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THE UNCONDITIONAL STABLE DIFFERENCE METHODS WITH INTRINSIC PARALLELISM FOR TWO DIMENSIONAL SEMILINEAR PARABOLIC SYSTEMS*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In this paper we are going to discuss the difference schemes with intrinsic parallelism for the boundary value problem of the two dimensional semilinear parabolic systems. The unconditional stability of the general finite difference schemes with intrinsic parallelism is justified in the sense of the continuous dependence of the discrete vector solution of the difference schemes on the discrete data of the original problems in the discrete $W_2^{(2,1)}$ norms. Then the uniqueness of the discrete vector solution of this difference scheme follows as the consequence of the stability.

Key words: Difference Scheme, Intrinsic Parallelism, Two Dimensional Semilinear Parabolic System, Stability.

1. Introduction

1. In this paper we consider the boundary value problems for the two dimensional semilinear parabolic systems of second order of the form

$$u_t = A(x,y,t)(u_{xx}+u_{yy}) + B(x,y,t,u)u_x + C(x,y,t,u)u_y + f(x,y,t,u) \tag{1}$$

where $(x,y) \in \Omega = (0,l_1)\times(0,l_2)$, $t\in(0,T]$, and $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t), \cdots, u_m(x,y,t))$ is a m–dimensional vector unknown function ($m \ge 1$); $A(x,y,t)$, $B(x,y,t,u)$ and $C(x,y,t,u)$ are given $m\times m$ matrix functions, and $f(x,y,t,u)$ is a m–dimensional vector function and $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{yy} = \frac{\partial^2 u}{\partial y^2}$ and $u_t = \frac{\partial u}{\partial t}$ are the corresponding m–dimensional vector derivatives of the m–dimensional unknown vector function $u(x,y,t)$.

In the domain $Q_T = \bar{\Omega}\times[0,T]$ the homogeneous boundary conditions and the initial condition for the system (1) are

$$u(x,y,t) = 0, \quad (x,y)\in\partial\Omega, 0 < t \le T, \tag{2}$$

$$u(x,y,0) = \varphi(x,y), \quad (x,y)\in\Omega. \tag{3}$$

In [1]–[9] the general finite difference schemes with intrinsic parallelism for the linear and quasilinear parabolic problems have been discussed. For the one-dimensional quasilinear parabolic systems, in [8] some general difference schemes with intrinsic parallelism are constructed and proved to be unconditional stable and convergent. For the two dimensional quasilinear parabolic systems, in [9] some fundamental behaviors of general finite difference schemes with intrinsic

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parallelism are studied, i.e., the existence of the discrete vector solutions of the nonlinear difference system with intrinsic parallelism is proved, and the convergence of the discrete vector solutions of the certain difference schemes with intrinsic parallelism to the unique generalized solution of the original quasilinear parabolic problem is proved.

2. Difference Schemes with Intrinsic Parallelism

2. Divide the domain $Q_T = \{0 \le x \le l_1, 0 \le y \le l_2, 0 \le t \le T\}$ into small grids by the parallel planes $x = x_i$ $(i = 0, 1, \cdots, I)$, $y = y_j$ $(j = 0, 1, \cdots, J)$ and $t = t^n$ $(n = 0, 1, \cdots, N)$ with $x_i = ih_1$, $y_j = jh_2$ and $t^n = n\tau$, where $Ih_1 = l_1$, $Jh_2 = l_2$ and $N\tau = T$, I, J and N are integers and h_1, h_2 and τ are the steplengths of grids. Denote $h^* = \max(h_1, h_2) = h$, $h_* = \min(h_1, h_2)$. Denote $v_\Delta = \{v_{ij}^n | i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ the m–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_i, y_j, t^n) | i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the grid points.

Let us now construct the general difference schemes with intrinsic parallelism for the boundary value problem (1), (2) and (3):

$$\frac{v_{ij}^{n+1} - v_{ij}^n}{\tau} = A_{ij}^{n+1} \overset{*}{\Delta} v_{ij}^{n+1} + B_{ij}^{n+1} \bar{\delta}_x^1 v_{ij}^{n+1} + C_{ij}^{n+1} \bar{\delta}_y^1 v_{ij}^{n+1} + f_{ij}^{n+1}, \qquad (1)_\Delta$$
$$(i = 1, 2, \cdots, I-1; j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1),$$

where

$$\begin{aligned} \overset{*}{\Delta} v_{ij}^{n+1} &= \overset{*}{\delta}{}_x^2 v_{ij}^{n+1} + \overset{*}{\delta}{}_y^2 v_{ij}^{n+1} \\ &= \frac{v_{i+1,j}^{n+\lambda_{ij}^n} - 2v_{ij}^{n+1} + v_{i-1,j}^{n+\mu_{ij}^n}}{h_1^2} + \frac{v_{i,j+1}^{n+\bar{\lambda}_{ij}^n} - 2v_{ij}^{n+1} + v_{i,j-1}^{n+\bar{\mu}_{ij}^n}}{h_2^2}, \end{aligned}$$

$$\begin{aligned} A_{ij}^{n+1} &= A(x_i, y_j, t^{n+1}), \\ B_{ij}^{n+1} &= B(x_i, y_j, t^{n+1}, \tilde{\delta}^0 v_{ij}^{n+1}), \\ C_{ij}^{n+1} &= C(x_i, y_j, t^{n+1}, \hat{\delta}^0 v_{ij}^{n+1}), \\ f_{ij}^{n+1} &= f(x_i, y_j, t^{n+1}, \bar{\delta}^0 v_{ij}^{n+1}). \end{aligned} \qquad (4)$$

In this difference scheme, the expressions $\tilde{\delta}^0 v_{ij}^{n+1}$, $\hat{\delta}^0 v_{ij}^{n+1}$, $\bar{\delta}^0 v_{ij}^{n+1}$, and $\bar{\delta}_x^1 v_{ij}^{n+1}, \bar{\delta}_y^1 v_{ij}^{n+1}$ can be taken in the following manner. We can take

$$\begin{aligned} \tilde{\delta}^0 v_{ij}^{n+1} = \; & \lambda_{ij}^n \alpha_{1ij}^n v_{i+1j}^{n+1} + \mu_{ij}^n \alpha_{2ij}^n v_{i-1j}^{n+1} + \bar{\lambda}_{ij}^n \alpha_{3ij}^n v_{ij+1}^{n+1} \\ & + \bar{\mu}_{ij}^n \alpha_{4ij}^n v_{ij-1}^{n+1} + \alpha_{5ij}^n v_{ij}^{n+1} + \bar{\alpha}_{1ij}^n v_{i+1j}^n \\ & + \bar{\alpha}_{2ij}^n v_{i-1j}^n + \bar{\alpha}_{3ij}^n v_{ij+1}^n + \bar{\alpha}_{4ij}^n v_{ij-1}^n + \bar{\alpha}_{5ij}^n v_{ij}^n \end{aligned} \qquad (5)$$

such that the sum of coefficients equals to unit, that is

$$\lambda_{ij}^n \alpha_{1ij}^n + \mu_{ij}^n \alpha_{2ij}^n + \bar{\lambda}_{ij}^n \alpha_{3ij}^n + \bar{\mu}_{ij}^n \alpha_{4ij}^n + \alpha_{5ij}^n + \bar{\alpha}_{1ij}^n + \bar{\alpha}_{2ij}^n + \bar{\alpha}_{3ij}^n + \bar{\alpha}_{4ij}^n + \bar{\alpha}_{5ij}^n = 1$$

and the sum of the absolute value of these coefficients is uniformly bounded by any given constant with respect to the indices i, j and n. The coefficients are dependent on the indices i, j and n, this means they are different for different layers and different grid points. This shows that the choice of the coefficients has great degree of freedom.

For the expressions $\bar{\delta}_x^1 v_{ij}^{n+1}$ and $\bar{\delta}_y^1 v_{ij}^{n+1}$, we can take for example as

$$\begin{aligned} \bar{\delta}_x^1 v_{ij}^{n+1} = \; & d_{1ij}^n \frac{v_{i+1j}^{n+\lambda_{ij}^n} - v_{ij}^{n+1}}{h_1} + d_{2ij}^n \frac{v_{ij}^{n+1} - v_{i-1j}^{n+\mu_{ij}^n}}{h_2} \\ & + d_{3ij}^n \delta_x v_{ij}^n + d_{4ij}^n \delta_x v_{i-1j}^n, \end{aligned}$$

where

$$d_{1ij}^n + d_{2ij}^n + d_{3ij}^n + d_{4ij}^n = 1$$

and the sum of absolute values of the coefficients is uniformly bounded by any given constant with respect to the indices i, j and n.

By the similar principal we have the expression for $\hat{\delta}^0 v_{ij}^{n+1}$, $\bar{\delta}^0 v_{ij}^{n+1}$ and $\bar{\delta}_y^1 v_{ij}^{n+1}$ with an analogous behaviors.

The finite difference boundary conditions are of the form

$$v_{0j}^n = v_{Ij}^n = v_{i0}^n = v_{iJ}^n = 0, \qquad (2)_\Delta$$
$$(i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N).$$

The finite difference initial condition is of the form

$$v_{ij}^0 = \varphi_{ij}, \qquad (3)_\Delta$$
$$(i = 0, 1, \cdots, I; j = 0, 1, \cdots, J),$$

where $\varphi_{ij} = \varphi(x_i, y_j)$ $(i = 0, 1, \cdots, I; j = 0, 1, \cdots, J)$.

3. Suppose that the following conditions are satisfied.

(I) $A(x, y, t)$, $B(x, y, t, u)$, $C(x, y, t, u)$and $f(x, y, t, u)$ are continuous functions with respect to $(x, y, t) \in Q_T$ and continuously differentiable with respect to $u \in R^m$; and there are constants $A_0 > 0$, $B_0 > 0, C_0 > 0$ and $C > 0$ such that $|A(x, y, t)| \le A_0$, $|B(x, y, t, u)| \le B_0, |C(x, y, t, u)| \le C_0$, and $|f(x, y, t, u)| \le |f(x, y, t, 0)| + C|u|$. And there is a constant C such that $|B_u| + |C_u| + |f_u| \le C$ for all $u_1, u_2 \in R^m$, and all $(x, y, t) \in Q_T$.

(II) There is a constant $\sigma_0 > 0$, such that, for any vector $\xi \in R^m$, and for $(x, y, t) \in Q_T$,

$$(\xi, A(x, y, t)\xi) \ge \sigma_0 |\xi|^2.$$

(III) The initial value m–dimensional vector function $\varphi(x, y) \in C^1(\Omega)$ and $\varphi(x, y) = 0$ for $(x, y) \in \partial\Omega$.

(IV) Suppose that h^*/h_* is uniformly bounded as h_1 and h_2 tend to zero. Let $\Lambda \equiv \tau\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right)$. Assume τ is small such that $\Lambda\tau \le \tau_0$ for a positive constant τ_0 depending only on the known data.

4. In the following we shall use the symbols and notations in [8]. The following estimates and existence results are proved in [9].

Lemma 1. *Suppose that the conditions* **(I)**–**(IV)** *are fulfilled. Then the general finite difference scheme (1)$_\Delta$–(3)$_\Delta$ with intrinsic parallelism corresponding to the original problem (1), (2) and (3) has at least one discrete solution v_Δ, and there hold*

$$\max_{n=0,1,\cdots,N} \|v_\Delta^n\|_2^2 \le K, \qquad (6)$$

$$\max_{n=0,1,\cdots,N} \left(\|\delta_x v_\Delta^n\|_2^2 + \|\delta_y v_\Delta^n\|_2^2\right) + \sum_{n=0}^{N-1} \left\| \overset{*}{\Delta} v_\Delta^{n+1} \right\|_2^2 \tau + \sum_{n=0}^{N-1} \left\| \frac{v_\Delta^{n+1} - v_\Delta^n}{\tau} \right\|_2^2 \tau \le K, \qquad (7)$$

and

$$\left(\sum_{n=0}^{N-1} \left\| \Delta v_\Delta^{n+1} \right\|_2^2 \tau \right)^{\frac{1}{2}} \le K. \qquad (8)$$

3. Stability and Uniqueness

5. Now we prove the stability and uniqueness of the solution for the difference schemes $(1)_\Delta$–$(3)_\Delta$ with intrinsic parallelism.

Let the $m \times m$ coefficient matrix functions $\tilde{A}(x, y, t)$, $\tilde{B}(x, y, t, u)$, $\tilde{C}(x, y, t, u)$ and the m–dimensional vector function $\tilde{f}(x, y, t, u)$ are different from $A(x, y, t)$, $B(x, y, t, u)$, $C(x, y, t, u)$

and $f(x,y,t,u)$ only by some errors respectively. And also the m–dimensional vector function $\tilde{\varphi}(x,y)$ is different from $\varphi(x,y)$ by some errors. These mentioned matrix functions and vector functions also satisfy the similar conditions as **(I)**–**(III)**.

Suppose that the discrete vector function $\tilde{v} = \{\tilde{v}_{ij}^n | i = 0,1,\cdots,I; j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$ satisfies the finite difference system

$$\frac{\tilde{v}_{ij}^{n+1} - \tilde{v}_{ij}^n}{\tau} = \tilde{A}_{ij}^{n+1} \overset{*}{\Delta} \tilde{v}_{ij}^{n+1} + \tilde{B}_{ij}^{n+1} \bar{\delta}_x^1 \tilde{v}_{ij}^{n+1} + \tilde{C}_{ij}^{n+1} \bar{\delta}_y^1 \tilde{v}_{ij}^{n+1} + \tilde{f}_{ij}^{n+1}, \tag{9}$$
$$(i = 1,2,\cdots,I-1; j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1)$$

and

$$\tilde{v}_{0j}^n = \tilde{v}_{Ij}^n = 0, \quad \tilde{v}_{i0}^n = \tilde{v}_{iJ}^n = 0, \tag{10}$$
$$(i = 0,1,\cdots,I; j = 0,1,\cdots,J; n = 0,1,\cdots,N)$$
$$\tilde{v}_{ij}^0 = \tilde{\varphi}_{ij} \tag{11}$$
$$(i = 0,1,\cdots,I; j = 0,1,\cdots,J),$$

where

$$\tilde{A}_{ij}^{n+1} = \tilde{A}(x_i, y_j, t^{n+1}),$$
$$\tilde{f}_{ij}^{n+1} = \tilde{f}(x_i, y_j, t^{n+1}, \bar{\delta}^0 \tilde{v}_{ij}^{n+1}), \tag{12}$$

and $\tilde{B}_{ij}^{n+1}$ and $\tilde{C}_{ij}^{n+1}$ are defined similarly.

The finite difference system satisfied by the difference vector function $w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_{ij}^n = v_{ij}^n - \tilde{v}_{ij}^n | i = 0,1,\cdots,I; j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$ of the two discrete vector solutions v_Δ and $\tilde{v}_\Delta$ is of the form

$$\frac{w_{ij}^{n+1} - w_{ij}^n}{\tau} = A_{ij}^{n+1} \overset{*}{\Delta} w_{ij}^{n+1} + S_{ij}^{n+1} + R_{ij}^{n+1}, \tag{13}$$

where

$$\begin{aligned} S_{ij}^{n+1} &= \tilde{B}_{ij}^{n+1} \bar{\delta}_x^1 w_{ij}^{n+1} + \tilde{C}_{ij}^{n+1} \bar{\delta}_y^1 w_{ij}^{n+1} + B(v,\tilde{v})_{ij}^{n+1} \bar{\delta}^0 w_{ij}^{n+1} \\ &\quad + C(v,\tilde{v})_{ij}^{n+1} \bar{\delta}^0 w_{ij}^{n+1} + f(v,\tilde{v})_{ij}^{n+1} \bar{\delta}^0 w_{ij}^{n+1}, \\ R_{ij}^{n+1} &= A\left[\tilde{v}\right]_{ij}^{n+1} \overset{*}{\Delta} \tilde{v}_{ij}^{n+1} + B\left[\tilde{v}\right]_{ij}^{n+1} \bar{\delta}_x^1 \tilde{v}_{ij}^{n+1} \\ &\quad + C\left[\tilde{v}\right]_{ij}^{n+1} \bar{\delta}_y^1 \tilde{v}_{ij}^{n+1} + F\left[\tilde{v}\right]_{ij}^{n+1}, \end{aligned}$$
$$(i = 1,2,\cdots,I-1; j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1),$$

and

$$\begin{aligned} A\left[\tilde{v}\right]_{ij}^{n+1} &= A\left(x_i, y_j, t^{n+1}\right) - \tilde{A}\left(x_i, y_j, t^{n+1}\right), \\ F\left[\tilde{v}\right]_{ij}^{n+1} &= f\left(x_i, y_j, t^{n+1}, \bar{\delta}^0 v_{ij}^{n+1}\right) - \tilde{f}\left(x_i, y_j, t^{n+1}, \bar{\delta}^0 \tilde{v}_{ij}^{n+1}\right), \end{aligned}$$
$$B(v,\tilde{v})_{ij}^{n+1} = (\tilde{B}_u)_{ij}^{n+1} \bar{\delta}_x^1 \tilde{v}_{ij}^{n+1},$$
$$C(v,\tilde{v})_{ij}^{n+1} = (\tilde{C}_u)_{ij}^{n+1} \bar{\delta}_y^1 \tilde{v}_{ij}^{n+1},$$
$$f(v,\tilde{v})_{ij}^{n+1} = (\tilde{f}_p)_{ij}^{n+1},$$

where

$$(\tilde{B}_u)_{ij}^{n+1} = \int_0^1 B_u(x_i, y_j, t^{n+1}, s\bar{\delta}^0 v_{ij}^{n+1} + (1-s)\bar{\delta}^0 \tilde{v}_{ij}^{n+1}) ds,$$
$$(\tilde{f}_u)_{ij}^{n+1} = \int_0^1 f_u(x_i, y_j, t^{n+1}, s\bar{\delta}^0 v_{ij}^{n+1} + (1-s)\bar{\delta}^0 \tilde{v}_{ij}^{n+1}) ds,$$

and there is similar expression for $(C_u)_{ij}^{n+1}$, $B\left[\tilde{v}\right]_{ij}^{n+1}$ and $C\left[\tilde{v}\right]_{ij}^{n+1}$.

6. Making the scalar product of the vector $\overset{*}{\Delta}\, w_{ij}^{n+1} h_1 h_2 \tau$ and the vector equation (13) and then summing up the resulting products for $i = 1, 2, \cdots, I-1$ and $j = 1, 2, \cdots, J-1$, then we have

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}\, w_{ij}^{n+1}, w_{ij}^{n+1} - w_{ij}^{n}\right) h_1 h_2 \\
= \;& \tau \sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}\, w_{ij}^{n+1}, A_{ij}^{n+1}\,\overset{*}{\Delta}\, w_{ij}^{n+1}\right) h_1 h_2 \\
&+\tau \sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}\, w_{ij}^{n+1}, S_{ij}^{n+1} + R_{ij}^{n+1}\right) h_1 h_2.
\end{aligned}$$

By the same argument as that in section 3 of [9] we have the following inequality

$$\begin{aligned}
&\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 - \|\delta_x w_\Delta^{n}\|_2^2 - \|\delta_y w_\Delta^{n}\|_2^2 \\
&+2\tau \sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}\, w_{ij}^{n+1}, A_{ij}^{n+1}\,\overset{*}{\Delta}\, w_{ij}^{n+1}\right) h_1 h_2 \\
\le\;& 2\tau \left|\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}\, w_{ij}^{n+1}, S_{ij}^{n+1} + R_{ij}^{n+1}\right) h_1 h_2\right|,
\end{aligned}$$

which yields

$$\begin{aligned}
&\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 - \|\delta_x w_\Delta^{n}\|_2^2 - \|\delta_y w_\Delta^{n}\|_2^2 + \sigma_0 \tau \left\|\overset{*}{\Delta}\, w_\Delta^{n+1}\right\|_2^2 \\
\le\;& \frac{4\tau}{\sigma_0}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\left|S_{ij}^{n+1}\right|^2 + \left|R_{ij}^{n+1}\right|^2\right) h_1 h_2. \qquad (14)
\end{aligned}$$

By the assumption **(I)** and Lemma 3 in [9], we have

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|S_{ij}^{n+1}\right|^2 h_1 h_2 \\
&\le C\left(\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 + \|\delta_x w_\Delta^{n}\|_2^2 + \|\delta_y w_\Delta^{n}\|_2^2 + \Lambda\tau \left\|\frac{w_\Delta^{n+1} - w_\Delta^{n}}{\tau}\right\|_2^2\right. \\
&\quad \left. +\|w_\Delta^{n+1}\|_\infty^2 (\|\delta_x \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_y \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_x \tilde{v}_\Delta^{n}\|_2^2 + \|\delta_y \tilde{v}_\Delta^{n}\|_2^2)\right),
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|R_{ij}^{n+1}\right|^2 h_1 h_2 \le C&\left(\left\|A\,[\tilde{v}]_\Delta^{n+1}\right\|_\infty^2 \left\|\overset{*}{\Delta}\, \tilde{v}_\Delta^{n+1}\right\|_2^2 + \left\|B\,[\tilde{v}]_\Delta^{n+1}\right\|_2^2 (\|\delta_x \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_x \tilde{v}_\Delta^{n}\|_2^2)\right. \\
&\left. + \left\|C\,[\tilde{v}]_\Delta^{n+1}\right\|_2^2 (\|\delta_y \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_y \tilde{v}_\Delta^{n}\|_2^2) + \left\|F\,[\tilde{v}]_\Delta^{n+1}\right\|_2^2\right),
\end{aligned}$$

which C is a constant independent of the steplengthes h_1, h_2 and τ; and $\Lambda = \tau\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right)$.

Then, by substituting the above inequalities into (14), we obtain

$$\begin{aligned}
&\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 - \|\delta_x w_\Delta^{n}\|_2^2 - \|\delta_y w_\Delta^{n}\|_2^2 + \sigma_0 \tau \left\|\overset{*}{\Delta}\, w_\Delta^{n+1}\right\|_2^2 \\
\le\;& C\tau \left\{\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 + \|\delta_x w_\Delta^{n}\|_2^2 + \|\delta_y w_\Delta^{n}\|_2^2\right. \\
&+\Lambda\tau \left\|\frac{w_\Delta^{n+1} - w_\Delta^{n}}{\tau}\right\|_2^2 + \|w_\Delta^{n+1}\|_\infty^2 (\|\delta_x \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_y \tilde{v}_\Delta^{n+1}\|_2^2 + \|\delta_x \tilde{v}_\Delta^{n}\|_2^2 + \|\delta_y \tilde{v}_\Delta^{n}\|_2^2))
\end{aligned}$$

$$+\left\|A\,[\tilde v]_\Delta^{n+1}\right\|_\infty^2\left\|\overset{*}{\Delta}\,\tilde v_\Delta^{n+1}\right\|_2^2+\left\|B\,[\tilde v]_\Delta^{n+1}\right\|_2^2(\|\delta_x\tilde v_\Delta^{n+1}\|_2^2+\|\delta_x\tilde v_\Delta^{n}\|_2^2)$$
$$+\left\|C\,[\tilde v]_\Delta^{n+1}\right\|_2^2(\|\delta_y\tilde v_\Delta^{n+1}\|_2^2+\|\delta_y\tilde v_\Delta^{n}\|_2^2)+\left\|F\,[\tilde v]_\Delta^{n+1}\right\|_2^2\Big\}. \tag{15}$$

Denote by $\overline{RH}$ those terms in the { } at the right hand of the above inequality.

7. From (13) it follows

$$\tau\left\|\frac{w_\Delta^{n+1}-w_\Delta^n}{\tau}\right\|_2^2\le C\tau\left(\left\|\overset{*}{\Delta}\,w_\Delta^{n+1}\right\|_2^2+\overline{RH}\right).$$

Take $\tau>0$ small such that $C\Lambda\tau\le\frac12$. Then

$$\tau\left\|\frac{w_\Delta^{n+1}-w_\Delta^n}{\tau}\right\|_2^2\le C\tau\left(\left\|\overset{*}{\Delta}\,w_\Delta^{n+1}\right\|_2^2+\overline{RH}_1\right) \tag{16}$$

where $\overline{RH}_1$ is different from $\overline{RH}$ only without the term $\Lambda\tau\left\|\frac{w_\Delta^{n+1}-w_\Delta^n}{\tau}\right\|_2^2$.

8. Substituting (16) into (15) and letting τ small such that $C\Lambda\tau\le\frac{\sigma_0}{4}$ we have

$$\|\delta_x w_\Delta^{n+1}\|_2^2+\|\delta_y w_\Delta^{n+1}\|_2^2-\|\delta_x w_\Delta^{n}\|_2^2-\|\delta_y w_\Delta^{n}\|_2^2$$
$$+\frac{3\sigma_0}{4}\tau\left\|\overset{*}{\Delta}\,w_\Delta^{n+1}\right\|_2^2\le C\tau\overline{RH}_1. \tag{17}$$

Multiplying (16) by $\varepsilon>0$ and combining the resulting product with (17), and then taking $C\varepsilon\le\frac{\sigma_0}{4}$ we obtain

$$\|\delta_x w_\Delta^{n+1}\|_2^2+\|\delta_y w_\Delta^{n+1}\|_2^2-\|\delta_x w_\Delta^{n}\|_2^2-\|\delta_y w_\Delta^{n}\|_2^2$$
$$+\frac{\sigma_0}{2}\tau\left\|\overset{*}{\Delta}\,w_\Delta^{n+1}\right\|_2^2+\varepsilon\tau\left\|\frac{w_\Delta^{n+1}-w_\Delta^n}{\tau}\right\|_2^2\le C\tau\overline{RH}_1. \tag{18}$$

From the estimate (7) and the Lemma 2 in [9], we obtain, for $0\le n\le N-1$

$$\|\delta_x w_\Delta^{n+1}\|_2^2+\|\delta_y w_\Delta^{n+1}\|_2^2+\frac{\sigma_0}{2}\sum_{k=0}^{n}\left\|\overset{*}{\Delta}\,w_\Delta^{k+1}\right\|_2^2\tau+\varepsilon\sum_{k=0}^{n}\left\|\frac{w_\Delta^{k+1}-w_\Delta^k}{\tau}\right\|_2^2\tau$$
$$\le C\left(\sum_{k=0}^{n}\|w_\Delta^{k+1}\|_\infty^2\tau+r_0\right), \tag{19}$$

where we denote

$$r_0\equiv\|\delta_x w_\Delta^0\|_2^2+\|\delta_y w_\Delta^0\|_2^2+\max_{0\le n\le N-1}\left\|A[\tilde v]_\Delta^{n+1}\right\|_\infty^2$$
$$+\sum_{n=0}^{N-1}\left(\left\|B[\tilde v]_\Delta^{n+1}\right\|_2^2+\left\|C[\tilde v]_\Delta^{n+1}\right\|_2^2+\left\|F[\tilde v]_\Delta^{n+1}\right\|_2^2\right)\tau.$$

By Lemma 3 in [9], for any $\varepsilon_1>0$ there holds

$$\|w_\Delta^{n+1}\|_\infty^2\le\varepsilon_1\|\Delta w_\Delta^{n+1}\|_2^2+\frac{C}{\varepsilon_1}(\|\delta_x w_\Delta^{n+1}\|_2^2+\|\delta_y w_\Delta^{n+1}\|_2^2)$$
$$\le\varepsilon_1\left(\|\overset{*}{\Delta}\,w_\Delta^{n+1}\|_2^2+8\Lambda\left\|\frac{w_\Delta^{n+1}-w_\Delta^n}{\tau}\right\|_2^2\right)$$
$$+\frac{C}{\varepsilon_1}(\|\delta_x w_\Delta^{n+1}\|_2^2+\|\delta_y w_\Delta^{n+1}\|_2^2).$$

By taking ε_1 small such that $C\varepsilon_1 \le \min\left(\frac{\sigma_0}{4}, \frac{\varepsilon}{2}\right)$, we get

$$\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 + \frac{\sigma_0}{4}\sum_{k=0}^{n}\left\|\overset{*}{\Delta} w_\Delta^{k+1}\right\|_2^2 \tau + \frac{\varepsilon}{2}\sum_{k=0}^{n}\left\|\frac{w_\Delta^{k+1} - w_\Delta^k}{\tau}\right\|_2^2 \tau$$
$$\le C\left(\sum_{k=0}^{n}(\|\delta_x w_\Delta^k\|_2^2 + \|\delta_y w_\Delta^k\|_2^2)\tau + r_0\right). \tag{20}$$

Then, by Lemma 2 (ii) in [9] we obtain, for $n = 0, 1, \cdots, N-1$,

$$\|\delta_x w_\Delta^{n+1}\|_2^2 + \|\delta_y w_\Delta^{n+1}\|_2^2 + \sum_{k=0}^{n}\left\|\overset{*}{\Delta} w_\Delta^{k+1}\right\|_2^2 \tau$$
$$+\sum_{k=0}^{n}\left\|\frac{w_\Delta^{k+1} - w_\Delta^k}{\tau}\right\|_2^2 \tau \le C r_0. \tag{21}$$

This shows that the discrete solution v_Δ of the finite difference system $(1)_\Delta$–$(3)_\Delta$ in the discrete functional space $W_2^{(2,1)}(Q_\Delta)$ is continuously dependent on the discrete initial vector function $\varphi(x)$ in the discrete functional space of the form H^1, and the coefficient matrixes $A(x,y,t)$, $B(x,y,t,u)$, $C(x,y,t,u)$ and the nonlinear vector function $f(x,y,t,u)$ in definite sense respectively. We have proved the following stability theorem.

Theorem. *Under the assumptions* **(I)**–**(IV)**, *the following estimates hold for the difference vector function* $w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_{ij}^n = v_{ij}^n - \tilde{v}_{ij}^n | i = 0,1,\cdots,I; j = 0,1,\cdots,J; n = 0,1,\cdots,N\}$,

$$\|v_\Delta - \tilde{v}_\Delta\|^2_{W_2^{(2,1)}(Q_\Delta)} \le K\Big\{\|\varphi_\Delta - \tilde{\varphi}_\Delta\|^2_{H^1_\Delta}$$
$$+ \max_{1\le i\le I-1, 1\le j\le J-1, 0\le n\le N-1}\left|A(x_i,y_j,t^{n+1}) - \tilde{A}(x_i,y_j,t^{n+1})\right|^2$$
$$+\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\sup_{u\in R^m}\left|B(x_i,y_j,t^{n+1},u) - \tilde{B}(x_i,y_j,t^{n+1},u)\right|^2 h_1 h_2 \tau$$
$$+\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\sup_{u\in R^m}\left|C(x_i,y_j,t^{n+1},u) - \tilde{C}(x_i,y_j,t^{n+1},u)\right|^2 h_1 h_2 \tau$$
$$+\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\sup_{u\in R^m}\left|f(x_i,y_j,t^{n+1},u) - \tilde{f}(x_i,y_j,t^{n+1},u)\right|^2 h_1 h_2 \tau\Big\},$$

where K is a constant independent of steplengths h_1, h_2 and τ; and

$$\|\varphi_\Delta\|^2_{H^1_\Delta} = \|\varphi_\Delta\|_2^2 + \|\delta_x\varphi_\Delta\|_2^2 + \|\delta_y\varphi_\Delta\|_2^2,$$

and

$$\|w_\Delta\|^2_{W_2^{(2,1)}(Q_\Delta)} \equiv \max_{n=0,1,\cdots,N}\|w_\Delta^n\|^2_{H^1_\Delta} + \sum_{n=0}^{N-1}\left\|\overset{*}{\Delta} w_\Delta^{n+1}\right\|_2^2 \tau$$
$$+\sum_{n=0}^{N-1}\left\|\frac{w_\Delta^{n+1} - w_\Delta^n}{\tau}\right\|_2^2 \tau.$$

Corollary. *Suppose that the conditions of the Theorem above are satisfied. Then the discrete solution of the difference scheme (1)$_\Delta$–(3)$_\Delta$ is unique.*

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