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GENERALIZED NEKRASOV MATRICES AND APPLICATIONS*

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Abstract

In this paper, the concept of generalized Nekrasov matrices is introduced, some properties of these matrices are discussed, obtained equivalent representation of generalized diagonally dominant matrices.

Key words: Nekrasov matrix, Generalized Nekrasov matrix, Generalized diagonally dominant matrix.

1. Introduction

In matrix computations, the investigation of Nekrasov matrices is both important in theory and applications. The concept of generalized Nekrasov matrices is introduced in this paper. Let the set of complex (real) $n \times n$ matrices be $C^{n\times n}(R^{n\times n})$, and denoted:

$$r_i(A) = \sum_{j\neq i} |a_{ij}|, \qquad \forall\, i \in <n> = \{1, 2, \ldots, n\}$$

$$R_1(A) = r_1(A), \qquad R_i(A) = \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}|, 2 \leq i \leq n$$

$$\alpha(A) = \{i \in <n> \mid |a_{ii}| = R_i(A)\}, \qquad \beta(A) = \{i \in <n> \mid |a_{ii}| = r_i(A)\}$$

$$J_\alpha(A) = \{i \in <n> \mid |a_{ii}| > R_i(A)\}, \qquad J_\beta(A) = \{i \in <n> \mid |a_{ii}| > r_i(A)\}$$

$\forall \alpha = \{i_1 < i_2 < \ldots < i_k\} \subseteq <n>$, denote $\alpha' = <n> \setminus \alpha$, $A[\alpha]$ is the principal submatrix whose rows and columns are indexed by α, and $A[\alpha'] = A(\alpha)$. Denote the directed graph of A by $\Gamma(A)$, the sets $V(A)$ and $E(A)$ are called the vertex set and arc set, respectively.

Definition 1.1. Suppose $A = (a_{ij}) \in C^{n\times n}$ satisfies

$$|a_{ii}| \geq R_i(A), \qquad \forall\, i \in <n> \tag{1}$$

then A is called the weak Nekrasov matrix and denote $A \in N_0$; if all inequalities in (1) are strict, then A is called the Nekrasov matrix and denote $A \in N$; if there exists a permutation matrix P such $PAP^T \in N$, then A is called the quasi-Nekrasov matrix and denote $A \in \widetilde{N}$; if there exists a positive diagonal matrix X such $AX \in N$, then A is called the generalized Nekrasov matrix and denote $A \in N^*$; if $<n> = \alpha(A) \cup J_\alpha(A)$, $J_\alpha(A) \neq \phi$ and for any $i \in \alpha(A)$ there exists a path in $\Gamma(A) : i \to i_1 \to \ldots \to i_p \to j$ such $j \in J_\alpha(A)$, then A is called the Nekrasov matrix with nonzero element chain and denote $A \in CN$.

Definition 1.2. Suppose $A = (a_{ij}) \in C^{n\times n}$ satisfies

$$|a_{ii}| \geq r_i(A), \qquad \forall\, i \in <n> \tag{2}$$

then A is called the diagonally dominant matrix and denote $A \in D_0$; if all inequalities in (2) are strict, then A is called strictly diagonally dominant matrix and denote $A \in D$; if $A \in D_0$, $J_\beta(A) \neq \phi$, and for any $i \in \beta(A)$ there exists a path in $\Gamma(A) : i \to i_1 \to \ldots \to i_p \to j$ such $j \in J_\beta(A)$, then A is called the diagonally dominant matrix with nonzero element chain and

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denote $A \in CD$; if there exists a positive diagonal matrix X such $AX \in D$,then A is called the generalized strictly diagonally dominant matrix and denote $A \in D^*$.

Clearly if $a_{ii} \neq 0 (i \in < n >)$ then $D_0 \subset N_0$, $D \subset N$.

2. Results

Lemma 2.1. Let $A = (a_{ij}) \in C^{n\times n} \cap N$, then there exist a positive diagonal matrix X and a matrix $B \in CD$ such $A = BX$.

Proof. Without loss of generality assume $r_i(A) > 0$ for $\forall i \in < n >$, (if not for example $r_1(A) = 0$, then only discuss $A(1) \in N$). Denote

$$X_1 = diag(r_1(A)/|a_{11}|, 1, \dots, 1), \qquad AX_1 = A^{(1)} = (a_{ij}^{(1)})$$
$$X_2 = diag(1, r_2(A^{(1)})/|a_{22}^{(1)}|, 1, \dots, 1), \qquad A^{(1)}X_2 = A^{(2)} = (a_{ij}^{(2)})$$
$$\dots \qquad \dots$$
$$X_{n-1} = diag(1, \dots, 1, r_{n-1}(A^{(n-2)})/|a_{n-1n-1}^{(n-2)}|, 1), \quad A^{(n-2)}X_{n-1} = A^{(n-1)} = (a_{ij}^{(n-1)})$$

Moreover denote $X^{-1} = X_1X_2 \dots X_{n-1} = diag(d_1, d_2, \dots, d_{n-1}, 1)$. Then

$$d_i = r_i(A^{(i-1)})/|a_{ii}^{(i-1)}| = R(A)/|a_{ii}| < 1, \quad 1 \le i \le n-1$$

where $A^{(0)} = A$. Therefore $A^{(n-1)} = AX^{-1} = (a_{ij}^{(n-1)})$ satisfies

$$|a_{11}^{(n-1)}| = r_1(A) = R_1(A) \ge r_1(A^{(n-1)})$$
$$|a_{22}^{(n-1)}| = r_2(A^{(1)}) = R_2(A) \ge r_2(A^{(n-1)})$$
$$\dots$$
$$|a_{n-1n-1}^{(n-1)}| = r_{n-1}(A^{(n-2)}) = R_{n-1}(A) \ge r_{n-1}(A^{(n-1)})$$
$$|a_{nn}^{(n-1)}| = |a_{nn}| > R_n(A) = r_n(A^{(n-1)})$$

so $A^{(n-1)} \in D_0$ and $\beta(A^{(n-1)}) \neq < n >$. For first row of $A^{(n-1)}$, since $r_1(A^{(n-1)}) = \sum_{n>j>1} |a_{1j}|\ d_j + |a_{1n}|$, if there exists $j_0 \in < n-1 > \backslash \{1\}$ such $a_{1j_0} \neq 0$, then $r_1(A^{(n-1)}) < r_1(A) = |a_{11}^{(n-1)}|$, hence $1 \notin \beta(A^{(n-1)})$. If $a_{1j} = 0, \forall j \in < n-1 > \backslash \{1\}$, then $a_{1n} \neq 0$. Since $|a_{11}^{(n-1)}| = r_1(A) = |a_{1n}| > 0$ and $n \notin \beta(A^{(n-1)})$, so vertex $1 \in \beta(A^{(n-1)})$ and vertex n are adjoin.

For second row of $A^{(n-1)}$, since $r_2(A^{(n-1)}) = \sum_{j\neq 2} |a_{2j}|d_j + |a_{2n}|$, if there exists $j_0 \in < n-1 > \backslash \{1,2\}$ such $a_{2j_0} \neq 0$, then $r_2(A^{(n-1)}) < R_2(A) = d_1|a_{21}| + \sum_{j>2} |a_{2j}| = |a_{22}^{(n-1)}|$, hence $2 \in \beta(A^{(n-1)})$. If $a_{2j} = 0, \forall j \in < n-1 > \backslash \{1,2\}$, then $r_2(A^{(n-1)}) = d_1|a_{21}| + a_{2n} = R_2(A) = |a_{22}^{(n-1)}|$, i.e. $2 \in \beta(A^{(n-1)})$. If $a_{21} \neq 0$, then vertex 2 and vertex 1 are adjoin, and vertex 1 either satisfies $1 \notin \beta(A^{(n-1)})$ or $1 \in \beta(A^{(n-1)})$ but is adjoin with vertex $n \notin \beta(A^{(n-1)})$, thus vertex 2 is adjoin with vertex in set $< n > \backslash \beta(A^{(n-1)})$. If $a_{21} = 0$, then must be $a_{2n} \neq 0$, hence vertex 2 and vertex $n \notin \beta(A^{(n-1)})$ are adjoin. Therefore if vertex $2 \in \beta(A^{(n-1)})$, then there exists a path in $\Gamma(A)$ such vertex 2 and some vertex of set $< n > \backslash \beta(A^{(n-1)})$ are adjoin.

In general, for any $i \in < n-1 > \backslash \{1\}$, by above deduction we have that vertices $1, 2, \dots, i-1$ either not belong in $\beta(A^{(n-1)})$ or belong in $\beta(A^{(n-1)})$ but there exists a path in $\Gamma(A)$ such these vertices are adjoin with vertices of set $< n > \backslash \beta(A^{(n-1)})$. For i-th row of $A^{(n-1)}$, since $r_i(A^{(n-1)}) = \sum_{j\neq i}^{n-1} |a_{ij}|d_j + |a_{in}|$, if there exists $i < j_0 \le n-1$ such $a_{ij_0} \neq 0$, then $r_i(A^{(n-1)}) < R_i(A) = \sum_{j<i} |a_{ij}|d_j + \sum_{j>i} |a_{ij}| = |a_{ii}^{(n-1)}|$, hence $i \notin \beta(A^{(n-1)})$. If $a_{ij} = 0$

for $\forall j \in< n-1 > \setminus\{i\}$, then $r_i(A^{(n-1)}) = \sum_{j<i} |a_{ij}|d_j + |a_{in}| = R_i(A) = |a_{ii}^{(n-1)}|$, hence $i \in \beta(A^{(n-1)})$. There exist two cases: if $a_{in} \neq 0$, then vertex i and vertex $n \notin \beta(A^{(n-1)})$ are adjoin; if $a_{in} = 0$, then must exist $j_0 \in< i-1 >$ such $a_{ij_0} \neq 0$, hence vertex i and some vertex j_0 of set $< i-1 >$ are adjoin, by assumptions we have that either $j_0 \notin \beta(A^{(n-1)})$ or there exists a path in $\Gamma(A)$ such vertex j_0 and some vertex of set $< n > \setminus\beta(A^{(n-1)})$ are adjoin, thus vertex i and some vertex of set $< n > \setminus\beta(A^{(n-1)})$ must be adjoin. Therefore we deduce $A^{(n-1)} \in CD$.

Let $B = A^{(n-1)}$, the proof is completed.

Corollary 2.2. Let $A \in C^{n\times n} \cap N$, then there exist a positive diagonal matrix X and a matrix $B \in CD$ such $XAX^{-1} = B$.

Proof. By Lemma 2.1 there exist a positive diagonal matrix X and a matrix $B_0 \in CD$ such $A = B_0X$, i.e.,$XAX^{-1} + XB_0$. Let $B = XB_0$, then $B \in CD$.

Theorem 2.3. Let $A = (a_{ij}) \in C^{n\times n}$, if $A \in \widetilde{N}$, then there exist a positive diagonal matrix X and a matrix $B \in CD$ such $A = BX$; conversely, for any $B \in CD$ there exists a positive diagonal matrix X such $BX \in N$.

Proof. Let $A \in \widetilde{N}$, then there exists a permutation matrix P such $PAP^T \in N$. By Lemma 2.1 there exist a positive diagonal matrix X_1 and a matrix $B_1 \in CD$ such $PAP^T = B_1X_1$. Let $B = P^TB_1P, X = P^TX_1P$, then X is also positive diagonal matrix and $A = BX$. Since $B_1 \in D^*$([3]), hence $B \in D^*$, additionally $B \in D_0$ so $B \in CD$([3]).

Conversely assume $B = (b_{ij}) \in CD$ and denote $\beta(B) = \{j_t, t \in< s >\} \subset< n >$. Since $B \in CD$ hence there exist $t_1 \in< s >$ and $i_1 \in \beta'(B) = j_\beta(B)$ such $b_{j_{t_1}i_1} \neq 0$. Take $1 > \epsilon_1 > 0$ such $r_{i_1}(B) < |b_{i_1i_1}|\epsilon_1$, and let $X_1 = diag(1, \cdots, 1, \overset{i_1}{\epsilon_1}, 1, \cdots, 1)$. Then $B^{(1)} = BX_1 = (b_{ij}^{(1)})$ satisfies $B^{(1)} \in CD$ and for $\forall i \in \beta'(B)$

$$
\begin{aligned}
|b_{j_{t_1 1}j_{t_1}}^{(1)}| &= |b_{j_{t_1 1}j_{t_1}}| = r_{j_{t_1}}(B) = \sum_{j\neq j_{t_1}} |b_{j_{t_1}j}| \\
&> \epsilon_1|b_{j_{t_1}i_1}| + \sum_{j\neq j_{t_1} j\neq i_1} |b_{j_{t_1}j}| = r_{j_{t_1}}(B^{(1)}), \\
|b_{ti}^{(1)}| &= |b_{ii}| > r_i(B) \geq r_i(B^{(1)}),
\end{aligned}
$$

thus $\beta(B^{(1)}) = \beta(B) \setminus \{j_{t_1}\}$. If $\beta(B^{(1)}) = \phi$, then conclusion holds. If $\beta'(B^{(1)}) \neq< n >$, since $B^{(1)} \in CD$, then there exist $t_2 \in< s > \setminus\{t_1\}$ and $i_2 \in \beta(B^{(1)}) = \beta'(B) \cup \{j_{t_1}\}$ such $b_{j_{t_2}i_2}^{(1)} \neq 0$, take $1 > \epsilon_2 > 0$ such $r_{i_2}(B^{(1)}) < |b_{i_2i_2}|\epsilon_2$, and let $X_2 = diag(1, \cdots, 1, \overset{i_2}{\epsilon_2}, 1, \cdots, 1)$. Then can deduce that $B^{(2)} = B^{(1)}X_2$ satisfies $B^{(2)} \in CD$ and $\beta(B^{(2)}) = \beta(B^{(1)}) \setminus \{j_{t_2}\} = \beta(B) \setminus \{j_{t_1}, j_{t_2}\}$. If $\beta(B^{(2)}) = \phi$, then conclusion holds. If $\beta(B^{(2)}) \neq \phi$, then for $B^{(2)}$ continuously make above analogical deductions, we have that there exist at most $|\beta(B)|$ positive diagonal matrices $X_1, X_2, \cdots, X_{|\beta(B)|}$ such $BX_1X_2\cdots X_{|\beta(B)|} \in D$, denote $X = X_1X_2\cdots X_{|\beta(B)|}$, then X is also a positive diagonal matrix, and since $D \subset N$ hence $BX \in N$.

Remark 1. By Theorem 1 easy deduce $CD \subset N^*$.

Lemma 2.4. Let $A = (a_{ij}) \in N_0 \cap N^*$, then $J_\alpha(A) \neq \phi$.

Proof. If conclusion is false, then we have that

$$
\begin{aligned}
|a_{11}| &= R_1(A) = r_1(A) \\
|a_{22}| &= R_2(A) = |a_{21}|\frac{R_1(A)}{|a_{11}|} + \sum_{j>2} |a_{2j}| = \sum_{j\neq 2} |a_{2j}| = r_2(A)
\end{aligned}
$$

In general, $\forall i \in< n > \setminus\{1\}$, assume $|a_{i-1i-1}| = R_{i-1}(A) = r_{i-1}(A)$ then

$$
|a_{ii}| = R_i(A) = \sum_{j<i} |a_{ij}|\frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}| = \sum_{j\neq i} |a_{ij}| = r_i(A)
$$

i.e., $A \in D_0$ and $J_\beta(A) = \phi$. But since $A \in N^*$, hence there exists a positive diagonal matrix X_1 such $AX_1 \in N$, since $N \subset D^*$, so there exists a positive diagonal matrix X_2 such $AX_1X_2 \in D$. Denote $X = X_1X_2$, then X is a positive diagonal matrix, so $AX \in D$, i.e., $A \in D^*$. By [4] and $A \in D^*$, hence $J_\beta(A) \neq \phi$, this is a contradiction. Therefor $J_\alpha(A) \neq \phi$.

Lemma 2.5. Let $A = (a_{ij}) \in C^{n\times n} \cap N_0$ satisfy $a_{ii} \neq 0$ for any $i \in < n >$, and $J_\alpha(A) \neq \phi$, for any $i_0 \in \alpha(A)$ there exists a path in $\Gamma(A) : i_0 \to i_1 \to \ldots \to i_p \to j$ such $j \in J_\alpha(A)$. Then $A \in N^*$.

Proof. Assume $\alpha(A) = \{i_1 < i_2 < \cdots < i_k\}$, then there exist $j_0 \in \alpha(A)$ and $j_1 \in J_\alpha(A)$ such $a_{j_0j_1} \neq 0$. Take $1 > \epsilon_1 > 0$ such $|a_{j_1j_1}|\epsilon_1 > R_{j_1}(A)$, and denote $X_1 = \{1, \cdot, 1, \overset{i_1}{\epsilon_1}, 1, \cdots, 1\}$, $A_1 = AX_1 = (a_{ij}^{(1)})$. Then $\forall i \in \alpha(A) \setminus \{j_0\}$, if $i \leq j_1$

$$\begin{aligned} R_i(A_1) &= \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + |a_{ij_1}|\epsilon_1 + \sum_{j>i, j\neq j_1} |a_{ij}| \\ &\leq \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}| = R_i(A) = |a_{ii}| = |a_{ii}^{(1)}| \end{aligned}$$

if $i > j_1$

$$\begin{aligned} R_i(A_1) &= \sum_{j<i, j\neq j_1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + |a_{ij_1}|\epsilon_1 \frac{R_{j_1}(A_1)}{|a_{j_1j_1}|\epsilon_1} + \sum_{j>i, j\neq j_1} |a_{ij}| \\ &\leq \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}| = R_i(A) = |a_{ii}| = |a_{ii}^{(1)}| \end{aligned}$$

And for $j_0 \in \alpha(A)$, assume $j_1 > j_0$(if $j_1 < j_0$ may analogically deduce) , then

$$\begin{aligned} R_{j_0}(A_1) &= \sum_{j<j_0} |a_{j_0j}| \frac{R_j(A)}{|a_{jj}|} + |a_{j_0j_1}|\epsilon_1 + \sum_{j>j_0, j\neq j_1} |a_{j_0j}| \\ &< \sum_{j<j_0} |a_{j_0j}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>j_0} |a_{j_0j}| = R_{j_0}(A) = |a_{j_0j_0}| = |a_{j_0j_0}^{(1)}| \end{aligned}$$

For any $i \in \alpha(A) \setminus \{j_1\}$, if $i < j_1$

$$\begin{aligned} R_i(A_1) &= \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + |a_{ij_1}|\epsilon_1 + \sum_{j>i, j\neq j_1} |a_{ij}| \\ &\leq \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}| = R_i(A) < |a_{ii}| = |a_{ii}^{(1)}| \end{aligned}$$

if $i = j_1$

$$\begin{aligned} R_{j_1}(A_1) &= \sum_{j<j_1} |a_{j_1j}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>j_1} |a_{j_1j}| \\ &\leq \sum_{j<j_1} |a_{j_1j}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>j_1} |a_{j_1j}| = R_{j_1}(A) < \epsilon_1 |a_{j_1j_1}| = |a_{j_1j_1}^{(1)}| \end{aligned}$$

if $i > j_1$

$$\begin{aligned} R_i(A_1) &= \sum_{j<i, j\neq j_1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + |a_{ij_1}|\epsilon_1 \frac{R_{j_1}(A_1)}{|a_{j_1j_1}|\epsilon_1} + \sum_{j>i} |a_{ij}| \\ &\leq \sum_{j<i} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j>i} |a_{ij}| = R_i(A) < |a_{ii}| = |a_{ii}^{(1)}| \end{aligned}$$

Therefore $\alpha(A_1) = \alpha(A) \setminus \{j_0\}$, $J_\alpha(A_1) = J_\alpha(A) \cup \{j_0\}$, and $A_1 \in CN$. By above analogical productions we have that there exist at most k positive diagonal matrices $X_1, X_2, \cdots, X_k$ such

$AX_1X_2\cdots X_k \in N$. Denote $X = X_1X_2\cdots X_k$ then X is also positive diagonal matrix, hence $A \in N^*$. The proof is completed.

Corollary 2.6. Let $A = (a_{ij}) \in C^{n\times n} \cap N_0$ be irreducible and $J_\alpha(A) \neq \phi$, then $A \in N^*$.

Theorem 2.7. Let $A = (a_{ij}) \in C^{n\times n} \cap N_0$. The following statements are equivalent:

(1) $A \in D^*$.

(2) $A \in N^*$.

(3) $A \in CN$.

Proof. (3) $\Rightarrow$ (2): From Lemma 2.5 can deduce conclusion.

(2) $\Rightarrow$ (3): By Lemma 2.4 we have $J_\alpha(A) \neq \phi$. Suppose $\alpha(A) = \{i_1 < i_2 < \cdots < i_k\}$, $k \in< n >$. If conclusion is false, then $a_{ij} = 0$, $\forall i \in \alpha(A)$, $j \in J_\alpha(A)$. Thus $r_i(A) = \sum\limits_{j\in J_\alpha(A)\setminus\{i\}} |a_{ij}|$, $\forall i \in \alpha(A)$. Moreover for any $i_t \in \alpha(A)$, $t \in< n >$ we have

$$|a_{i_ti_t}| = R_{i_t}(A) = \sum_{j<i_t, j\in\alpha(A)} |a_{i_tj}|\frac{R_j(A)}{|a_{jj}|} + \sum_{j>i_t, j\in\alpha(A)} |a_{i_tj}| = r_{i_t}(A)$$

hence $A[\alpha(A)] \in D_0$ and either $\beta(A[\alpha(A)]) = \alpha(A)$ or $J_\beta(A[\alpha(A)]) = \phi$. But since $A \in D^*$ so $A[\alpha(A)] \in D^*$ ([5]), and $J_\beta(A[\alpha(A)]) \neq \phi$ ([4]). This is a contradiction, hence (3) holds.

(2) $\Rightarrow$ (1): By $A \in N^*$ there exists a positive diagonal matrix X_1 such $AX_1 \in N$. Moreover since $N \subset D^*$, so there exists a positive diagonal matrix X_2 such $AX_1X_2 \in D$. Let $X = X_1X_2$ then X is also positive diagonal matrix and by $AX \in D$ we have $A \in D^*$.

(1) $\Rightarrow$ (3): If $\alpha(A) =< n >$, by Lemma 2.4 we have $A \in D_0$ and $\beta(A) =< n >$ or $J_\beta(A) = \phi$ this contradits to $A \in D^*$. Thus $J_\beta(A) \neq \phi$. And if $A \notin CN$ by deduction of (2) $\Rightarrow$ (3) we deduce $A[\alpha(A)] \in D_0$ and $\beta(A[\alpha(A)]) = \alpha(A)$, i.e.,$J_\beta(A[\alpha(A)]) = \phi$, this contradicts to $A \in D^*$, so (3) holds. The proof is completed.

Lemma 2.8. Let $A = (a_{ij}) \in C^{n\times n}\cap N^*$, $\alpha = \{i_1 < i_2 < \cdot < i_k\} \subseteq< n >$, then $A[\alpha] \in N^*$.

Proof. Firstly prove that $A[\alpha] \in N$, $\forall \alpha \subset< n >$, when $A \in N$. Since

$$R_{i_1}(A[\alpha]) \leq \sum_{j<i_1} |a_{i_1j}|\frac{R_j(A)}{|a_{jj}|} + \sum_{j>j_1} |a_{i_1j}| = R_{i_1}(A) < |a_{i_1i_1}|,$$

and if suppose

$$R_{i_{t-1}}(A[\alpha]) \leq R_{i_{t-1}}(A) < |a_{i_{t-1}i_{t-1}}| \quad \forall t \in< k > \setminus\{1\},$$

then can deduce

$$R_{i_t}(A[\alpha]) = \sum_{j\in\alpha, j<i_t} |a_{i_tj}|\frac{R_j(A)}{|a_{jj}|} + \sum_{j\in\alpha, j>i_t} |a_{i_tj}| = R_{i_t}(A) < |a_{i_ti_t}|$$

thus $A[\alpha] \in N$. When $A \in N^*$ then exists a positive diagonal matrix X such $AX \in N$. Notice that $(AX)[\alpha] = A[\alpha]\cdot X[\alpha]$, $\forall \alpha \subseteq< n >$, and $(AX)[\alpha] \in N$, i.e., $A[\alpha]\cdot X[\alpha] \in N$, hence $A[\alpha] \in N^*$.

Lemma 2.9. Let $A = (a_{ij}) \in C^{n\times n} \cap N_0$, and $a_{ii} \neq 0$, for all $i \in< n >$. If $A[\alpha(A)] \in N^*$, then $A \in N^*$.

Proof. Must have $|\alpha(A)| < n$, if not, $J_\alpha(A) = \phi$ but $A[\alpha(A)] = A \in N^*$, this is a contraction. Denote $\alpha(A) = \{i_1 < i_2 < \cdots < i_k\}$. Since $R_i(A) \geq R_i(A[\alpha(A)])$,$\forall i \in \alpha(A)$, and $A \in N_0$, hence $A[\alpha(A)] \in N_0$. Moreover, since $A[\alpha(A)] \in N^*$, so $A[\alpha(A)] \in LN$, and $\phi \neq J_\alpha(A[\alpha(A)]) = \{i'_1 < i'_2 < \cdots, i'_t\} \subseteq \alpha(A)$, and there exist $j_1, j_2, \cdots, j_t \in J_\alpha(A)$ such $\alpha_{i'_1j_1}, \alpha_{i'_2j_2}, \cdots, \alpha_{i'_tj_t} \neq 0$ (if not then $R_i(A) = R_i(A[\alpha(A)]) = |a_{ii}|$,$\forall i \in \alpha(A)$ i.e., $J_\alpha(A[\alpha(A)]) = \phi$, this contradicts to $A[\alpha(A)] \in N^*$). If $t = k$,i.e., $J_\alpha(A[\alpha(A)]) = \alpha(A)$, then each vertex in $\alpha(A)$ is adjoin with some vertex in set $J_\alpha(A)$, hence $A \in LN$, i.e., $A \in N^*$.

If $t < k$, since $A[\alpha(A)] \in LN$, hence for any $i \in \alpha(A) \setminus J_\alpha(A[\alpha(A)])$ (i.e.,$i \in \alpha(A[\alpha(A)])$) there exists a path in $\Gamma(A[\alpha(A)])$ such vertex i and vertex in set $J_\alpha(A[\alpha(A)])$ are adjoin. Thus, for any $i \in \alpha(A)$, there exists a path in $\Gamma(A)$ such vertex i and vertex in set $J_\alpha(A)$ are adjoin, i.e., $A \in LN$, hence $A \in N^*$.

Theorem 2.10. Let $A = (a_{ij}) \in C^{n\times n} \cap N_0$. The following statements are equivalent:

(1) $A[\alpha(A)] \in D^*$

(2) $A[\alpha(A)] \in N^*$

(3) $A \in D^*$

(4) $A \in N^*$

(5) $A \in CN$

(6) there exists a positive integer $k \in< n >$ such set $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ satisfies $\phi = \alpha_k \subseteq \cdots \subseteq \alpha_1 \subseteq< n >$ and $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_k \neq< n >$, where $\alpha_1 = \alpha(A)$, $\alpha_j = \alpha(A_{j-1})$, $A_{j-1} = A[\alpha_{j-1}], j \in< k > \backslash\{1\}$.

Proof. By Theorem 2.7 can deduce (1) and (2) are equivalent, (3) and (4) with (5) are equivalent. By Lemma 2.8 and Lemma 2.9 we have (4) and (2) are equivalent.

(1) $\Rightarrow$ (6): Since $\alpha_1 = \alpha(A)$ and (1) and (3) are equivalent, hence $\alpha_1 \neq< n >$ ([4]). If $\alpha_1 = \phi$, the conclusion holds. If $\alpha_1 \neq \phi$, since $\alpha_2 = \alpha(A_1)$ and $A_1 \in D^*$ hence $\alpha_1 \neq \alpha_2$, i.e.,$\alpha_2 \subset \alpha_1$. If $\alpha_2 = \phi$, the conclusion holds, if not, since $A_2 \in D^*$([5]) so $\alpha_3 \subset \alpha_2, cdots$, continuously make above deductions, we have that there exist at most $k \leq |\alpha(A)|$ deductions, then must have $A_{k-1} \in N$ or $\alpha_k = \phi$.

(6) $\Rightarrow$ (4): By $A \in N_0$ we have $A_j \in N_0$,$\forall j \in< k-1 >$, again by $\alpha_k = \phi$ have $A_{k-2} \in N$, by (2) and (4) are equivalent, have $A_{k-1} \in N^*$, moreover have $A_{k-3} \in N^*, \cdots, A_1 = A[\alpha(A)] \in N^*$, finally deduce $A \in N^*$. The proof is completed.

Remark 2. When $A \in D_0$, for any $i \in \alpha(A)$ we have $|a_{ii}| = R_i(A) \leq r_i(A) \leq |a_{ii}|$, hence $i \in \beta(A)$, i.e., $\alpha \subset \beta(A)$. Hence Theorem 2.10 improved corresponding results in [1] and [2].

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