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THE PREDICTION-CORRECTION LEGENDRE COLLOCATION METHOD FOR NONLINEAR EVOLUTIONARY PROBLEMS *1)

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Abstract

The initial-boundary value problem of Burgers equation is considered. A prediction-correction Legendre collocation scheme is presented, which is easy to be performed. Its numerical solution possesses the accuracy of second-order in time and higher order in space. Numerical results are reported, which show the high accuracy of this approach. The techniques used in this paper are also applicable to other nonlinear evolutionary problems.

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Key words: Burgers equation, Prediction-correction operator, Legendre collocation approximation, Convergence, Numerical results.

1. Introduction

Since spectral methods possess the accuracy of "infinite" order, they have been widely applied to computational fluid dynamics, e.g., see [1-3]. As we know, the Burgers equation plays an important role in fluid dynamics. Maday and Quarteroni [4] studied Legendre and Chebyshev spectral approximations to the steady problem. Recently Bialecki and Karageorghis [10] considered Legendre collocation method and He Li-ping and Sun Shun-kai [11] provided a fast direct Legendre collocation algorithm for linear elliptic problems. Ma He-ping and Guo Ben-yu [6] developed Chebyshev spectral methods for unsteady problem. In some existing work, the temporal discretizations for nonlinear evolutionary problems is of the first order, and so limits the merit of spectral approximations in space, e.g., see [7]. To remedy this trouble, a prediction-correction Legendre spectral scheme was proposed in [8,9], which produces precise numerical results, in which the linear leading terms are approximated implicitely and the nonlinear terms are approximated explicitely at each step. Thus we can solve it explicitely by using the algorithm in [11]. However, how to analyze the high accuracy for such an approach is still an open problem. Indeed, so far, there has been no result on error estimate of prediction-correction operator Legendre collocation method. Since we can not derive an explicit relationship between the numerical solution and its predicted one, and so the analysis is very difficult.

In this paper, we take the unsteady Burgers equation as an example to show how to construct a reasonable Legendre spectral collocation approximation using prediction-correction operator in time and how to analyze the errors. Let $T > 0$, $\Lambda = (-1, 1)$, $\partial\Lambda = \{-1, 1\}$ and $\mu > 0$ be the kinetic viscosity. $f(x)$ and u_0 describe the source term and the initial state. Then the unsteady Burgers equation is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{2}\dfrac{\partial}{\partial x}u^2 = f, & \text{in } \Lambda \times (0, T], \\ u = 0, & \text{on } \partial\Lambda \times (0, T], \\ u(x, 0) = u_0(x), & \text{in } \Lambda \cup \partial\Lambda. \end{cases} \tag{1.1}$$

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In Section 2, we construct the scheme and present the convergence. In Section 3, we present the numerical results which show the high accuracy of this method. We list some lemmas in Section 4 and prove the accuracy of second-order in time and high order in space in the final section. The technique provided in this paper are also applicable to other nonlinear evolutionary problems.

2. The Prediction-correction Legendre Collocation Scheme

Throughout the paper we use Sobolev spaces $H^r(\Lambda)$ and $H_0^r(\Lambda)$. For simplicity, let $L^2(\Lambda) = H^0(\Lambda)$. Their definitions and properties can be found in [12]. The inner product, the semi-norm and the norm of $H^r(\Lambda)$, $r \geq 0$ are denoted by $(\cdot,\cdot)_r$, $|\cdot|_r$, $\|\cdot\|_r$ respectively. If $r=0$, then the index r is omitted. We recall that the usual semi-norm $|\cdot|_r$ is equivalent to the norm $\|\cdot\|_r$ in $H_0^r(\Lambda)$. Further let $H^{-r}(\Lambda)$ be the dual space of $H_0^r(\Lambda)$, and $<\cdot,\cdot>_{L(H^{-r},H_0^r)}$ be the duality parting between $H^{-r}(\Lambda)$ and $H_0^r(\Lambda)$. Define the bilinear form $A(\cdot,\cdot) : H^1(\Lambda)\times H^1(\Lambda) \longmapsto R$ and the trilinear form $B(\cdot,\cdot,\cdot) : L^4(\Lambda)\times L^4(\Lambda)\times H^1(\Lambda) \longmapsto R$ as follows

$$\begin{aligned} A(u,v) &= (\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}), \quad \forall u, \quad v\in H^1(\Lambda),\\ B(u,v,w) &= -\frac{1}{2}(uv,\frac{\partial w}{\partial x}), \quad \forall u,v\in L^4(\Lambda), \quad w\in H^1(\Lambda). \end{aligned} \tag{2.1}$$

So the weak formulation of (1.1) is to find a function $u\in L^2(0,T;H_0^1(\Lambda))\bigcap L^\infty(0,T,\ L^2(\Lambda))$ such that

$$\begin{cases} (\frac{\partial u}{\partial t},v)+\mu A(u,v)+B(u,u,v) = < f,v >_{L(H^{-1},H_0^1)}, \quad \forall v\in H_0^1(\Lambda).\\ u(x,0)=u_0(x). \end{cases} \tag{2.2}$$

It can be proved that if $f\in L^2(0,T;H^{-1}(\Lambda))$ and $u_0\in L^2(\Lambda)$, then (2.2) has a unique solution.

Let $P_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N and $P_N^0(\Lambda) = P_N(\Lambda)\bigcap H_0^1(\Lambda)$. We define the orthogonal projection operator $P_N^1 : H_0^1(\Lambda)\longmapsto P_N^0(\Lambda)$ such that

$$A(u-P_N^1u, v_N)=0, \quad \forall v_N\in P_N^0(\Lambda).$$

Also, we shall use the orthogonal projection operator $P_N : L^2(\Lambda)\longmapsto P_N(\Lambda)$ defined as

$$(u-P_Nu, v_N)=0, \quad \forall v_N\in P_N(\Lambda).$$

Denote by $\{x_j\}_{j=0,1,\cdots,N}$ and $\{\omega_j\}_{j=0,1,\cdots,N}$ the nodes and weights of the Gauss-Lobatto-Legendre quadratuare formula on $\overline{\Lambda}$=[-1,1]. Let $I_N : C(\overline{\Lambda})\longmapsto P_N(\Lambda)$ be the interpolation operator on $\{x_j\}_{j=0,1,\cdots,N}$. It is obvious that

$$I_Nu(x_j)=u(x_j), \quad j=0,1,\cdots,N.$$

The discrete inner product and norm are defined as

$$(u,v)_N=\sum_{j=0}^N u(x_j)v(x_j)\omega_j, \| u \|_N=(u,u)_N^{\frac{1}{2}}.$$

Also, we shall use the norm

$$\| u \|_{L^q(\Lambda),N}=\begin{cases} (\sum_{j=0}^N | u(x_j) |^q\omega_j)^{\frac{1}{q}}, & 1\leq q<\infty,\\ \max_{0\leq j\leq N} | u(x_j) |, & q=\infty. \end{cases}$$

We now consider the discretization only in space. The semi-discrete Legendre spectral approximation solution of (1.1) is a function $u_N(x,t)\in P_N^0(\Lambda)$ such that

$$\begin{cases} \dfrac{\partial u_N}{\partial t}-\mu\dfrac{\partial^2 u_N}{\partial x^2}+\dfrac{1}{2}\dfrac{\partial}{\partial x}I_N u_N^2\mid_{x=x_j}=f(x_j,t), \quad j=1,2,\cdots,N-1, t\in(0,T], \\ u(x_j,0)=u_0(x_j), \quad j=1,2,\cdots,N. \end{cases} \tag{2.3}$$

Indeed, (2.3) is easy to be coded. To show this, we consider its vector form. Let

$$U_0=(u_0(x_0),u_0(x_1),\cdots,u_0(x_N))^T, \quad U_N(t)=(u_N(x_0,t),u_N(x_1,t),\cdots,u_N(x_N,t))^T,$$
$$F_N(t)=(0,f(x_1,t),\cdots,f(x_{N-1},t),0)^T, \quad Z_N=\mathrm{diag}(0,1,\cdots,1,0)$$

and D_N be the differentiation matrix(see[3]). Denote the Hadamard product of the vectors U and V by $U\Diamond V$. It is clearly that the jth component of $U\Diamond V$ is the product of the jth component of U and V. Then the vector form of (2.3) is as follows:

$$\begin{cases} Z_N(\dfrac{\partial U_N}{\partial t}-\mu D_N^2 U_N+\dfrac{1}{2}D_N(U_N\Diamond U_N))=F_N, \quad t\in(0,T], \\ U_N(0)=U_0. \end{cases}$$

We shall use the variation form of (2.3). Let $v_N\in P_N^0(\Lambda)$. By multiplying it by $v_N(x_j)\omega_j$ and summing it for $j=0,1,\cdots,N$, we obtain that

$$(\frac{\partial u_N}{\partial t},v_N)_N-\mu(\frac{\partial^2 u_N}{\partial x^2},v_N)+\frac{1}{2}(\frac{\partial}{\partial x}I_N u_N^2,v_N)_N=(f,v_N)_N.$$

Since

$$(\frac{\partial}{\partial x}I_N u_N^2,v_N)_N=-(I_N u_N^2,\frac{\partial v_N}{\partial x})=-(u_N^2,\frac{\partial v_N}{\partial x})_N,$$

we introduce the trilinear form $B_N(\cdot,\cdot,\cdot):C(\overline{\Lambda})\times C(\overline{\Lambda})\times C^1(\overline{\Lambda})\longmapsto R$, satisfying

$$B_N(u,v,w)=-\frac{1}{2}(uv,\frac{\partial w}{\partial x})_N, \forall u,v\in C(\overline{\Lambda}), w\in C^1(\overline{\Lambda}).$$

Thus, the semi-discrete Legendre spectral collocation can be written as

$$\begin{cases} (\dfrac{\partial u_N}{\partial t},v_N)_N+\mu A(u_N,v_N)+B_N(u_N,u_N,v_N)=(f,v_N)_N, \quad \forall v_N\in P_N^0(\Lambda), \\ u_N(0)=I_N u_0. \end{cases}$$

We now describe the fully discrete scheme. Divide the interval $[0,T]$ into M uniform subintervals and then let $\tau=\frac{T}{M}$. Set $u^k(x)=u(x,k\tau)$, also denoted by u^k for simplicity. Let $u_N^k\in P_N^0(\Lambda)$ be the approximation to the solution of (2.2) at time $t_k=k\tau$. Denote by $u_{N,pre}^{k+1}$ the predicted value of u_N^{k+1}. Then the prediction-correction Legendre collocation scheme for (2.2) is of the form

$$\begin{cases} \dfrac{1}{\tau}(u_{N,pre}^{k+1}-u_N^k,v_N)_N+\dfrac{1}{2}\mu A(u_N^k+u_{N,pre}^{k+1},v_N)+B_N(u_N^k,u_N^k,v_N)=(f^k,v_N)_N, \\ \dfrac{1}{\tau}(u_N^{k+1}-u_N^k,v_N)_N+\dfrac{1}{2}\mu A(u_N^k+u_N^{k+1},v_N)+\dfrac{1}{2}B_N(u_N^k,u_N^k,v_N) \\ \quad +\dfrac{1}{2}B_N(u_{N,pre}^{k+1},u_{N,pre}^{k+1},v_N)=\dfrac{1}{2}(f^k+f^{k+1},v_N)_N, \forall v_N\in P_N^0(\Lambda), 1\le k\le M-1, \\ u_N^0=I_N u_0. \end{cases} \tag{2.4}$$

Obviously the approximate solution on the initial level is well defined. Now assume that the numerical solution on the kth level has been calculated. Let

$$O_{\mathrm{pc}}(u_N,v_N)=(u_N,v_N)+\frac{1}{2}\mu\tau A(u_N,v_N), \quad \forall u_N,v_N\in P_N^0(\Lambda)$$

be the prediction-correction operator. Clearly $O_{\rm pc}(u_N, v_N)$ is a bilinear continuous and coercive form on $P_N^0(\Lambda)\times P_N^0(\Lambda)$. Hence by Lax-Milgram theorem, the numerical solution on the $(k+1)$th level is determined uniquely. So this scheme has a unique solution on each level as long as $f \in C(0,T;L^2(\Lambda))$ and $u_0 \in H_0^1(\Lambda)$.

We now present the convergence. Let $u^\star$ be the weak solution of (2.2) and

$$E(w^n) = \| w^n \|^2 + \frac{1}{2}\mu\tau \sum_{k=0}^{n-1} | w^n + w^{n+1} |_1^2.$$

THEOREM 2.1. *Assume that*
(i). $u^\star \in H^1(0,T;H_0^1(\Lambda)) \cap H^m(\Lambda)) \cap H^2(0,T;H^1(\Lambda)) \cap H^3(0,T;H^{-1}(\Lambda))$, *and* $m > 1$,
(ii). $f \in H^1(0,T;L^2(\Lambda) \cap H^m(\Lambda)) \cap H^2(0,T;H^{-1}(\Lambda)), u_0 \in H_0^1(\Lambda) \bigcap H^m(\Lambda)$,
(iii). $\tau = O(N^{-\lambda}), \lambda > \frac{4}{5}$.
Then we have

$$E(u^{\star,n} - u_N^n) \le M^\star(\tau^4 + \tau^5 N^2 + N^{-2m}), \quad \forall n \ge 1$$

where $M^\star > 0$ *depends only on* μ *and the norms of* $f, u^\star, u_0$ *in the spaces mentioned above.*

3. Numerical Results

In this section we report some numerical results and compare the results of scheme (2.4) with the following scheme(see[7])

$$\begin{cases} \frac{1}{\tau}(u_N^{k+1} - u_N^k, v_N)_N + \mu A(\sigma u_N^{k+1} + (1-\sigma)u_N^k, v_N) \\ +B_N(\delta u_N^{k+1} + (1-\delta)u_N^k, u_N^k, v_N) = (f^k, v_N)_N, \quad \forall v_N \in P_N^0(\Lambda), \quad 1 \le k \le M-1, \\ u_N^0 = I_N u_0 \end{cases} \tag{3.1}$$

where σ, δ are parameters, $0 \le \sigma, \delta \le 1$. In all calculations, we take $\sigma = 0.5, \delta = 0$. In this case, (3.1) is just the prediction scheme of (2.4). Hence, the efficiency of the correction scheme is obviously. For descibing the errors, we denote the relative error by

$$E(u^\star(t)) = \frac{\| u_N - u^\star \|_N}{\| u^\star \|_N}.$$

Example 1. Consider Bergers equation with the exact solution

$$u^\star(x,t) = Ae^{Bt}\sin \pi x.$$

Take $A = B = 0.1$. The numerical results are presented in Table 1-Table 3. Table 1 shows the high accuracy of scheme (2.4). Table 2 indicates that the numerical solution of scheme (2.4) converges to the exact solution as $N \to \infty, \tau \to 0$. It also shows that this scheme is very stably for long time calculation even for small N and big τ. In particular, the accuracy of scheme (2.4) with $\tau = 0.05$ is as good as scheme (3.1) with $\tau = 0.0005$. So we can save the work. Table 3 shows that the calculation are still stable for very small μ, even if $\mu = 0$. It shows again that scheme (2.4) provides more accurate numerical results than (3.1).

Table 1 The error $E(u^\star(t))$ of scheme (2.4) $\mu = 0.5, N = 16$

τ	T=1	T=2	T=3	T=4	T=5
τ=0.1	1.764382E-5	1.949775E-5	2.154612E-5	2.380920E-5	2.630932E-5
τ=0.01	2.139889E-7	2.364836E-7	2.613397E-7	2.888052E-7	3.191528E-7
τ=0.001	2.378627E-9	2.628647E-9	2.904912E-9	3.210177E-9	3.547475E-9

Table 2 The error $E(u^\star(30))$, $\mu = 0.5$,

	scheme (2.4)			scheme (3.1)		
	N=8	N=16	N=32	N=8	N=16	N=32
τ=0.5	4.107021E-3	3.865909E-3	3.865913E-3	2.520100E-3	2.247803E-3	2.247349E-3
τ=0.05	5.952485E-3	8.106889E-5	8.106945E-5	7.146238E-3	2.216810E-3	2.214106E-3
τ=0.005	6.018622E-3	9.264174E-7	9.266689E-7	6.106448E-3	2.213275E-4	2.213272E-4
τ=0.0005	6.019403E-3	1.903300E-8	9.412395E-9	6.028449E-4	2.212974E-5	2.212953E-5

Table 3 The error $E(u^\star(20))$, N=16

	scheme (2.4)		scheme (3.1)	
	τ=0.05	τ=0.025	τ=0.05	τ=0.025
μ=0.001	3.225754E-5	1.572462E-6	8.012451E-4	4.050147E-4
μ=0.0001	6.155348E-6	1.611909E-6	1.267925E-3	6.261861E-4
μ=0	6.190734E-6	1.663028E-6	1.414238E-3	6.969674E-4

Example 2. Consider Burgers equation with the exact solution

$$u^\star(x,t) = (1-x^2)\ln(H + (x-Gt)^2).$$

The numerical results are listed in Table 4. It also shows that scheme (2.4) is stable even for long time calculation.

Table 4 The error $E(u^\star(t))$, $\mu = 0.05, H = 1, G = 0.01, N = 16$

	scheme (2.4)			scheme (3.1)		
	τ =0.1	τ =0.01	τ =0.001	τ =0.1	τ =0.01	τ =0.001
t=60	1.610350E-5	7.539710E-7	5.532477E-8	1.298524E-3	1.294839E-4	1.295397E-5
t=120	2.570979E-6	3.373624E-8	2.659552E-8	2.586907E-4	2.584729E-5	2.583722E-6

4. Some Lemmas

Lemma 4.1(see[2]). *For any $u \in P_N(\Lambda)$,*

$$\| u \| \le \| u \|_N \le \sqrt{2 + \frac{1}{N}} \| u \| .$$

Lemma 4.2 (see[2]). *If $q \ge 2$ and $u \in P_N(\Lambda)$, then*

$$\| u \|_{L^q(\Lambda),N} \le (\frac{N+1}{\sqrt{2}})^{1-\frac{2}{q}} \| u \|_N .$$

Lemma 4.3. *If $w \in H_0^1(\Lambda) \bigcap C^1(\Lambda)$, then*

$$| B_N(u,v,w) | \le \begin{cases} \frac{1}{2} \| u \|_N \| \frac{\partial v}{\partial x} \| \| \frac{\partial w}{\partial x} \|_N, & \forall u \in C(\Lambda), v \in H_0^1(\Lambda) \bigcap C^1(\Lambda), \\ \frac{1}{2} \| \frac{\partial u}{\partial x} \| \| v \|_N \| \frac{\partial w}{\partial x} \|_N, & \forall v \in C(\Lambda), u \in H_0^1(\Lambda) \bigcap C^1(\Lambda). \end{cases}$$

Proof. By imbedding theorem, we have that

$$| u(y) |^2 \le c(\Lambda) \| \frac{\partial u}{\partial x} \|^2 .$$

It is not difficult to show that the constant $c(\Lambda) \le 1$ which indicates that

$$\| u \|_{C(\Lambda)} \le \| \frac{\partial u}{\partial x} \|, \quad \| u \| \le \sqrt{2} \| \frac{\partial u}{\partial x} \|, \quad \| u \|_1 \le \sqrt{3} | u |_1 . \tag{4.1}$$

Hence, the Cauchy inequality gives

$$\begin{aligned} | B_N(u,v,w) | &\leq \frac{1}{2}\sum_{j=0}^{N} | u(x_j) || v(x_j) || \frac{\partial w}{\partial x}(x_j) | w_j \\ &\leq \frac{1}{2}\| v \|_{C(\Lambda)} \sum_{j=0}^{N} | u(x_j) || \frac{\partial w}{\partial x}(x_j) | w_j \\ &\leq \frac{1}{2} \| \frac{\partial v}{\partial x} \| \| u \|_N \| \frac{\partial w}{\partial x} \|_N . \end{aligned}$$

Thus, the first conclusion follows. The above statements also leads to the second conclusion.

Lemma 4.4. *If $u, v \in C(\Lambda)$ and $w \in H_0^1(\Lambda) \bigcap C^1(\Lambda)$, then*

$$| B_N(u,v,w) | \leq \frac{N+1}{2\sqrt{2}} \| u \|_N \| v \|_N \| \frac{\partial w}{\partial x} \|_N .$$

Proof. By Lemma 4.2 and the Hölder inequality, we get that

$$\begin{aligned} | B_N(u,v,w) | &\leq \frac{1}{2}\| u \|_{L^4(\Lambda),N} \| v \|_{L^4(\Lambda),N} \| \frac{\partial w}{\partial x} \|_N \\ &\leq \frac{N+1}{2\sqrt{2}} \| u \|_N \| v \|_N \| \frac{\partial w}{\partial x} \|_N . \end{aligned}$$

Lemma 4.5. *Let $u, v \in L^2(\Lambda)$, and $w \in H_0^1(\Lambda)$. We have*

$$| B(u,v,w) | \leq \begin{cases} \frac{1}{2} \| u \|| v |_1 | w |_1, & \forall v \in H_0^1(\Lambda), \\ \frac{1}{2} | u |_1 \| v \|_1 | w |_1, & \forall u \in H_0^1(\Lambda), \\ | u |_1 | v |_1 \| w \|, & \forall u, v \in H_0^1(\Lambda). \end{cases}$$

Proof. It suffice to prove it for any $u, v, w \in C_0^\infty(\Lambda)$. Using the Cauchy inequality and (4.1), we deduce that

$$\begin{aligned} | B(u,v,w) | &= \frac{1}{2} \left| \int_{-1}^{1} uv \frac{\partial w}{\partial x} dx \right| \leq \frac{1}{2} \| v \|_{C(\Lambda)} \int_{-1}^{1} | u \frac{\partial w}{\partial x} | dx \\ &\leq \frac{1}{2} \| u \| \|| v \|_{C(\Lambda)} | w |_1 \leq \frac{1}{2} \| u \| | v |_1 | w |_1 . \end{aligned}$$

The above statements also leads to the second conclusion. Finally, by integration by parts,

$$\begin{aligned} \left| \int_{-1}^{1} uv \frac{\partial w}{\partial x} dx \right| &\leq \left| \int_{-1}^{1} \frac{\partial u}{\partial x} vw dx \right| + \left| \int_{-1}^{1} u \frac{\partial v}{\partial x} w dx \right| \\ &\leq | u |_1 \| v \|_{C(\Lambda)} \| w \| + \| u \|_{C(\Lambda)} | v |_1 \| w \| \leq 2 | u |_1 | v |_1 \| w \| . \end{aligned}$$

Lemma 4.6 (see[2]). *If $0 \leq r \leq 1 \leq s$, then*

$$\| u - P_N^1 u \| \leq c(r,s) N^{r-s} \| u \|_s, \quad \forall u \in H^s(\Lambda) \bigcap H_0^1(\Lambda),$$

$c(r,s)$ being certain positive constant depending only on r and s.

Lemma 4.7 (see[5]). *If $0 \leq r \leq 1, s > \frac{1}{2} + \frac{r}{2}$, then*

$$\| u - I_N u \|_r \leq c(r,s) N^{r-s} \| u \|_s, \forall u \in H^s(\Lambda).$$

Lemma 4.8 (see[13]). *Assume that*

(a). E^k and F^k are non-negative functions for $k = 0, 1, \cdots$;

(b). C_0, C_1, a_l, b_l and q are non-negative constants;

(c). $A(\cdot)$ is a real-valued function such that,if $z \le C_1$, then $A(z) \le 0$;
(d). For certain non-negative integer p,

$$E^n \le q + C_0\tau \sum_{k=0}^{n-1} \left(\left(1 + \sum_{l=1}^{p} (E^k)^{a_l} N^{b_l} \right) E^k + A(E^k)F^k \right);$$

(e). $E^0 \le q$, $n\tau \le T$ and

$$q \exp\{C_0 T(p+1)\} \le \min\left(\min_{1\le l\le p} N^{-\frac{b_l}{a_l}}, C_1 \right).$$

then

$$E^n \le q \exp\{C_0 T(p+1)\}$$

In particular,if $p = 0$, and for all $\forall z, A(z) \le 0$, then the above inequality holds for all n and any q.

5. The Proof of Convergence Theorem

In this section, we prove Theorem 2.1 by five steps.

Step I. Let $u_N^\star(x,t) = P_N^1 u^\star(x,t)$, $E_{1,N}^k = \frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}) - \frac{\partial u}{\partial t}^{\star,k}$,

$$E_{2,N}^k = \frac{1}{2}(u_N^{\star,k} + u_N^{\star,k+1}) - u^{\star,k}, \quad G(u,v,w) = B(u,u,w) - B(v,v,w).$$

Clearly

$$G(u,v,w) = G(u,z,w) + G(z,v,w). \tag{5.1}$$

We have from (2.2) that

$$\begin{cases} \frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}, v_N)_N + \frac{1}{2}\mu A(u_N^{\star,k} + u_N^{\star,k+1}, v_N) \\ = (f^k, v_N) + (E_{1,N}^k, v_N) + \mu A(E_{2,N}^k, v_N) - B(u^{\star,k}, u^{\star,k}, v_N), \quad \forall v_N \in P_N^0(\Lambda), \\ u_N^{\star,0} = I_N u_0. \end{cases} \tag{5.2}$$

Let $e_N^k = u_N^k - u_N^{\star,k}$ and $e_{N,pre}^{k+1} = u_{N,pre}^{\star,k+1} - u_N^{\star,k+1}$. By substracting the above equation from the first formula of (2.4), we find that

$$\begin{cases} \frac{1}{\tau}(e_{N,pre}^{k+1} - e_N^k, v_N)_N + \frac{1}{2}\mu A(e_N^k + e_{N,pre}^{k+1}, v_N) \\ = (f^k, v_N)_N - (f^k, v_N) + \frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}, v_N) - \frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}, v_N)_N \\ -(E_{1,N}^k, v_N) - \mu A(E_{2,N}^k, v_N) + B(u^{\star,k}, u^{\star,k}, v_N) - B_N(u_N^k, u_N^k, v_N), \\ e_N^0 = 0. \end{cases} \tag{5.3}$$

Let $E_3^k = \frac{1}{2}(f^{k+1} - f^k)$. By substracting (5.2) from the second formula of (2.4), we get that

$$\begin{aligned} \frac{1}{\tau}(e_N^{k+1} - e_N^k, v_N)_N + \frac{1}{2}\mu A(e_N^k + e_N^{k+1}, v_N) &= \frac{1}{2}(f^k + f^{k+1}, v_N)_N - \frac{1}{2}(f^k + f^{k+1}, v_N) \\ +\frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}, v_N) &- \frac{1}{\tau}(u_N^{\star,k+1} - u_N^{\star,k}, v_N)_N - (E_{1,N}^k, v_N) - \mu A(E_{2,N}^k, v_N) \\ +(E_3^k, v_N) - \frac{1}{2}\sum_{m=k,k+1} G(u_N^{\star,m}, u^{\star,k}, v_N) &- \frac{1}{2}B_N(u_N^k, u_N^k, v_N) + \frac{1}{2}B(u_N^{\star,k}, u_N^{\star,k}, v_N) \\ -\frac{1}{2}B_N(u_{N,pre}^{k+1}, u_{N,pre}^{k+1}, v_N) &+ \frac{1}{2}B(u_N^{\star,k+1}, u_N^{\star,k+1}, v_N). \end{aligned} \tag{5.4}$$

The combination of (5.3) with (5.4) brings

$$\begin{aligned}\frac{1}{\tau}(e_{N,pre}^{k+1}-e_N^{k+1},v_N)_N+\frac{1}{2}\mu A(e_{N,pre}^{k+1}-e_N^{k+1},v_N) &= -(E_3^k,v_N)_N-\frac{1}{2}G(u_N^{\star,k},u^{\star,k},v_N)\\ &+\frac{1}{2}G(u_N^{\star,k+1},u^{\star,k},v_N)-\frac{1}{2}(B_N(u_N^k,u_N^k,v_N)-B(u_N^{\star,k},u_N^{\star,k},v_N))\\ &-\frac{1}{2}(B_N(u_{N,pre}^{k+1},u_{N,pre}^{k+1},v_N)-B(u_N^{\star,k+1},u_N^{\star,k+1},v_N)).\end{aligned} \tag{5.5}$$

Step II. We rewrite (5.4) as

$$\frac{1}{\tau}(e_N^{k+1}-e_N^k,v_N)_N+\frac{1}{2}\mu A(e_N^k+e_N^{k+1},v_N)=\sum_{l=1}^{5}X_l^k(v_N),\forall v_N\in P_N^0(\Lambda) \tag{5.6}$$

where

$$\begin{aligned}X_1^k(v_N) &= -(E_{1,N}^k,v_N)-\mu A(E_{2,N}^k,v_N)+(E_3^k,v_N)-\frac{1}{2}\sum_{m=k,k+1}G(u_N^{\star,m},u^{\star,k},v_N),\\ X_2^k(v_N) &= \frac{1}{2}(f^k+f^{k+1},v_N)_N-\frac{1}{2}(f^k+f^{k+1},v_N),\\ X_3^k(v_N) &= \frac{1}{\tau}(u_N^{\star,k+1}-u_N^{\star,k},v_N)-\frac{1}{\tau}(u_N^{\star,k+1}-u_N^{\star,k},v_N)_N,\\ X_4^k(v_N) &= -\frac{1}{2}B_N(u_N^k,u_N^k,v_N)+\frac{1}{2}B(u_N^{\star,k},u_N^{\star,k},v_N),\\ X_5^k(v_N) &= -\frac{1}{2}B_N(u_{N,pre}^{k+1},u_{N,pre}^{k+1},v_N)+\frac{1}{2}B(u_N^{\star,k+1},u_N^{\star,k+1},v_N).\end{aligned}$$

Taking $v_N=e_N^k+e_N^{k+1}$ in (5.6), we derive that

$$\frac{1}{\tau}(\| e_N^{k+1}\|_N^2-\| e_N^{k+1}\|_N^2)+\frac{1}{2}\mu| e_N^k+e_N^{k+1}|_1^2=\sum_{l=1}^{5}X_l^k(e_N^k+e_N^{k+1}). \tag{5.7}$$

We first estimate $| X_1^k(v_N)|$. We assert that

$$X_1^k(v_N)=\sum_{l=1}^{7}B_l^k(v_N) \tag{5.8}$$

where

$$\begin{aligned}B_1^k(v_N) &= \frac{1}{2}<\int_{k\tau}^{(k+1)\tau}\frac{\partial^2 f}{\partial t^2}((k+1)\tau-t)dt,v_N>_{L(H^{-1},H_0^1)},\\ B_2^k(v_N) &= -(E_{1,N}^k-\frac{1}{2}\tau\frac{\partial^2 u^{\star,k}}{\partial t^2},v_N),\\ B_3^k(v_N) &= -\mu A(E_{2,N}^k-\frac{1}{2}\tau\frac{\partial u^{\star,k}}{\partial t},v_N),\\ B_4^k(v_N) &= -\frac{1}{2}B(u_N^{\star,k+1},u^{\star,k+1},v_N),\\ B_5^k(v_N) &= -\frac{1}{2}B(u_N^{\star,k},u^{\star,k},v_N),\\ B_6^k(v_N) &= -\frac{1}{2}B(u^{\star,k+1}-u^{\star,k},u^{\star,k+1}-u^{\star.k},v_N),\\ B_7^k(v_N) &= -B(u^{\star,k},\int_{k\tau}^{(k+1)\tau}\frac{\partial^2 u^{\star}}{\partial t^2}((k+1)\tau-t)dt,v_N).\end{aligned}$$

In order to prove (5.8), we differentiate (2.2) with respect to the variable t to obtain

$$<\frac{\partial f}{\partial t},v>_{L(H^{-1},H_0^1)}=(\frac{\partial^2 u^{\star}}{\partial t^2},v)+\mu A(\frac{\partial u^{\star}}{\partial t},v)+2B(u^{\star},\frac{\partial u^{\star}}{\partial t},v). \tag{5.9}$$

On the other hand, we have that

$$f^{k+1} - f^k = \tau \frac{\partial f}{\partial t}(t_k) + \int_{k\tau}^{(k+1)\tau} \frac{\partial^2 f}{\partial t^2}(t)((k+1)\tau - t)dt. \qquad (5.10)$$

Since

$$< f, v >_{L(H^{-1}, H_0^1)} = (f, v), \quad \forall f \in L^2(\Lambda), \quad \forall v \in H_0^1(\Lambda).$$

the combination of (5.9) with (5.10) indicates that

$$(E_3^k, v_N) = \frac{1}{2}\tau(\frac{\partial^2 u}{\partial t^2}^{\star,k}, v_N) + \frac{\mu}{2}\tau A(\frac{\partial u}{\partial t}^{\star,k}, v_N) + \tau B(u^{\star,k}, \frac{\partial u}{\partial t}^{\star,k}, v_N) + B_1^k(v_N).$$

Thus

$$-(E_1^k, v_N) - \mu A(E_{2,N}^k, v_N) + (E_3^k, v_N) = \sum_{l=1}^{3} B_l^k(v_N) + \tau B(u^{\star,k}, \frac{\partial u^{\star,k}}{\partial t}, v_N). \qquad (5.11)$$

On the other hand, thanks to (5.11), we get

$$G(u_N^{\star,k+1}, u^{\star,k}, v_N) = G(u^{\star,k+1}, u^{\star,k}, v_N) - 2B_4^k(v_N). \qquad (5.12)$$

Adding $-2B_5^k(v)$ to the both sides of this identity yields,

$$\sum_{m=k,k+1} G(u_N^{\star,m}, u^{\star,k}, v_N) = G(u^{\star,k+1}, u^{\star,k}, v_N) - 2B_4^k(v_N) - 2B_5^k(v_N). \qquad (5.13)$$

Furthermore, (5.1) gives

$$G(u^{\star,k+1}, u^{\star,k}, v_N) = 2B(u^{\star,k}, u^{\star,k+1} - u^{\star,k}, v_N) - 2B_6^k(v_N) \qquad (5.14)$$

and (5.10) brings

$$B(u^{\star,k}, u^{\star,k+1} - u^{\star,k}, v_N) = \tau B(u^{\star,k}, \frac{\partial u}{\partial t}^{\star,k}, v_N) - B_7^k(v_N).$$

This identity together with (5.14) imply that

$$G(u^{\star,k+1}, u^{\star,k}, v_N) = 2\tau B(u^{\star,k}, \frac{\partial u}{\partial t}^{\star,k}, v_N) - 2B_6^k(v_N) - 2B_7^k(v_N). \qquad (5.15)$$

The combination of (5.13) with (5.15) leads to

$$\sum_{m=k,k+1} G(u_N^{\star,m}, u^{\star,k}, v_N) = 2\tau B(u^{\star,k}, \frac{\partial u}{\partial t}^{\star,k}, v_N) - 2\sum_{l=4}^{7} B_l^k(v_N).$$

This equality together with (5.11), lead to (5.8).

We now turn to estimate $| B_l^k(e_N^k + e_N^{k+1}) |$ for $1 \le l \le 7$. Let $D > 0$ be an undetermined constant. Firstly by the property of Bochner intergral and Poincaré inequality,

$$\begin{aligned} | B_1^k(v_N) | &\le \frac{1}{2}\int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 f}{\partial t^2} \|_{-1} ((k+1)\tau - t)dt \| v_N \|_1 \\ &\le \frac{1}{2}\tau^{\frac{3}{2}}(\int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 f}{\partial t^2} \|_{-1}^2 dt)^{\frac{1}{2}} | v_N |_1 \\ &\le D | v_N |_1^2 + C_1(D)\tau^3 \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 f}{\partial t^2} \|_{-1}^2 dt \end{aligned}$$

where $C_1(D) = \dfrac{1}{16D}$. Next, since

$$u^{\star,k+1} = u^{\star,k} + \tau \frac{\partial u}{\partial t}^{\star,k} + \frac{1}{2}\tau^2 \frac{\partial^2 u}{\partial t^2}^{\star,k} + \frac{1}{2}\int_{k\tau}^{(k+1)\tau} \frac{\partial^3 u^\star}{\partial t^3}(t)((k+1)\tau - t)^2 dt,$$

we can write

$$-B_2^k(v) = (Y_1, v) + < Y_2, v >_{L(H^{-1}, H_0^1)}, \qquad \forall v \in H_0^1(\Lambda)$$

where

$$\begin{aligned} Y_1 &= \frac{u_N^{\star,k+1} - u_N^{\star,k}}{\tau} - \frac{u^{\star,k+1} - u^{\star,k}}{\tau} = -\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} \frac{\partial}{\partial t}(u^\star(t) - u_N^\star(t))dt, \\ Y_2 &= \frac{1}{2\tau}\int_{k\tau}^{(k+1)\tau} \frac{\partial^3 u^\star}{\partial t^3}(t)((k+1)\tau - t)^2 dt. \end{aligned}$$

By the Cauchy inequality,

$$\| Y_1 \| \leq \frac{1}{\tau^{\frac{1}{2}}}\left(\int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u^\star(t) - u_N^\star(t)) \|^2 dt\right)^{\frac{1}{2}}. \tag{5.16}$$

Similarly,

$$\| Y_2 \|_{-1} \leq \frac{1}{2\sqrt{5}}\tau^{\frac{3}{2}}\left(\int_{k\tau}^{(k+1)\tau} \| \frac{\partial^3 u^\star}{\partial t^3}(t) \|_{-1}^2 dt\right)^{\frac{1}{2}}. \tag{5.17}$$

So, by (5.16), (5.17) and the Poincaré inequality (4.1),

$$\begin{aligned} | B_2^k(v_N) | &\leq \| Y_1 \| \| v_N \| + \| Y_2 \|_{-1} \| v_N \|_1 \\ &\leq 2D \mid v_N \mid_1^2 + \frac{1}{2D} \| Y_1 \|^2 + \frac{3}{4D} \| Y_2 \|_{-1}^2 \\ &\leq 2D \mid v_N \mid_1^2 + C_2(D)\tau^3 \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^3 u^\star}{\partial t^3}(t) \|_{-1}^2 \, dt \\ &\quad + \frac{C_2(D)}{\tau}\int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u^\star(t) - u_N^\star(t)) \|^2 \, dt \end{aligned}$$

where $C_2(D) = \dfrac{1}{2D} + \dfrac{3}{80D}$. Since $A(u^{\star,k} - u_N^{\star,k}, v_N) = 0$ for any $v_N \in P_N^0(\Lambda)$, we have that

$$-B_3^k(v_N) = \frac{1}{2}\mu\tau A\left(\frac{u^{\star,k+1} - u^{\star,k}}{\tau} - \frac{\partial u^{\star,k}}{\partial t}, v_N\right).$$

Furthermore,

$$\begin{aligned} \left| \frac{u^{\star,k+1} - u^{\star,k}}{\tau} - \frac{\partial u^{\star,k}}{\partial t} \right|_1 &= \left| \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} \frac{\partial^2 u^\star}{\partial t^2}(t)((k+1)\tau - t)dt \right|_1 \\ &\leq \frac{1}{\sqrt{3}}\tau^{\frac{1}{2}}\left(\int_{k\tau}^{(k+1)\tau} \mid \frac{\partial^2 u^\star}{\partial t^2}(t) \mid_1^2 \, dt\right)^{\frac{1}{2}}, \end{aligned} \tag{5.18}$$

and so

$$| B_3^k(v_N) | \leq D \mid v_N \mid_1^2 + C_3(D)\tau^3 \int_{k\tau}^{(k+1)\tau} \mid \frac{\partial^2 u^\star}{\partial t^2}(t) \mid_1^2 dt$$

where $C_3(D) = \dfrac{\mu^2}{3}C_1(D)$. Thanks to (5.1), we get

$$B_4^k(v_N) = -\frac{1}{2}B(u_N^{\star,k+1} + u^{\star,k+1}, u_N^{\star,k+1} - u^{\star,k+1}, v_N).$$

Thus we obtain from Lemma 4.5 that

$$\begin{aligned} B_4^k(v_N) &\le \frac{1}{4} \mid u_N^{\star,k+1} + u^{\star,k+1} \mid_1 \| u_N^{\star,k+1} - u^{\star,k+1} \| \| v_N \mid_1 \\ &\le \frac{1}{2} \| u^\star \|_{C(0,T;H_0^1(\Lambda))} \| u_N^{\star,k+1} - u^{\star,k+1} \| \| v_N \mid_1 \\ &\le D \mid v_N \mid_1^2 + C_4(D) \| u_N^{\star,k+1} - u^{\star,k+1} \|^2 \end{aligned} \tag{5.19}$$

where $C_4(D) = \dfrac{1}{16D} \| u^\star \|^2_{C(0,T;H_0^1(\Lambda))}$. Similarly, we have

$$\mid B_5^k(v_N) \mid \le D \mid v_N \mid_1^2 + C_4(D) \| u_N^{\star,k} - u^{\star,k} \|^2 . \tag{5.20}$$

By the Hölder inequality with $p = 4$ and $q = \dfrac{4}{3}$, we have

$$\int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1 dt \le \tau^{\frac{3}{4}} \left(\int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1^4 dt \right)^{\frac{1}{4}}.$$

Along with Lemma 4.5, this gives

$$\begin{aligned} \mid B_6^k(v_N) \mid &\le \frac{1}{2} \mid u^{\star,k+1} - u^{\star,k} \mid_1 \| u^{\star,k+1} - u^{\star,k} \| \| v_N \mid_1 \\ &\le \frac{1}{\sqrt{2}} \tau^{\frac{3}{2}} \left(\int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1^4 dt \right)^{\frac{1}{2}} \mid v_N \mid_1 \\ &\le D \mid v_N \mid_1^2 + C_5(D) \tau^3 \int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1^4 dt \end{aligned}$$

where $C_5(D) = 2C_1(D)$. Also, we deduce that

$$\begin{aligned} \mid B_7^k(v_N) \mid &\le \frac{1}{2} \mid u^{\star,k} \mid_1 \| \int_{k\tau}^{(k+1)\tau} \frac{\partial^2 u^\star}{\partial t^2}(t)((k+1)\tau - t) dt \| \| v_N \mid_1 \\ &\le D \mid v_N \mid_1^2 + C_6(D) \tau^3 \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 u^\star}{\partial t^2}(t) \|^2 dt \end{aligned}$$

where $C_6(D) = \dfrac{1}{3} C_4(D)$.

We now estimate $\mid X_2^k(v_N) \mid$ and $\mid X_3^k(v_N) \mid$. By the property of Gauss-Legendre-Lobatto quadrature formula, we have

$$(f^k, v_N)_N - (f^k, v_N) = (I_N f^k - P_{N-1} f^k, v_N)_N - (f^k - P_{N-1} f^k, v_N).$$

Moreover, from the Cauchy ineqality and Lemma 4.1, we deduce that

$$\begin{aligned} \mid (I_N f^k - P_{N-1} f^k, v_N)_N \mid &\le \| I_N f^k - P_{N-1} f^k \|_N \| v_N \|_N \\ &\le (2 + \frac{1}{N}) \| I_N f^k - P_{N-1} f^k \| \| v_N \| . \end{aligned}$$

Hence, we see from the Poincaré inequality that

$$\begin{aligned} \mid (f^k, v_N)_N - (f^k, v_N) \mid &\le \sqrt{3} (3 \| f^k - I_N f^k \| + 4 \| f^k - P_{N-1} f^k \|) \mid v_N \mid_1 \\ &\le D \mid v_N \mid_1^2 + C_7(D) (\| f^k - I_N f^k \|^2 + \| f^k - P_{N-1} f^k \|^2) \end{aligned}$$

where $C_7(D) = \dfrac{24}{D}$. Thus we get

$$\mid X_2^k(v_N) \mid \le 2D \mid v_N \mid_1^2 + C_7(D) \sum_{m=k,k+1} (\| f^m - I_N f^m \|^2 + \| f^m - P_{N-1} f^m \|^2). \tag{5.21}$$

Since

$$X_3^k(v_N) = \frac{1}{\tau}\left((\int_{k\tau}^{(k+1)\tau} \frac{\partial u_N}{\partial t}^\star dt, v_N) - (\int_{k\tau}^{(k+1)\tau} \frac{\partial u_N}{\partial t}^\star dt, v_N)_N \right)$$
$$= \frac{1}{\tau}\left((\int_{k\tau}^{(k+1)\tau} \frac{\partial}{\partial t}(u_N^\star - P_{N-1}u^\star)dt, v_N) - (\int_{k\tau}^{(k+1)\tau} \frac{\partial}{\partial t}(u_N^\star - P_{N-1}u^\star)dt, v_N)_N \right),$$

we have

$$\begin{aligned} | X_3^k(v_N) | &\leq (3+\frac{1}{N})\frac{1}{\tau^{\frac{1}{2}}}(\int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u_N^\star - \mathrm{P}_{N-1}u^\star) \|^2 dt)^{\frac{1}{2}} \| v_N \| \\ &\leq D \mid v_N \mid_1^2 + 2C_7(D)\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} (\| \frac{\partial}{\partial t}(u^\star - u_N^\star) \|^2 + \| \frac{\partial}{\partial t}(u^\star - \mathrm{P}_{N-1}u^\star) \|^2)dt. \end{aligned} \tag{5.22}$$

Finally we estimate $| X_4^k(v_N) |$ and $| X_5^k(v_N) |$. We rewrite $| X_4^k(v_N) |$ and $| X_5^k(v_N) |$ as

$$X_l^k(v_N) = X_{l,1}^k(v_N) + X_{l,2}^k(v_N) + X_{l,3}^k(v_N) + X_{l,4}^k(v_N), \quad l = 4, 5,$$

where

$$\begin{aligned} X_{4,1}^k(v_N) &= -\frac{1}{2}B_N(e_N^k, e_N^k, v_N), & X_{4,2}^k(v_N) &= -\frac{1}{2}B_N(e_N^k, u_N^{\star,k}, v_N), \\ X_{4,3}^k(v_N) &= -\frac{1}{2}B_N(u_N^{\star,k}, e_N^k, v_N), & X_{4,4}^k(v_N) &= -\frac{1}{4}((u_N^{\star,k})^2 - I_N(u_N^{\star,k})^2, \frac{\partial v_N}{\partial x}), \\ X_{5,1}^k(v_N) &= -\frac{1}{2}B_N(e_{N,pre}^{k+1}, e_{N,pre}^{k+1}, v_N), & X_{5,2}^k(v_N) &= -\frac{1}{2}B_N(e_{N,pre}^{k+1}, u_{N,pre}^{\star,k+1}, v_N), \\ X_{5,3}^k(v_N) &= -\frac{1}{2}B_N(u_N^{k+1}, e_{N,pre}^{k+1}, v_N), & X_{5,4}^k(v_N) &= -\frac{1}{4}((u_N^{\star,k+1})^2 - I_N(u_N^{\star,k+1})^2, \frac{\partial v_N}{\partial x}). \end{aligned}$$

By Lemma 4.1 and Lemma 4.3, we see that

$$\begin{aligned} | X_{4,1}^k(v_N) | &\leq \frac{N+1}{4\sqrt{2}} \| e_N^k \|_N^2 | v_N |_1 \leq D | v_N |_1^2 + C_{8,1}(D)N^2 \| e_N^k \|^4, \\ | X_{4,2}^k(v_N) | &\leq \frac{1}{4} \| e_N \|_N | u_N^{\star,k} |_1 | v_N |_1 \leq D | v |_1^2 + C_{8,2}(D) \| e_N \|^2, \\ | X_{4,3}^k(v_N) | &\leq \frac{1}{4} | u_N^{\star,k} |_1 \| e_N \|_N | v_N |_1 \leq D | v |_1^2 + C_{8,3}(D) \| e_N \|^2, \\ | X_{4,4}^k(v_N) | &\leq \frac{1}{4} \| (u_N^{\star,k})^2 - I_N(u_N^{\star,k})^2 \| | v_N |_1 \\ &\leq \frac{1}{4}(\| (I_N - I)((u^{\star,k})^2 - (u_N^{\star,k})^2) + (u^{\star,k})^2 - I_N(u^{\star,k})^2) \|) | v_N |_1 \\ &\leq D | v_N |_1^2 + C_{8,4}(D)(\| (I - I_N)((u^{\star,k})^2 - (u_N^{\star,k})^2) \|^2 \\ &+ \| (u^{\star,k})^2 - I_N(u^{\star,k})^2 \|^2) \end{aligned}$$

where $C_{8,1}(D) = \frac{9}{2}C_1(D), C_{8,2} = C_{8,3}(D) = \frac{3}{4}C_4(D), C_{8,4}(D) = \frac{1}{2}C_1(D)$. Thus,

$$\begin{aligned} | X_4^k(v_N) | &\leq 4D | v_N |_1^2 + 2C_{8,2}(D) \| e_N^k \|^2 + C_{8,1}(D)N^2 \| e_N^k \|^4 \\ &+ C_{8,4}(D)(\| (I - I_N)((u^{\star,k})^2 - (u_N^{\star,k})^2) \|^2 + \| (u^{\star,k})^2 - I_N(u^{\star,k})^2 \|^2). \end{aligned} \tag{5.23}$$

Similarly, we have that

$$\begin{aligned} | X_{5,1}^k(v_N) | &\leq D | v_N |_1^2 + C_{8,1}(D)N^2 \| e_{N,pre}^{k+1} \|^4, \\ | X_{5,2}^k(v_N) | &\leq 2D | v_N |_1^2 + C_{8,2}(D) \| e_N^{k+1} \|^2 + C_{8,2}(D) \| e_{N,pre}^{k+1} - e_N^{k+1} \|_N^2, \\ | X_{5,3}^k(v_N) | &\leq 2D | v_N |_1^2 + C_{8,2}(D) \| e_N^{k+1} \|^2 + C_{8,2}(D) \| e_{N,pre}^{k+1} - e_N^{k+1} \|_N^2, \\ | X_{5,4}^k(v_N) | &\leq D | v_N |_1^2 + C_{8,4}(D)(\| (I - I_N)((u^{\star,k})^2 - (u_N^{\star,k})^2) \|^2 \\ &+ \| (u^{\star,k})^2 - I_N(u^{\star,k})^2 \|^2), \end{aligned}$$

and thus

$$\begin{array}{rcl} | X_5^k(v_N) | & \leq & 6D \mid v_N |_1^2 + 2C_{8,2}(D) \parallel e_N^{k+1} \parallel^2 + 2C_{8,2}(D) \parallel e_{N,pre}^{k+1} - e_N^{k+1} \parallel^2 \\ & + & C_{8,1}(D)N^2 \parallel e_{N,pre}^{k+1} \parallel^4 + C_{8,4}(D)(\parallel (I-I_N)((u^{\star,k})^2 - (u_N^{\star,k})^2) \parallel^2 \\ & + & \parallel (u^{\star,k})^2 - I_N(u^{\star,k})^2 \parallel^2). \end{array} \tag{5.24}$$

Step III. So far, we have estimated the right-hand side of (5.6). It remains to estimate the terms $\parallel e_{N,pre}^{k+1} - e_N^{k+1} \parallel_N^2$ and $N^2 \parallel e_{N,pre}^{k+1} \parallel^4$ in (5.24). We now estimate the first term. Using the notations in the previous paragraphs and (5.12), We rewrite (5.5) as

$$\begin{array}{l} (e_{N,pre}^{k+1} - e_N^{k+1}, v_N)_N + \dfrac{1}{2}\mu\tau A(e_{N,pre}^{k+1} - e_N^{k+1}, v_N) = -\tau(E_3^k, v_N)_N \\ \qquad - \tau B_4^k(v_N) + \tau B_5^k(v_N) + \tau X_4^k(v_N) - \tau X_5^k(v_N) + \dfrac{1}{2}\tau G(u^{\star,k+1}, u^{\star,k}, v_N). \end{array} \tag{5.25}$$

Hereafter let D_i be some undetermined positive constants. We have from (5.1) and Lemma 4.5 that

$$\begin{array}{rcl} \dfrac{1}{2}\tau \mid G(u^{\star,k+1}, u^{\star,k}, v_N) \mid & \leq & \dfrac{1}{2}\tau \mid u^{\star,k+1} + u^{\star,k} |_1 | u^{\star,k+1} - u^{\star,k} |_1 \parallel v_N \parallel \\ & \leq & \tau^{\frac{3}{2}} \parallel u^\star \parallel_{C(0,T;H_0^1(\Lambda))} (\displaystyle\int_{k\tau}^{(k+1)\tau} | \dfrac{\partial u^\star}{\partial t}(t) |_1^2 \, dt)^{\frac{1}{2}} \parallel v_N \parallel_N \\ & \leq & D_2 \parallel v_N \parallel_N^2 + C_4(D_2)\tau^3 \displaystyle\int_{k\tau}^{(k+1)\tau} | \dfrac{\partial u^\star}{\partial t}(t) |_1^2 \, dt. \end{array}$$

Clearly,

$$\begin{array}{l} \tau \mid (E_3^k, v_N)_N \mid = \dfrac{\tau}{2} \mid (\displaystyle\int_{k\tau}^{(k+1)\tau} \dfrac{\partial}{\partial t} I_N f dt, v_N)_N \mid \leq \dfrac{\tau}{2} \displaystyle\int_{k\tau}^{(k+1)\tau} \parallel \dfrac{\partial}{\partial t} I_N f \parallel_N dt \parallel v_N \parallel_N \\ \quad \leq \dfrac{\sqrt{3}\tau}{2} \displaystyle\int_{k\tau}^{(k+1)\tau} (\parallel \dfrac{\partial}{\partial t}(f - I_N f) \parallel + \parallel \dfrac{\partial f}{\partial t} \parallel) dt \parallel v_N \parallel_N \\ \quad \leq \dfrac{\sqrt{3}}{2}\tau^{\frac{3}{2}}((\displaystyle\int_{k\tau}^{(k+1)\tau} \parallel \dfrac{\partial}{\partial t}(f - I_N f) \parallel^2 dt)^{\frac{1}{2}} + (\displaystyle\int_{k\tau}^{(k+1)\tau} \parallel \dfrac{\partial f}{\partial t} \parallel^2 dt)^{\frac{1}{2}}) \parallel v_N \parallel_N \\ \quad \leq D_2 \parallel v_N \parallel_N^2 + 6C_1(D_2)\tau^3 \displaystyle\int_{k\tau}^{(k+1)\tau} (\parallel \dfrac{\partial}{\partial t}(f - I_N f) \parallel^2 + \parallel \dfrac{\partial f}{\partial t} \parallel^2) dt. \end{array}$$

Set $D = D_1 = \dfrac{\mu}{40}$ in (5.19), (5.20), (5.23) and (5.24) and $D_2 = \dfrac{1}{4}$ in above estimations. Then by taking $v_N = e_{N,pre}^{k+1} - e_N^{k+1}$ in (5.25), we find that

$$\begin{array}{rcl} \parallel e_{N,pre}^{k+1} - e_N^{k+1} \parallel_N^2 & \leq & 4C_{8,2}(D_1)\tau(\parallel e^k \parallel^2 + \parallel e_{N,pre}^{k+1} \parallel^2) \\ & + & 2C_{8,1}(D_1)\tau N^2(\parallel e^k \parallel^4 + \parallel e_{N,pre}^{k+1} \parallel^4) + R_{k,N}^{(1)} \end{array} \tag{5.26}$$

where

$$\begin{array}{rcl} R_{k,N}^{(1)} & = & 2\tau \displaystyle\sum_{m=k,k+1} (C_4(D_1) \parallel u^{\star,m} - u_N^{\star,m} \parallel^2 + C_{8,4}(D_1)(\parallel (I-I_N)((u^{\star,m})^2 - (u_N^{\star,m})^2) \parallel^2 \\ & + & \parallel (u^{\star,m})^2 - I_N(u^{\star,m})^2 \parallel^2)) + 2\tau^3(C_4(D_2) \displaystyle\int_{k\tau}^{(k+1)\tau} | \dfrac{\partial u^\star}{\partial t}(t) |_1^2 \, dt \\ & + & 6C_1(D_2)(\displaystyle\int_{k\tau}^{(k+1)\tau} \parallel \dfrac{\partial}{\partial t}(f - I_N f) \parallel^2 dt + \displaystyle\int_{k\tau}^{(k+1)\tau} \parallel \dfrac{\partial f}{\partial t} \parallel^2 dt)). \end{array}$$

Step IV. Now, we estimate the terms involving $e_{N,pre}^{k+1}$. For this purpose, we take $v_N =$

$e_N^k + e_{N,pre}^{k+1}$ in (5.3), and get

$$\begin{aligned} &\| e_{N,pre}^{k+1} \|_N^2 - \| e_N^k \|_N^2 + \frac{1}{2}\mu\tau \mid e_N^k + e_{N,pre}^{k+1} \mid_1^2 = \tau((f^k, e_N^k + e_{N,pre}^{k+1})_N \\ &\quad - (f^k, e_N^k + e_{N,pre}^k)) + \tau X_3^k(e_N^k + e_{N,pre}^{k+1}) - \tau(E_{1,N}^k, e_N^k + e_{N,pre}^{k+1}) \\ &\quad - \mu\tau A(E_{2,N}^k, e_N^k + e_{N,pre}^{k+1}) + 2\tau X_4^k(e_N^k + e_{N,pre}^{k+1}) + 2\tau B_5^k(e_N^k + e_{N,pre}^{k+1}). \end{aligned} \tag{5.27}$$

Let $D_3 > 0$ be another undetermined constante. By the same procedure as that used to obtain (5.16) and (5.18), we obtain

$$\begin{aligned} \tau \mid (E_{1,N}^k, v_N) \mid &\le \tau(\| Y_1 \| + \| \frac{u^{\star,k+1} - u^{\star,k}}{\tau} - \frac{\partial u^{\star,k}}{\partial t} \|) \| v_N \| \\ &\le 2D_3\tau \mid v_N \mid_1^2 + C_2(D_3)\int_{k\tau}^{(k+1)\tau} (\| \frac{\partial}{\partial t}(u^\star(t) - u_N^\star) \|^2 + \tau^2 \| \frac{\partial^2 u^\star}{\partial t} \|^2)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \mu\tau \mid A(E_{2,N}^k, v_N) \mid &\le \frac{1}{2}\mu\tau \mid u^{\star,k+1} - u^{\star,k} \mid_1 \mid v_N \mid_1 \\ &\le D_3\tau \mid v_N \mid_1^2 + 3C_3(D_3)\tau^2 \int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1^2 dt. \end{aligned}$$

By putting $D = D_3 = \dfrac{\mu}{32}$ in (5.20), (5.21), (5.23), (5.24) and in above estimations and taking $v_N = e_N^k + e_{N,pre}^{k+1}$ in (5.27), we see that

$$\| e_{N,pre}^{k+1} \|_N^2 \le (3 + 4C_{8,2}(D_3)\tau) \| e_N^k \|^2 + 2C_{8,1}(D_3)\tau N^2 \| e_N^k \|^4 + R_{k,N}^{(2)}, \tag{5.28}$$

where

$$\begin{aligned} R_{k,N}^{(2)} &= C_7(D_3)\tau(\| f^k - I_N f^k \|^2 + \| f^k - P_{N-1} f^k \|^2) + 2C_4(D_3)\tau \| u^{\star,k} - u_N^{\star,k} \|^2) \\ &+ 2C_{8,4}(D_3)\tau(\| (I - I_N)((u^{\star,k})^2 - (u^{\star,k})^2) \|^2 + \| (u^{\star,k})^2 - I_N(u^{\star,k})^2 \|^2) \\ &+ 4C_7(D_3)(\int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u^\star - u_N^\star) \|^2 dt + \int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u^\star - P_{N-1}u^\star) \|^2 dt) \\ &+ C_2(D_3)\int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t}(u^\star - u_N^\star) \|^2 dt + C_2(D_3)\tau^2 \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 u^\star}{\partial t^2} \|^2 dt) \\ &+ 3C_3(D_3)\tau^2 \int_{k\tau}^{(k+1)\tau} \mid \frac{\partial u^\star}{\partial t}(t) \mid_1^2 dt. \end{aligned}$$

Moreover

$$\begin{aligned} N^2 \| e_{N,pre}^{k+1} \|_N^4 &\le 3(3 + 4C_{8,2}(D_3)\tau)^2 N^2 \| e_N^k \|^4 \\ &+ 12C_{8,1}^2(D_3)\tau^2 N^6 \| e_N^k \|^8 + 3N^2(R_{k,N}^{(2)})^2. \end{aligned} \tag{5.29}$$

Step V. Now we substitute (5.26),(5.28),(5.29) into (5.24), and then obtain that

$$\begin{aligned} \sum_{l=1}^5 \mid X_l^k(v_N) \mid &\le 21D \mid v_N \mid_1^2 + 2C_{8,2}(D) \| e_N^{k+1} \|^2 + M_1 \| e_N^k \|^2 \\ &+ M_2 N^2 \| e_N^k \|^4 + M_3\tau^2 N^6 \| e_N^k \|^8 + R_{k,N}^{(3)}, \end{aligned}$$

where

$$\begin{aligned} M_1 &= 4C_{8,2}(D) + 32C_{8,2}(D)C_{8,2}(D_1)\tau + 32C_{8,2}(D)C_{8,2}(D_1)C_{8,2}(D_3)\tau^2, \\ M_2 &= (C_{8,1}(D) + 16C_{8,2}(D)C_{8,2}(D_1)C_{8,1}(D_1)\tau)(1 + 3(3 + 4C_{8,2}(D_3)\tau)^2) \\ &\quad + 16C_{8,2}(D)C_{8,2}(D_1)C_{8,1}(D_3)\tau^2, \\ M_3 &= 12C_{8,1}^2(D_3)(C_{8,1}(D) + 4C_{8,2}(D)C_{8,1}(D_1)\tau), \end{aligned}$$

$$
\begin{aligned}
R_{k,N}^{(3)} = \ & \tau^3 \left\{ C_1(D) \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 f}{\partial t^2} \|_{-1}^2 \, dt + C_2(D) \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^3 u^\star}{\partial t^3} \|_{-1}^2 \, dt \right. \\
& + C_3(D) \int_{k\tau}^{(k+1)\tau} | \frac{\partial^2 u^\star}{\partial t^2} |_1^2 \, dt + C_5(D) \int_{k\tau}^{(k+1)\tau} | \frac{\partial u^\star}{\partial t} |_1^4 \, dt \\
& \left. + \ C_6(D) \int_{k\tau}^{(k+1)\tau} \| \frac{\partial^2 u^\star}{\partial t^2} \|^2 \, dt \right\} + C_4(D) \sum_{m=k,k+1} \| u^{\star,m} - u_N^{\star,m} \|^2 \\
& + C_7(D) \sum_{m=k,k+1} (\| f^m - I_N f^m \|^2 + \| f^m - P_{N-1} f^m \|^2) \\
& + C_{8,4}(D)(\| (I - I_N)((u^{\star,k})^2 - (u_N^{\star,k})^2) \|^2 + \| (u^{\star,k})^2 - I_N (u^{\star,k})^2 \|^2) \\
& + \frac{1}{\tau} \left\{ 4C_7(D) \int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t} (u^\star(t) - P_{N-1} u^\star(t)) \|^2 \, dt \right. \\
& \left. + \ (4C_7(D) + C_2(D)) \int_{k\tau}^{(k+1)\tau} \| \frac{\partial}{\partial t} (u^\star(t) - u_N^\star(t)) \|^2 \, dt \right\} + 2C_{8,2}(D) R_{k,N}^{(1)} \\
& + 8C_{8,2}(D) C_{8,2}(D_1) \tau R_{k,N}^{(2)} + 3(C_{8,1}(D) + 4C_{8,2}(D) C_{8,1}(D_1)\tau) N^2 (R_{k,N}^{(2)})^2.
\end{aligned}
$$

By taking $v_N = e_N^k + e_N^{k+1}$ and putting $D = \dfrac{\mu}{84}$, we obtain from (5.7) that

$$
\begin{aligned}
& \frac{1}{\tau}(\| e_N^{k+1} \|_N^2 - \| e_N^k \|_N^2) + \frac{1}{4}\mu \, | e_N^k + e_N^{k+1} |_1^2 \\
& \le 2C_4(D) \| e_N^{k+1} \|^2 + M_1 \| e_N^k \|^2 + M_2 N^2 \| e_N^k \|^4 + M_3 \tau^2 N^6 \| e_N^k \|^8 + R_{k,N}^{(3)}.
\end{aligned} \tag{5.30}
$$

Let $2C_4(D)\tau \le \dfrac{1}{2}$ and

$$
q^{(n)} = 2\tau \sum_{k=0}^{n} R_{k,N}^{(3)}.
$$

Using Lemma 4.1 and summing (5.30) for all $0 \le k \le n-1$, we obtain that

$$
\begin{aligned}
& \| e^n \|^2 + \frac{1}{2}\mu\tau \sum_{k=0}^{n-1} | e^k + e^{k+1} |_1^2 \le q^{(M)} \\
& \quad + 2\tau \sum_{k=0}^{n-1} \left\{ (2C_4(D) + M_1) \| e^k \|^2 + M_2 N^2 \| e^k \|^4 + M_3 \tau^2 N^6 \| e^k \|^8 \right\}.
\end{aligned}
$$

If $\tau = O(N^{-\lambda})$ with $\lambda > 0$, then there exists a positive constant $L_1 > 0$ such that $\tau \le L_1 N^{-\lambda}$. Let $C_0 = 2(2C_4(D) + M_1) + 2M_2 + 2M_3 L_1$. Then

$$
E(e^n) \le q^{(M)} + C_0 \tau \sum_{k=0}^{n} (E(e_N^k) + N^2 E^2(e_N^k) + N^{6-2\lambda} E^4(e_N^k)).
$$

Let $C^\star$ be a positive constant independent of τ and N. We know from Lemma 4.8 that if $q^{(M)} \exp\{C_0 T\} \le C^\star N^{-2}$, then $E(e_N^k) \le q^{(M)} \exp\{C_0 T\}$ valids for all $n\tau \le T$.

In order to get the rate of convergence, we need only to estimate $q^{(M)}$. For instance, we use Lemma 4.6 and Lemma 4.7 to find that

$$
\begin{aligned}
\sum_{k=0}^{n-1} (R_{k,N}^{(2)})^2 \ \le \ & C^\star \left\{ (\| \frac{\partial u^\star}{\partial t} \|_{L^2(0,T;H_0^1(\Lambda))}^4 + \| \frac{\partial^2 u^\star}{\partial t^2} \|_{L^2(0,T;H_0^1(\Lambda))}^4) \tau^4 \right. \\
& \left. + \ (\| u^\star \|_{L^2(0,T;H^m(\Lambda))}^4 + \| \frac{\partial u^\star}{\partial t} \|_{L^2(0,T;H^m(\Lambda))}^4) N^{-4m} \right\}.
\end{aligned}
$$

We can estimate the other terms in $R_{k,N}^{(3)}$ similarly and so

$$q^{(M)} \le M_1^\star(\tau^4 + N^{-2m} + \tau^5 N^2 + \tau N^{2-4m}), m \ge 1$$

where $M_1^\star > 0$ depends only on μ and the norms of $f, u^\star, u_0$ in the spaces mentioned above. Therefor,if

$$N^2(\tau^4 + N^{-2m} + \tau^5 N^2 + \tau N^{2-4m}) \le M_2^\star, \tag{5.31}$$

$M_2^\star$ being a positive constant independent of τ and N, then $q^{(M)} \exp\{C_0 T\} \le C^\star N^{-2}$. A sufficiet conditon for (5.31) is that

$$\tau = O(N^{-\lambda}), \quad \lambda = \frac{4}{5}, \quad m > 1.$$

Finally the above arguments together with the triangle inequality complete the proof.

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