Journal of Computational Mathematics, Vol.22, No.6, 2004, 873–880.

THE NEAREST BISYMMETRIC SOLUTIONS OF LINEAR MATRIX EQUATIONS *1)

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Abstract

The necessary and sufficient conditions for the existence of and the expressions for the bisymmetric solutions of the matrix equations (I) $A_1X_1B_1+A_2X_2B_2+\cdots+A_kX_kB_k=D$, (II) $A_1XB_1+A_2XB_2+\cdots+A_kXB_k=D$ and (III) $(A_1XB_1, A_2XB_2, \cdots, A_kXB_k)=(D_1, D_2, \cdots, D_k)$ are derived by using Kronecker product and Moore-Penrose generalized inverse of matrices. In addition, in corresponding solution set of the matrix equations, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm is given. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

Mathematics subject classification: 5A24, 65F30, 65F05.
Key words: Bisymmetric matrix, Matrix equation, Matrix nearness problem, Kronecker product, Frobenius norm, Moore-Penrose generalized inverse.

1. Introduction

Denote by R^n the set of all real n-component vectors, $R^{m\times n}$ the set of all $m\times n$ real matrices and $BSR^{n\times n}$ the set of all $n\times n$ real bisymmetric matrices (A symmetric matrix $A=(a_{ij})\in R^{n\times n}$ is called bisymmetric if $a_{ij}=a_{n+1-j,n+1-i}$ for all $1\le i,j\le n$). I_n represents the $n\times n$ identity matrix. $\|A\|_F, A^+$ and A^T stand for the Frobenius norm, Moore-Penrose generalized inverse and transpose of a matrix A, respectively. On $R^{m\times n}$ we define inner product: $\langle A,B\rangle = trace(B^TA)$ for all $A,B\in R^{m\times n}$, then $R^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm. For $A=(a_{ij})\in R^{m\times n}, B=(b_{ij})\in R^{p\times q}$, let $A\bigotimes B\in R^{mp\times nq}$ be the Kronecker product of A and B.

Various aspects for the solution of linear matrix equations have been investigated. For example, Baksalary and Kala [1], Chu [4], He [8], and Xu, Wei and Zheng [13] considered the nonsymmetric solution of the matrix equation $AXB+CXD=E$ by using Moore-Penrose generalized inverse and the generalized singular value decomposition of matrices, while Chang and Wang [3], Jameson [9] and Dai [6] considered the symmetric conditions on the solution of the matrix equations: $AXA^T+BYB^T=C$, $AX+YA=C$, $AX=YB$ and $AXB=C$. Zietak [14, 15] discussed the l_p-solution and chebyshev-solution of the matrix equation $AX+YB=C$. Dobovisek [7] discussed the minimal solution of the matric equation $AX-YB=0$. Chu [5], and Kucera [11] and Jameson [10] are, respectively, studied the nonsymmetric solution of the matrix equation $AXB+CXD=E$ and its special case $AX+XB=C$. Mitra [12], Chu [4] and the references therein studied the nonsymmetric solution of the matrix equation $(AXB, CXD)=(E,F)$.

In this paper, the following problems are considered

* Received January 7, 2003.
1) Research supported by National Natural Science Foundation of China.

Problem I. Given $X_i^* \in R^{n_i\times n_i}, A_i \in R^{p\times n_i}, B_i \in R^{n_i\times q}\ \ (i=1,2,\cdots,k)$ and $D\in R^{p\times q}$. Let

$$H_1 = \{[X_1, X_2, \cdots, X_k] : A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D, X_i \in BSR^{n_i\times n_i}\}, \quad (1.1)$$

find $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k] \in H_1$ such that

$$\begin{aligned}\|[\hat{X}_1, \cdots, \hat{X}_k] - [X_1^*, \cdots, X_k^*]\|_F \equiv & \ (\|\hat{X}_1 - X_1^*\|_F^2 + \|\hat{X}_2 - X_2^*\|_F^2 + \cdots + \|\hat{X}_k - X_k^*\|_F^2)^{\frac{1}{2}} \\ = & \ \min_{[X_1,\cdots,X_k]\in H_1} \|[X_1, \cdots, X_k] - [X_1^*, \cdots, X_k^*]\|_F. \quad (1.2)\end{aligned}$$

Problem II. Given $X^* \in R^{n\times n}, A_i \in R^{p\times n}, B_i \in R^{n\times q}\ \ (i=1,2,\cdots,k)$ and $D\in R^{p\times q}$. Let

$$H_2 = \{X \in BSR^{n\times n} : A_1XB_1 + A_2XB_2 + \cdots + A_kXB_k = D\}, \quad (1.3)$$

find $\hat{X} \in H_2$ such that

$$\|\hat{X} - X^*\|_F = \min_{X\in H_2} \|X - X^*\|_F. \quad (1.4)$$

Problem III. Given $X^* \in R^{n\times n}, A_i \in R^{p_i\times n}, B_i \in R^{n\times q_i}$ and $D_i \in R^{p_i\times q_i}\ \ (i=1,2,\cdots,k)$. Let

$$H_3 = \{X \in BSR^{n\times n} : A_1XB_1 = D_1, A_2XB_2 = D_2, \cdots, A_kXB_k = D_k\}, \quad (1.5)$$

find $\hat{X} \in H_3$ such that

$$\|\hat{X} - X^*\|_F = \min_{X\in H_3} \|X - X^*\|_F. \quad (1.6)$$

Using Kronecker product and Moore-Penrose generalized inverse of matrices, the necessary and sufficient conditions for the existence of and the explicit expressions for the solution of Problem I, II and III are derived. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

2. Solving Problems I, II and III

For matrix $A \in R^{m\times n}$, denotes by $vec(A)$ the following vector containing all the entries of matrix A:

$$vec(A) = [A(1,:), A(2,:), \cdots, A(n,:)]^T \in R^{mn}, \quad (2.1)$$

where $A(i,:)$ denote ith row of matrix A. For vector $\mathbf{x} \in R^{n^2}$, denote by $vec_n^{-1}(\mathbf{x})$ the following matrix containing all the entries of vector $\mathbf{x}$:

$$vec_n^{-1}(\mathbf{x}) = \begin{pmatrix} \mathbf{x}(1:n)^T \\ \mathbf{x}(n+1:2n)^T \\ \vdots \\ \mathbf{x}[(n-1)n+1:n^2]^T \end{pmatrix} \in R^{n\times n}, \quad (2.2)$$

where $\mathbf{x}(i:j)$ denotes elements i to j of vector $\mathbf{x}$.

Let

$$vec(BSR^{n\times n}) = \{vec(A) : A \in BSR^{n\times n}\} \subset R^{n^2}, \quad (2.3)$$

then the dimension of the subspace $vec(BSR^{n\times n})$ is $r = (n+1)^2/4$ when n is add or $r = n(n+2)/4$ when n is even. Suppose $w_1, w_2, \cdots, w_r$ is an orthonormal basis-set for $vec(BSR^{n\times n})$. For example, suitable w_i for $n = 3$ might be $w_1 = [\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}]^T$, $w_2 = [0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0]^T$, $w_3 = [0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0]^T$, $W_4 = [0, 0, 0, 0, 1, 0, 0, 0, 0]^T$. Consequently

$$W = [w_1, w_2, \cdots, w_r] \in R^{n^2\times r} \quad (2.4)$$

is a basis-matrix for $vec(BSR^{n\times n})$ and

$$W^TW = I_r, R[W] = vec(BSR^{n\times n}), R[W^T] = R^r, \tag{2.5}$$

where $R[W]$ (or $R[W^T]$) denotes the range of W (or W^T).

For bisymmetric A, let $\widetilde{vec}(A)$ denotes the vector of coordinates of $vec(A)$ with respect to the basis-set $w_1, w_2, \cdots, w_r$, i.e., with respect to the columns of W. Then, in view of (2.3),(2.4) and (2.5),

$$vec(A) = W\widetilde{vec}(A) \in R^{n^2}, \widetilde{vec}(A) = W^T vec(A) \in R^r \tag{2.6}$$

and

$$vec_n^{-1}(Wx) \in BSR^{n\times n} \tag{2.7}$$

for any vector $x \in R^r$.

The following lemma 2.1 and 2.2 are well known results, see, for instance, Baksalary and Kala [1] and Ben-Israel and Greville [2].

Lemma 2.1. *For any matrices A, B and X in suitable size, one gets*

$$vec(AXB) = (A \bigotimes B^T)vec(X). \tag{2.8}$$

Lemma 2.2. *The matrix equation $AX = B$, with $A \in R^{p\times m}$ and $B \in R^{p\times n}$, has a solution $X \in R^{m\times n}$ if and only if $AA^+B = B$. In that case it has the general solution $X = A^+B + (I_m - A^+A)G$, where $G \in R^{m\times n}$ is arbitrary matrix.*

Lemma 2.3. *Given $A \in R^{p\times m}$ and $B \in R^{p\times n}$, then the optimal approximation problem*

$$\min_{X\in R^{m\times n}} \|(I_m - A^+A)X - B\|_F \tag{2.9}$$

has a solution which can be expressed as $X = B + A^+G$, where $G \in R^{p\times n}$ is arbitrary matrix.

Proof. Applying the properties of Moore-Penrose generalized inverse and the inner product defined in space $R^{m\times n}$, we have

$$\|(I_m - A^+A)X - B\|_F^2 = \langle (I_m - A^+A)X - B, (I_m - A^+A)X - B\rangle$$

$$= \langle (I_m - A^+A)(X - B), (I_m - A^+A)(X - B)\rangle + \langle A^+AB, A^+AB\rangle$$

$$= \|(I_m - A^+A)(X - B)\|_F^2 + \|A^+AB\|_F^2.$$

Hence,

$$\min_{X\in R^{m\times n}} \|(I_m - A^+A)X - B\|_F$$

if and only if

$$\min_{X\in R^{m\times n}} \|(I_m - A^+A)(X - B)\|_F.$$

It is clear that $X = B + A^+G$, with $G \in R^{p\times n}$ be arbitrary, is the solution of the above optimal approximation problem. So, the solution of the optimal approximation problem (2.9) can be expressed as $X = B + A^+G$.

The following theorems are the mainly results in this paper.

Theorem 2.1. *Given $X_i^* \in R^{n_i\times n_i}, A_i \in R^{p\times n_i}, B_i \in R^{n_i\times q}$ $(i = 1, 2, \cdots, k)$ and $D \in R^{p\times q}$. Assume $W_i \in R^{n_i^2\times r_i}$ is the basis-matrix for subspace $vec(BSR^{n_i\times n_i})$, let*

$$A_0 = \left(\ (A_1 \bigotimes B_1^T)W_1, (A_2 \bigotimes B_2^T)W_2, \cdots, (A_k \bigotimes B_k^T)W_k\ \right), \tag{2.10}$$

then the set H_1 *is nonempty if and only if*

$$A_0 A_0^+ vec(D) = vec(D). \tag{2.11}$$

When the condition (2.11) is satisfied, H_1 *can be expressed as*

$$H_1 = \Big\{ \ [vec_{n_1}^{-1}(W_1\alpha(1:r_1)), \cdots, vec_{n_k}^{-1}(W_k\alpha(r_1+\cdots+r_{k-1}+1:r_1+\cdots+r_k))]: \\ \alpha = A_0^+ vec(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+A_0)G, G \in R^{r_1+r_2+\cdots+r_k} \ \Big\}. \tag{2.12}$$

In H_1*, there exists an unique* $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k]$ *that makes (1.2) held and*

$$\hat{X}_i = vec_{n_i}^{-1}[W_i\hat{\alpha}(r_1+r_2+\cdots+r_{i-1}+1:r_1+r_2+\cdots+r_i)], i = 1,2,\cdots,k, \tag{2.13}$$

where

$$\hat{\alpha} = A_0^+ vec(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+A_0)\begin{pmatrix} W_1^T vec(X_1^*) \\ W_2^T vec(X_2^*) \\ \vdots \\ W_k^T vec(X_k^*) \end{pmatrix}. \tag{2.14}$$

Proof. If the set H_1 is nonempty, then the matrix equation

$$A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D \tag{2.15}$$

has a solution $X_i \in BSR^{(n_i\times n_i)}$ $(i = 1,2,\cdots,k)$. From Lemma 2.1, we have

$$(A_1 \bigotimes B_1^T)vec(X_1) + (A_2 \bigotimes B_2^T)vec(X_2) + \cdots + (A_k \bigotimes B_k^T)vec(X_k) = vec(D), \tag{2.16}$$

which is, in view of (2.6), equivalent to

$$(A_1 \bigotimes B_1^T)W_1\widetilde{vec}(X) + (A_2 \bigotimes B_2^T)W_2\widetilde{vec}(X_2) + (A_k \bigotimes B_k^T)W_k\widetilde{vec}(X_k) = vec(D),$$

i.e.,

$$\Big(\ (A_1 \otimes B_1^T)W_1, (A_2 \otimes B_2^T)W_2, \cdots, (A_k \otimes B_k^T)W_k \ \Big)\begin{pmatrix} \widetilde{vec}(X_1) \\ \widetilde{vec}(X_2) \\ \vdots \\ \widetilde{vec}(X_k) \end{pmatrix} = vec(D). \tag{2.17}$$

Hence, we have from Lemma 2.2 and (2.10) that (2.11) is held.

Conversely, if (2.11) is held, we have from Lemma 2.2 that (2.17) has a solution which has explicit expression as

$$\alpha = \begin{pmatrix} \widetilde{vec}(X_1) \\ \widetilde{vec}(X_2) \\ \vdots \\ \widetilde{vec}(X_k) \end{pmatrix} = A_0^+ vec(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+A_0)G, \tag{2.18}$$

where $G \in R^{r_1+r_2+\cdots+r_k}$ is arbitrary. Hence, $vec(X_i) = W_i\widetilde{vec}(X_i) = W_1\alpha(r_1+r_2+\cdots+r_{i-1}+1:r_1+r_2+\cdots+r_i)$ $(i = 1,2,\cdots,k)$, where the first equality gets from (2.6), are the general solution of the equation (2.16), i.e., $X_i = vec_{n_i}^{-1}[W_i\alpha(r_1+r_2+\cdots+r_{i-1}+1:r_1+r_2+\cdots+r_i)] \in BSR^{n_i\times n_i}$, which gets from (2.7), are the general solution of the matrix equation (2.15). This implies that the set H_1 is nonempty.

In addition, H_1 is a close convex set, so there is an unique $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k] \in H_1$ that makes (1.2) held. Choose $\widetilde{W}_i$ such that $(W_i, \widetilde{W}_i)$ $(i = 1, 2, \cdots, k)$ is an $n_i^2 \times n_i^2$ orthogonal matrix. Using the invariance of the Frobenius norm under orthogonal transformations, we have from (2.6) and (2.18) that

$$\begin{aligned}
&\|[X_1, X_2, \cdots, X_k] - [X_1^*, X_2^*, \cdots, X_k^*]\|_F^2 \\
&= \left\| \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ \vdots \\ vec(X_k) \end{pmatrix} - \begin{pmatrix} vec(X_1^*) \\ vec(X_2^*) \\ \vdots \\ vec(X_k^*) \end{pmatrix} \right\|_F^2 \\
&= \left\| \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & W_k \end{pmatrix} \begin{pmatrix} \widetilde{vec}(X_1) \\ \widetilde{vec}(X_2) \\ \vdots \\ \widetilde{vec}(X_k) \end{pmatrix} - \begin{pmatrix} vec(X_1^*) \\ vec(X_2^*) \\ \vdots \\ vec(X_k^*) \end{pmatrix} \right\|_F^2 \\
&= \left\| \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & W_k \end{pmatrix} \left[A_0^+ vec(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0)G \right] - \begin{pmatrix} vec(X_1^*) \\ vec(X_2^*) \\ \vdots \\ vec(X_k^*) \end{pmatrix} \right\|_F^2 \\
&= \left\| (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0)G + A_0^+ vec(D) - \begin{pmatrix} W_1^T vec(X_1^*) \\ W_2^T vec(X_2^*) \\ \vdots \\ W_k^T vec(X_k^*) \end{pmatrix} \right\|_F^2 + \left\| \begin{pmatrix} \widetilde{W}_1^T vec(X_1^*) \\ \widetilde{W}_2^T vec(X_2^*) \\ \vdots \\ \widetilde{W}_k^T vec(X_k^*) \end{pmatrix} \right\|_F^2 .
\end{aligned}$$

Hence, there exists $[X_1, X_2, \cdots, X_k] \in H_1$ such that $\|[X_1, X_2, \cdots, X_k] - [X_1^*, X_2^*, \cdots, X_k^*]\|_F^2 = \min$ is equivalent to exist $G \in R^{r_1+r_2+\cdots+r_k}$ such that

$$\left\| (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0)G + A_0^+ vec(D) - \begin{pmatrix} W_1^T vec(X_1^*) \\ W_2^T vec(X_2^*) \\ \vdots \\ W_k^T vec(X_k^*) \end{pmatrix} \right\|_F = \min .$$

From Lemma 2.3, we know that the solution of the above optimal approximate problem is

$$G = \begin{pmatrix} W_1^T vec(X_1^*) \\ W_2^T vec(X_2^*) \\ \vdots \\ W_k^T vec(X_k^*) \end{pmatrix} - A_0^+ vec(D) + A_0^+ G_0,$$

where $G_0 \in R^{r_1+r_2+\cdots+r_k}$ is arbitrary. Takeing G into (2.18) and furthermore using (2.6) and (2.7) we know that the solution of Problem I can expressed as (2.13).

Similar to the proof of Theorem 2.1, we can prove the following theorems 2.2 and 2.3.

Theorem 2.2. *Given $X^* \in R^{n\times n}, A_i \in R^{p\times n}, B_i \in R^{n\times q}$ $(i = 1, 2, \cdots, k)$ and $D \in R^{p\times q}$. Assume $W \in R^{n^2\times r}$ is the basis-matrix for subspace $vec(BSR^{n\times n})$, let*

$$A_0 = \left(\ (A_1 \bigotimes B_1^T) + (A_2 \bigotimes B_2^T) + \cdots + (A_k \bigotimes B_k^T) \ \right) W,$$

then the set H_2 is nonempty if and only if

$$A_0A_0^+vec(D) = vec(D).$$

Under this condition, H_2 can be expressed as

$$H_2 = \{vec_n^{-1}(W\alpha) \in BSR^{n\times n} : \alpha = A_0^+vec(D) + (I_r - A_0^+A_0)G, G \in R^r\}.$$

In H_2, there exists an unique $\hat{X}$ that makes (1.4) held and

$$\hat{X} = vec_n^{-1}[WA_0^+vec(D) + W(I_r - A_0^+A_0)W^Tvec(X^*)].$$

Theorem 2.3. *Given $X^* \in R^{n\times n}, A_i \in R^{p_i\times n}, B_i \in R^{n\times q_i}\quad (i = 1, 2, \cdots, k)$ and $D_i \in R^{p_i\times q_i}$. Assume $W \in R^{n^2\times r}$ is the basis-matrix for subspace $vec(BSR^{n\times n})$, let*

$$A_0 = \begin{pmatrix} A_1 \bigotimes B_1^T \\ A_2 \bigotimes B_2^T \\ \vdots \\ A_k \bigotimes B_k^T \end{pmatrix} W, \quad D_0 = \begin{pmatrix} vec(D_1) \\ vec(D_2) \\ \vdots \\ vec(D_k) \end{pmatrix},$$

then the set H_3 is nonempty if and only if

$$A_0A_0^+D_0 = D_0.$$

Under this condition, H_3 can be expressed as

$$H_3 = \left\{\ vec_n^{-1}(W\alpha) \in BSR^{n\times n} : \alpha = A_0^+D_0 + (I_r - A_0^+A_0)G, G \in R^r\ \right\}.$$

In H_3, there exists an unique $\hat{X}$ that makes (1.6) held and

$$\hat{X} = vec_n^{-1}[WA_0^+D_0 + W(I_r - A_0^+A_0)W^Tvec(X^*)].$$

3. The Algorithm Description and Numerical Examples

The discussion in section 2 provides us with a recipe for finding the solutions of Problem I, II and III. In summary, the method, for example Problem I, is as follows.

(1). Taking an orthogonal basis-set $\xi_1, \xi_2, \cdots, \xi_{r_i} \in R^{n_i^2}$ for $vec(BSR^{n_i\times n_i})\quad (i = 1, 2, \cdots, k)$, then construction a basis-matrix $W_i = (\xi_1, \xi_2, \cdots, \xi_{r_i}) \in R^{n_i^2\times r_i}$ for $vec(BSR^{n_i\times n_i})$.

(2). Let $A_0 = \left(\ (A_1 \bigotimes B_1^T)W_1, (A_2 \bigotimes B_2^T)W_2, \cdots, (A_k \bigotimes B_k^T)W_k\ \right)$.

(3). If $A_0^+A_0vec(D) = vec(D)$, go to step (4), otherwise go to step (5).

(4). According to (2.14) and (2.15) calculate $\hat{X}_i\quad (i = 1, 2, \cdots, k)$.

(5). Stop.

Using the above compute steps for solving Problem I, we now give an example to illustrate that the results obtained in this paper are correct.

Example 1. For Problem I, taking $k = 3, p = 4, q = 3, n_1 = 3, n_2 = 4, n_3 = 5$ and

$$X_1^* = \begin{pmatrix} -6 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & 7 & -9 \end{pmatrix}, X_2^* = \begin{pmatrix} 3 & -2 & 3 & 5 \\ -2 & 6 & -7 & 3 \\ 3 & -5 & 0 & -2 \\ 7 & 3 & -2 & 3 \end{pmatrix}, X_3^* = \begin{pmatrix} 3 & 1 & -7 & 6 & -2 \\ 5 & 5 & 7 & 5 & -9 \\ -4 & 3 & 5 & 4 & -1 \\ -5 & 4 & 3 & 1 & 8 \\ -3 & 6 & -1 & 7 & -9 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 4 & -3 & -2 \\ -2 & 4 & 1 \\ 1 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 2 & 3 & -4 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & -4 \\ 1 & 2 & 3 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ -2 & 3 & -4 & 5 & 1 \\ 2 & -3 & 2 & 4 & 5 \\ 1 & 2 & 2 & -2 & 2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 2 & -1 \\ 4 & 1 & -2 \\ 3 & 2 & -1 \end{pmatrix}, B_3 = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 4 & 3 \\ 2 & 1 & -1 \\ 2 & -3 & 2 \\ 1 & 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 49 & -52 & 29 \\ 84 & 21 & -6 \\ 75 & 6 & 60 \\ 33 & 19 & 32 \end{pmatrix}.$$

It is easy to verify that these given matrices satisfy the conditions of the Theorem 2.1. After calculating on the microcomputer through making program, we have an unique $\hat{X}_1 \in BSR^{3\times 3}, \hat{X}_2 \in BSR^{4\times 4}$ and $\hat{X}_3 \in BSR^{5\times 5}$ as follow:

$$\hat{X}_1 = \begin{pmatrix} -2.5545 & 2.9599 & -1.4397 \\ 2.9599 & 1.4630 & 2.9599 \\ -1.4397 & 2.9599 & -2.5545 \end{pmatrix}, \hat{X}_2 = \begin{pmatrix} 2.7080 & -2.2338 & 2.2628 & -0.1725 \\ -2.2338 & 2.0599 & -4.0876 & 2.2628 \\ 2.2628 & -4.0876 & 2.0599 & -2.2338 \\ -0.1725 & 2.2628 & -2.2338 & 2.7080 \end{pmatrix},$$

$$\hat{X}_3 = \begin{pmatrix} -1.5011 & 1.1569 & -1.4325 & -2.0473 & -1.4126 \\ 1.1569 & 1.3217 & 2.3337 & -1.0599 & -2.0473 \\ -1.4325 & 2.3337 & 7.8856 & 2.3337 & -1.4325 \\ -2.0473 & -1.0599 & 2.3337 & 1.3217 & 1.1569 \\ -1.4126 & -2.0473 & -1.4325 & 1.1569 & -1.5011 \end{pmatrix}.$$

Conclusions. This paper considered the bisymmetric conditions on the solution of the matrix equations: $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D, A_1XB_1 + A_2XB_2 + \cdots + A_kXB_k = D$ and $(A_1XB_1, A_2XB_2, \cdots, A_kXB_k) = (D_1, D_2, \cdots, D_k)$. Using the ideal that any real $n \times n$ bisymmetric matrix A can be described by an $(n+1)^2/4$-component vector when n is add or an $n(n+2)/4$-component vector when n is even, together with Kronecker product of matrices, solve these equations can be transformed into solves equation $Ax = b$. This method can also be used to finding Toeplitz, Hankel, Circulant or Browian matrix solutions of these equations. Of course, this method can only explore the small size of matrices because the size of the resulting matrices will, in general, be very large by using Kronecker products.

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