^{-1} & \tilde{Y}_{13} \\ S_1^{-1}E_{21} & S_1^{-1}(E_{22}-S_2\hat{Z}_{22}S_4)S_3^{-1} & \tilde{Y}_{23} \\ \tilde{Y}_{31} & \tilde{Y}_{32} & \tilde{Y}_{33} \end{pmatrix}\tilde{P}^T, \tag{25}$$

$$\hat{G}_2 = \tilde{V}\begin{pmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} & \tilde{Z}_{13} \\ \tilde{Z}_{21} & \hat{Z}_{22} & S_2^{-1}E_{23} \\ \tilde{Z}_{31} & E_{32}S_4^{-1} & E_{33} \end{pmatrix}\tilde{Q}^T, \tag{26}$$

$$\tilde{Z}_{22} = \Phi * [\tilde{Z}_{22} + S_1^{-1}S_2(\tilde{Y}_{22} - S_1^{-1}E_{22}S_3^{-1})S_4S_3^{-1}] \tag{27}$$

with $\Phi = (\varphi_{ij}) \in R^{r_4\times r_6}, \varphi_{ij} = \frac{\alpha_i^2\delta_j^2}{\alpha_i^2\delta_j^2+\beta_i^2\sigma_j^2}$*,* U_2, P_2 *are the same as (5), and* $\tilde{U}, \tilde{V}, \tilde{P}$ *and* $\tilde{Q}$ *are the same as (14).*

Proof. Because X, B and A_0 satisfy the conditions of the Theorem 1, the solution set S_E of Problem 1 is nonempty. According to the proof of theorem 7.4 in [7], it is easy to verify that S_E is a closed convex cone. Hence, the corresponding Problem 2 has an unique optimal approximate solution. Choose U_1 and P_1 such that $U = (U_1, U_2)$ and $P = (P_1, P_2)$ are orthogonal matrices. Attention to $U, P, D, \tilde{U}, \tilde{V}, \tilde{P}$ and $\tilde{Q}$ are orthogonal matrices, and $R(U_2) = N(X_1^T), R(P_2) = N(X_2^T)$, we have from (21)-(23) that

$$\begin{aligned}
\|A - A^*\|^2 &= \|B_1X_1^+ + G_1U_2^T - A_{11}^*\|^2 + \|B_2X_2^+ + G_2P_2^T - A_{22}^*\|^2 + \|A_{12}^*\|^2 + \|A_{21}^*\|^2 \\
&= \|G_1 - A_{11}^*U_2\|^2 + \|G_2 - A_{22}^*P_2\|^2 + \|(B_1X_1^+ - A_{11}^*)U_1\|^2 \\
&+ \|(B_2X_2^+ - A_{22}^*)P_1\|^2 + \|A_{12}^*\|^2 + \|A_{21}^*\|^2 \\
&= \left\| \begin{pmatrix} E_{11} & E_{12}S_3^{-1} & Y_{13} \\ S_1^{-1}E_{21} & S_1^{-1}(E_{22} - S_2Z_{22}S_4)S_3^{-1} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} - \tilde{U}^TA_{11}^*U_2\tilde{P} \right\|^2 \\
&+ \left\| \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & S_2^{-1}E_{23} \\ Z_{31} & E_{32}S_4^{-1} & E_{33} \end{pmatrix} - \tilde{V}^TA_{22}^*P_2\tilde{Q} \right\|^2 \\
&+ \|B_1X_1^+U_1 - A_{11}^*U_1\|^2 + \|B_2X_2^+P_1 - A_{22}^*P_1\|^2 + \|A_{12}^*\|^2 + \|A_{21}^*\|^2.
\end{aligned}$$

Hence, there exists $A \in S_E$ such that $\|A - A^*\| = \min$ is equivalent to

$$\begin{cases} \|Y_{i3} - \tilde{Y}_{i3}\| = \min, \|Y_{3j} - \tilde{Y}_{3j}\| = \min, \|Z_{i1} - \tilde{Z}_{i1}\| = \min, \ (i = 1, 2, 3, j = 1, 2), \\ \|Z_{12} - \tilde{Z}_{12}\| = \min, \|Z_{22} - \tilde{Z}_{22}\|^2 + \|S_1^{-1}(E_{22} - S_2Z_{22}S_4)S_3^{-1} - \tilde{Y}_{22})\|^2 = \min. \end{cases} \tag{28}$$

It follow from Lemma 5 that

$$Y_{i3} = \tilde{Y}_{i3}, Y_{3j} = \tilde{Y}_{3j}, Z_{i1} = \tilde{Z}_{i1}, \ (i = 1, 2, 3, j = 1, 2), Z_{12} = \tilde{Z}_{12} \tag{29}$$

and

$$\tilde{Z}_{22} = \Phi * (\tilde{Z}_{22} + S_1^{-1}S_2(\tilde{Y}_{22} - S_1^{-1}E_{22}S_3^{-1})S_4S_3^{-1}). \tag{30}$$

Substituting (29) and (30) into (21)-(23), we obtain (24)-(27).

4. The Algorithm Description and Numerical Example

According to discuss in section 2 and 3, we now give a method for solving Problem 2 as following steps:

step 1. According to (4) calculate X_i, B_i $(i = 1, 2)$, furthermore calculate $B_iX_i^+X_i$ $(i = 1, 2)$.

step 2. If $B_iX_i^+X_i = B_i (i = 1, 2)$, then the set Γ is nonempty and we continue. Otherwise we stop.

step 3. According to (11) calculate W_1, W_2. According to (12) calculate $\tilde{E}$.

step 4. Find unit column orthogonal matrices U_2, P_2 basis of linear equations $X_1^Tu = 0$ and $X_2^Tv = 0$. Chosen U_1, P_1 such that $U = (U_1, U_2), P = (P_1, P_2)$ are orthogonal matrices.

step 5. According to (14) decomposing the matrix pairs (W_1^T, W_2^T) and (N_1, N_2) using the $GSVD$.

step 6. Partitioning the matrix E according to the right side of (20).

step 7. If $E_{13}=0, E_{31}=0, E_{14}=0, E_{24}=0, E_{34}=0, E_{41}=0, E_{42}=0, E_{43}=0, E_{44}=0$, then the solution set S_E to Problem 1 is nonempty and we continue. Otherwise we stop.

step 8. Partition the matrices $D_n^T A^* D_n, \tilde{U}^T A_{11}^* U_2 \tilde{P}$ and $\tilde{V}^T A_{22}^* P_2 \tilde{Q}$ into block matrices according to $D_n^T A D_n$ in (21), $\tilde{U}^T G_1 \tilde{P}$ in (22) and $\tilde{V}^T G_2 \tilde{Q}$ in (23), respectively.

step 9. According to (24)–(27) calculate $\hat{A}$.

Example 1. Taking

$$X=\begin{pmatrix} 8.3 & 9.2 & -8.3 & 9.2\\ 9.9 & 7.7 & -9.9 & 7.7\\ -7.9 & -7.3 & 7.9 & -7.3\\ 6.3 & 8.3 & -6.3 & 8.3\\ 7.5 & 8.3 & -7.5 & 8.3\\ -6.1 & 9.4 & 6.1 & 9.4\\ 5.3 & -9.2 & -5.3 & -9.2\\ 8.3 & 7.3 & -8.3 & 7.3 \end{pmatrix}, B=\begin{pmatrix} -2.9 & 15.1 & 2.9 & 15.1\\ 12.3 & 15.3 & -12.3 & 15.3\\ 40.8 & 37.5 & -40.8 & 37.5\\ 39.3 & 20.6 & -39.3 & 20.6\\ 42.1 & 26.5 & -42.1 & 26.5\\ 30.2 & 49.6 & -30.2 & 49.6\\ 9.9 & 77.7 & -9.9 & 77.7\\ -6.7 & 62.9 & 6.7 & 62.9 \end{pmatrix},$$

$$A_0=\begin{pmatrix} 1 & 2 & 6\\ 2 & 3 & 7\\ 3 & 4 & 5 \end{pmatrix}, A^*=\begin{pmatrix} -1.5 & 2.4 & -2.2 & 1.7 & 1.3 & 2.4 & 1.9 & 2.7\\ 3.5 & 1.9 & -2.5 & -1.4 & 1.4 & 2.1 & -1.5 & 0.8\\ -1.4 & -1.5 & 1.9 & 1.5 & -0.8 & 1.5 & -2.1 & -2.4\\ 1.5 & -1.2 & 2.4 & -1.5 & -2.7 & -1.9 & -2.4 & -1.3\\ -1.3 & -1.3 & -0.6 & -2.5 & -0.5 & 1.5 & -1.5 & 1.7\\ 1.6 & 1.4 & -2.7 & -1.2 & -1.5 & -1.7 & -2.7 & -1.8\\ 1.2 & 0.7 & -3.4 & -1.6 & -1.8 & -2.7 & -1.7 & -1.5\\ 2.5 & 1.6 & 1.3 & 1.3 & 1.7 & -1.5 & 1.5 & -2.5 \end{pmatrix}.$$

we obtain X_1^+, X_2^+, B_1 and B_2 are,respectively,

$$\begin{pmatrix} 0.0043 & 0.0220 & -0.0211 & 0.0004\\ 0.0176 & -0.0198 & 0.0198 & 0.0208\\ -0.0043 & -0.0220 & 0.0211 & -0.0004\\ 0.0176 & -0.0198 & 0.0198 & 0.0208 \end{pmatrix}, \begin{pmatrix} -0.0473 & 0.1829 & 0.1798 & -0.1576\\ 0.0113 & -0.0137 & -0.0549 & 0.0299\\ 0.0473 & -0.1829 & -0.1798 & 0.1576\\ 0.0113 & -0.0137 & -0.0549 & 0.0299 \end{pmatrix},$$

$$\begin{pmatrix} -6.7882 & 55.1543 & 6.7882 & 55.1543\\ 15.6978 & 65.7609 & -15.6978 & 65.7609\\ 50.2046 & 61.5890 & -50.2046 & 61.5890\\ 57.5585 & 33.3047 & -57.5585 & 33.3047 \end{pmatrix}, \begin{pmatrix} 2.6870 & -33.7997 & -2.6870 & -33.7997\\ 1.6971 & -44.1235 & -1.6971 & -44.1235\\ 7.4953 & -8.5560 & -7.4953 & -8.5560\\ -1.9799 & -4.1719 & 1.9799 & -4.1719 \end{pmatrix}.$$

By a direct computing, we know that $B_i X_i^+ X_i = B_i \quad (i=1,2)$. According to step 4, we get two orthogonal matrices $U=(U_1,U_2), P=(P_1,P_2)$ as follow

$$U_1=\begin{pmatrix} 0.6637 & 0.2642\\ 0.3353 & -0.6353\\ -0.2971 & 0.6231\\ 0.5990 & 0.3719 \end{pmatrix}, \quad U_2=\begin{pmatrix} 0.6976 & 0.0555\\ -0.1328 & 0.6829\\ -0.0109 & 0.7234\\ -0.7040 & -0.0850 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0.0769 & 0.1568 \\ 0.7197 & -0.5875 \\ -0.6899 & -0.6024 \\ -0.0093 & 0.5170 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.9296 & -0.3244 \\ -0.0860 & -0.3597 \\ 0.0187 & -0.4010 \\ -0.3578 & -0.7775 \end{pmatrix}.$$

After calculating according to step 5 and 6, the condition (b) of Theorem 1 is satisfied. Hence, the conditions of Theorem 1 are satisfied, and the corresponding problem 2 has an unique solution $\hat{A}$. After calculating by (24)-(27), we have $\hat{A}$ as follow:

$$\begin{pmatrix} 1.0000 & 2.0000 & 6.0000 & -1.7316 & 5.5018 & 2.6940 & 1.7672 & -0.8298 \\ 2.0000 & 3.0000 & 7.0000 & -1.5539 & 5.6900 & 2.4985 & 2.6065 & -1.2239 \\ 3.0000 & 4.0000 & 5.0000 & -2.0757 & 3.9990 & 2.0306 & 2.1184 & 0.0052 \\ 3.5291 & 0.1945 & -0.4371 & 0.8850 & -1.5600 & -1.6861 & -0.3234 & 0.2630 \\ 0.2630 & -0.3234 & -1.6861 & -1.5600 & 0.8850 & -0.4371 & 0.1945 & 3.5291 \\ 0.0052 & 2.1184 & 2.0306 & 3.9990 & -2.0757 & 5.0000 & 4.0000 & 3.0000 \\ -1.2239 & 2.6065 & 2.4985 & 5.6900 & -1.5539 & 7.0000 & 3.0000 & 2.0000 \\ -0.8298 & 1.7672 & 2.6940 & 5.5018 & -1.7316 & 6.0000 & 2.0000 & 1.0000 \end{pmatrix}.$$

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