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CONSERVATION OF THREE-POINT COMPACT SCHEMES ON SINGLE AND MULTIBLOCK PATCHED GRIDS FOR HYPERBOLIC PROBLEMS*[1)]

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Abstract

For nonlinear hyperbolic problems, conservation of the numerical scheme is important for convergence to the correct weak solutions. In this paper the conservation of the well-known compact scheme up to fourth order of accuracy on a single and uniform grid is studied, and a conservative interface treatment is derived for compact schemes on patched grids. For a pure initial value problem, the compact scheme is shown to be equivalent to a scheme in the usual conservative form. For the case of a mixed initial boundary value problem, the compact scheme is conservative only if the rounding errors are small enough. For a patched grid interface, a conservative interface condition useful for mesh refinement and for parallel computation is derived and its order of local accuracy is analyzed.

Key words: Conservation, Compact scheme, Uniform grid, Multiblock patched grid.

1. Introduction

In recent years, the compact finite difference method receives an increasing interest due to its high order of accuracy. The compact finite difference method was developed in 1970s within a variety of frameworks. It was then realized that all these methods can be constructed in a systematic way [17]. In [21], a detailed exposition of compact schemes and derivation techniques was given. The history of the development of compact schemes including some works in 1980s is also briefly reviewed in [12]. Recent works in this direction were emphasized in discretization on nonuniform grid [8, 9, 19] or on non-staggered grids[20], method with spectral-like resolution [14], stability of initial-boundary value problem [2, 10] and optimal accuracy for a given grid and initial data [11], parallel treatment of compact schemes[18], control of the group velocity [7], nonlinear compact schemes[4, 5, 6], and mixing with other methods [3].

The main feature of the compact scheme is that the space derivative of the differential equation is computed implicitly and that it is not written in the usual conservative form. For nonlinear hyperbolic problem, conservation of the numerical scheme is required to ensure the solution to converge to a weak solution for vanishing mesh sizes[13]. A compact scheme is however not in the usual conservation form. An important and fundamental question is whether the compact scheme yields numerical solutions which, when the solution converges, converge to weak solutions for vanishing mesh size.

A second important issue is that compact schemes are mainly applied to compute flows in simple geometries with a structured grid. It is very hard to find study of compact schemes for complex geometries. A natural way to apply the compact schemes to complex geometries is obviously by domain decomposition. However, applying domain decomposition to compact schemes is not simple since the compact schemes are inherently implicit. It is not clear how

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to construct interface conditions to ensure independent (and hence parallel) solution of the compact schemes in each subdomain. Besides, for nonlinear hyperbolic problems, conservation at grid interfaces is very important [1, 24].

In this paper we will address these two important issues. First, we want to establish the equivalence between a compact scheme and a usual scheme in conservative form. The usual conservative scheme expresses the increment of the numerical solution at each grid point as the difference between two adjacent numerical fluxes. According to Lax-Wendroff [13], if the numerical flux is consistent with the exact flux function of the hyperbolic system and if the numerical flux involves a finite number of grid points, then the numerical solution is a weak solution if it converges boundedly almost everywhere to some function for vanishing mesh size. However, we will see that the compact scheme, when made equivalent to a scheme in usual conservative form, has a numerical flux involving an infinite number of grid points. Hence we have to extend the convergence theorem of Lax and Wendroff to such a case. This will be done in Section 2.

In Section 3 we will establish the equivalence between a compact scheme and a scheme in usual conservative form. Both initial value problem and initial-boundary-value problems will be considered. For the case of initial-boundary-value problem, it is interesting to note that rounding errors play an important role, especially in the case of a shock wave.

In Section 4, we construct interface conditions for patched grids with and without grid continuity. This is important for mesh refinement and for treating complex geometries. The interface treatment is required to be conservative and accurate, and to ensure independent or parallel solution of the implicit schemes in each subdomain.

In Appendix A, we show that it is not evident that compact schemes with non-constant coefficients are conservative.

2. Conservation for a Nonlinear Hyperbolic Equation

2.1. Hyperbolic Equation and Numerical Solution

One of the great advantage of the compact scheme is that each space direction can be treated independently of the others. Hence one can just consider a one-dimensional problem. The case of multidimensions for patched grid will be considered in the end of this paper.

Let us consider the scalar hyperbolic equation

$$u_t + f(u)_x = 0, \;\; x \in \Omega, \;\; t > 0 \tag{2.1}$$

together with the initial condition

$$u(x, 0) = u_0(x), \;\; x \in \Omega \tag{2.2}$$

If $\Omega = \mathbb{R}$, then (2.1)-(2.2) define a pure initial-value problem. If $\Omega = (-1, 1)$, then (2.1)-(2.2) together with the boundary condition

$$u(-1, t) = g(t) \;\; \text{if} \;\; a(u) = f'(u) > 0 \tag{2.3}$$

define a mixed initial-boundary-value problem.

The result derived under the assumption of a scalar equation remains valid for a system of hyperbolic equations, since the compact scheme is component-invariant. When we treat problems with shock waves, an upwinding compact scheme is necessary, which can be simply done, in the case of a system, by splitting the flux f into a positive part f^+ and a negative part f^- according to the positive eigenvalues and negative eigenvalues of the Jacobian matrix $A =. \partial_u f(u)$. Then the positive part f^+ is approximated through a left-sided compact scheme, and the negative part f^- is approximated through a right-sided compact scheme. See [7] for more details. Then the following analysis can be proceeded in a similar way, and the same conclusions hold. Hence we restrict the analysis to a scalar equation.

The solution of (2.1)-(2.2) is not always differentiable and in the case of discontinuous solutions, it is more convenient to consider weak solutions. A weak solution u^* of the hyperbolic equation (2.1) with initial condition (2.2) satisfies the following relation

$$\int\int u^*\phi_t dxdt + \int\int f(u^*)\phi_x dxdt = -\int u(x,0)\phi(x,0)dx \tag{2.4}$$

for any test function $\phi(x,t)$ which is continuously differentiable and which has compact support. A weak solution is equivalent to the classical solution in smooth regions and satisfies the Rankine-Hugoniot relation at a discontinuity.

Let x_j for $j \in \mathbb{Z}$ denote the abscise of the grid point j of a generally nonuniform grid. The mesh size $h_j = \frac{1}{2}(x_{j+1} - x_{j-1})$ is generally a function of j. The numerical solution, for semidiscrete schemes, at point j is denoted u_j.

2.2. Conservative Scheme in Terms of Numerical Flux

On a uniform grid with $h_j = h$ for all j, if there is a function $f^{(nu)}_{j+\frac{1}{2}}$, called numerical flux and defined by

$$f^{(nu)}_{j+\frac{1}{2}} = f^{(nu)}(u_{j-l}, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_{j+r}; h),\ \forall j \tag{2.5}$$

$$f^{(nu)}(u, \dots, u, u, u, \dots, u; h) = f(u) \qquad \text{(consistency)} \tag{2.6}$$

such that the semidiscrete scheme can be written as:

$$\frac{du_j}{dt} = -\frac{1}{h}(f^{(nu)}_{j+\frac{1}{2}} - f^{(nu)}_{j-\frac{1}{2}}), \qquad \forall j, \tag{2.7}$$

then the scheme is said to be conservative. Here l and r are two non-negative integers so that the scheme (2.7) involves $(l + r + 1)$ points in space.

In a usual scheme, both l and r are finite. The most frequently used schemes for engineering purpose involve three points, i.e., $l = r = 1$. Modern high resolution schemes, such as TVD (total variation diminishing) schemes and ENO (essentially non-oscillatory schemes) schemes for computing shock flows in gas dynamic problems, involve 5 points in space, i.e., $l = r = 2$.

However, the compact scheme considered in this paper, when made equivalent to (2.7), involves an infinite number of grid points in space, i.e., $l \to \infty$, $r \to \infty$ (see Section 3).

For a three-point scheme, Lax and Wendroff [13] proved the following theorem which remains valid for schemes with more than three points in space.

Theorem 2.1. *If the solution of the conservative scheme (2.7) converges boundedly almost everywhere to some function u^* for $h \to 0$, then the solution u^* is a weak solution.*

Note that the above theorem was proved in [13] for the case of a fully discrete scheme. It is trivial to repeat the proof for the semidiscrete scheme.

It is not yet known whether Theorem 2.1 remains valid for $l \to \infty$ and $r \to \infty$. For this case, we need the following two conditions for convergence

$$f^{(nu)}(u, \dots, u, u, u, \dots, u; h) = f(u) \tag{2.8}$$

$$\lim_{h\to 0} f^{(nu)}(u_{j-\infty}, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_{j+\infty}; h) = f(u(x_{j-\frac{1}{2}})) \tag{2.9}$$

The condition (2.9) is new in compasrison with a scheme for a finite number of grid points. Thus the following theorem would be new.

Theorem 2.2. *Consider the difference equation (2.7) involving an infinite number of grid points, i.e., $l \to \infty$ and $r \to \infty$. The corresponding numerical flux is defined by*

$$f^{(nu)}_{j+\frac{1}{2}} = f^{(nu)}(u_{j-\infty}, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_{j+\infty}; h),\ \forall j$$

which satisfies the constraints (2.8)-(2.9). If the solution of the conservative scheme (2.7) converges boundedly almost everywhere to some function u^* for $h \to 0$, then the solution u^* is a weak solution of the hyperbolic equation (2.1) with initial condition (2.2).

Proof. Multiply (2.7) by a test function $\phi_j = \phi(x_j, t)$ which is continuously differentiable and which has compact support, integrate this equation with respect to time, and sum the resulting equation over all points in space, we obtain

$$\sum_{j=-\infty}^{j=\infty} h \int_0^\infty \frac{du_j}{dt}\phi_j dt = -\sum_{j=-\infty}^{j=\infty} \int_0^\infty (f_{j+\frac{1}{2}}^{(nu)} - f_{j-\frac{1}{2}}^{(nu)})\phi_j dt \tag{2.10}$$

Using integration by parts for the left-hand side of (2.10), we have for $h \to 0$,

$$\begin{aligned}
\sum_{j=-\infty}^{j=\infty} h \int_0^\infty \frac{du_j}{dt}\phi_j dt &= \sum_{j=-\infty}^{j=\infty} h \int_0^\infty \frac{d}{dt}(u_j\phi_j)dt - \sum_{j=-\infty}^{j=\infty} h \int_0^\infty u_j \frac{d\phi_j}{dt}dt \\
&= -\sum_{j=-\infty}^{j=\infty} hu_j(0)\phi_j(0) - \sum_{j=-\infty}^{j=\infty} h \int_0^\infty u_j \frac{d\phi_j}{dt}dt \\
&= -\int u(x,0)\phi(x,0)dx - \int_{-\infty}^{+\infty}\int_0^\infty u^*\phi_t dxdt
\end{aligned} \tag{2.11}$$

For $h \to 0$, the right-hand side of (2.10) can be rewritten as

$$\begin{aligned}
-\sum_{j=-\infty}^{j=\infty} \int_0^\infty (f_{j+\frac{1}{2}}^{(nu)} - f_{j-\frac{1}{2}}^{(nu)})\phi_j dt &= -\sum_{j=-\infty}^{j=\infty} \int_0^\infty f_{j+\frac{1}{2}}^{(nu)}\phi_j dt + \sum_{j=-\infty}^{j=\infty} \int_0^\infty f_{j-\frac{1}{2}}^{(nu)}\phi_j dt \\
&= -\sum_{j=-\infty}^{j=\infty} \int_0^\infty f_{j-\frac{1}{2}}^{(nu)}\phi_{j-1} dt + \sum_{j=-\infty}^{j=\infty} \int_0^\infty f_{j-\frac{1}{2}}^{(nu)}\phi_j dt \\
&= \sum_{j=-\infty}^{j=\infty} h \int_0^\infty f_{j-\frac{1}{2}}^{(nu)} \frac{\phi_{j+1} - \phi_j}{h} dt
\end{aligned} \tag{2.12}$$

Through using (2.8)-(2.9), we have

$$\lim_{h\to 0} \sum_{j=-\infty}^{j=\infty} h f_{j-\frac{1}{2}}^{(nu)} \frac{\phi_{j+1} - \phi_j}{h} = \int_{-\infty}^{\infty} f(u^*)\phi_x dx$$

Hence from (2.12) we have

$$-\sum_{j=-\infty}^{j=\infty} \int_0^\infty (f_{j+\frac{1}{2}}^{(nu)} - f_{j-\frac{1}{2}}^{(nu)})\phi_j dt = \int_{-\infty}^{\infty}\int_0^\infty f(u^*)\phi_x dxdt \tag{2.13}$$

In view of (2.11) and (2.13), the equation (2.10) yields for $h \to 0$,

$$\int\int u^*\phi_t dxdt + \int\int f(u^*)\phi_x dxdt = -\int u(x,0)\phi(x,0)dx$$

That is, u^* is a weak solution.

Hence, for a given numerical flux, we must verify whether (2.9) is satisfied.

3. Compact Scheme on a Uniform Grid

3.1. The Compact Scheme

For the hyperbolic equation (2.1), the semidiscrete compact scheme with three points in space can be written as

$$\frac{du_j}{dt} = F_j \tag{3.1}$$

Here $F = f_x(u)$, in the case of a constant coefficient scheme, is given by the implicit formula

$$a_{-1}hF_{j-1} + a_0hF_j + a_1hF_{j+1} = b_{-1}f_{j-1} + b_0f_j + b_1f_{j+1} \tag{3.2}$$

Most of the compact schemes used in the past have constant coefficients. Only recently we see the appearance of nonlinear schemes[4, 5, 6]. The present paper focuses on constant coefficient schemes. In Appendix A, we give a remark on the conservation for schemes with non-constant coefficients. Further study for such schemes should be considered in the future.

For the scheme (3.2) to be consistent with the equation $F = f_x$, the coefficients a_i, b_i with $i = -, 0, +$, after properly normalized, must satisfy the relations

$$a_{-1} + a_0 + a_1 = 1, \quad b_{-1} + b_0 + b_1 = 0. \tag{3.3}$$

Furthermore, the right-hand side of (3.2) is an approximation to $f(u)_x$, that is,

$$b_{-1}f_{j-1} + b_0f_j + b_1f_{j+1} = f(u)_x + O[h^{1+p}],\ p > 0.$$

Table 3.1 gives these coefficients for various orders of accuracy.

Table 3.1. Coefficients for the compact scheme: the second-order two-parameter family, third-order one-parameter family and fourth-order one. The parameters α and β are arbitrary.

accuracy	a_{-1}	a_0	a_1	b_{-1}	b_0	b_1
2^{th}-order	$\frac{\beta}{1+\alpha+\beta}$	$\frac{1}{1+\alpha+\beta}$	$\frac{\alpha}{1+\alpha+\beta}$	$\frac{\alpha-3\beta-1}{2+2\alpha+2\beta}$	$\frac{4(\beta-\alpha)}{2+2\alpha+2\beta}$	$\frac{1-\beta+3\alpha}{2+2\alpha+2\beta}$
3^{th}-order	$\frac{1}{3(1+\alpha)}$	$\frac{2}{3}$	$\frac{\alpha}{3(1+\alpha)}$	$\frac{-5-\alpha}{6(1+\alpha)}$	$\frac{2(1-\alpha)}{3(1+\alpha)}$	$\frac{5\alpha+1}{6(1+\alpha)}$
4^{th}-order	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{2}$	0	$\frac{1}{2}$

3.2. Conservation for a Pure Initial Value Problem

The scheme (3.1)-(3.2) is not in the usual conservative form and it is not clear whether it is conservative. Consider for example a shock at $x = 0$ which locates at the middle of two adjacent cells, say $j = 0$ and $j = 1$. If the flux derivatives are computed "exactly", then $(f_x)_0 = 0$ and $(f_x)_1 = 0$ so that the shock remains motionless independent of its real speed. In order to capture the correct shock speed, we must show that it can be made equivalent to a scheme in the usual conservative form (2.7).

In order to have conservation in the usual sense, we must require that there exit a function $f^{(nu)}_{j+\frac{1}{2}}$ defined by (2.5)-(2.6) such that the grid function F_j be related to this flux function by

$$F_j = -\frac{1}{h}(f^{(nu)}_{j+\frac{1}{2}} - f^{(nu)}_{j-\frac{1}{2}})\ \forall j. \tag{3.4}$$

Under a reasonable invertible assumption (which we should always impose), the relation (3.2) can be solved to yield

$$F_j = \sum_{l=-\infty}^{\infty} z_l f_{j-l} \tag{3.5}$$

with $z_l \to 0$ for $| l | \to \infty$. By consistency requirement, we must have $\sum_{l=-\infty}^{\infty} z_l = 0$. For the initial-value problem, the coefficients z_l are independent of j. The exact form of z_l is not needed here.

Let us assume the numerical flux in the following form:

$$f^{(nu)}_{j+\frac{1}{2}} = \sum_{l=-\infty}^{\infty} y_l f_{j-l} \tag{3.6}$$

where y_l , which is independent of j, is to be related to z_l and is subjected to the following consistency constraint:

$$\sum_{l=-\infty}^{\infty} y_l = 1. \tag{3.7}$$

By translation we write

$$f^{(nu)}_{j-\frac{1}{2}} = \sum_{l=-\infty}^{\infty} y_l f_{j-l-1} = \sum_{l=-\infty}^{\infty} y_{l-1} f_{j-l}, \tag{3.8}$$

Substituting (3.6) and (3.8) in (3.4), we obtain

$$F_j = -\sum_{l=-\infty}^{\infty} \frac{y_l - y_{l-1}}{h} f_{j-l} \tag{3.9}$$

Comparing (3.9) with (3.5) yields

$$-\frac{y_l - y_{l-1}}{h} = z_l \quad \forall l. \tag{3.10}$$

The relation (3.10) can be solved to yield

$$y_l = y_0 - h \sum_{l'=1}^{l} z_{l'}$$

which, when substituted in (3.7), leads to the following unique value for b_0

$$y_0 = \lim_{l\to\infty} \left(\frac{h}{2l+1} \sum_{l'=-l}^{l} (l - l' + 1) z_{l'} + \frac{1}{2l+1} \right)$$

Hence there is a unique set of y_l such that the flux function (3.6) satisfies the consistency assumption.

The above result can be summarized below.

Lemma 3.1. *Let the left-hand side of (3.2) be invertible. Then the scheme defined by (3.1)-(3.2) can be made equivalent to the usual conservative form (3.4).*

Now we must show that the corresponding numerical flux satisfies the consistency conditions (2.8)-(2.9). By Lemma 3.1, there is a unique set of y_l subjected to the constraint (3.7). Hence

$$f^{(nu)}(u, \dots, u, u, u, \dots, u; h) = \sum_{l=-\infty}^{\infty} y_l f(u) = f(u)$$

so that the constraint (2.8) is indeed satisfied.

Introducing (3.4) into (3.2) leads to the following relation for $f^{(nu)}_{j+\frac{1}{2}}$:

$$a_{-1} f^{(nu)}_{j-\frac{3}{2}} + (a_0 - a_{-1}) f^{(nu)}_{j-\frac{1}{2}} + (a_1 - a_0) f^{(nu)}_{j+\frac{1}{2}} - a_1 f^{(nu)}_{j+\frac{3}{2}} = b_{-1} f_{j-1} + b_0 f_j + b_1 f_{j+1}$$

Let $h \to 0$, then at each point j, we have two situations: either the three points $j-1$, j and $j+1$ all lie in a smooth region of the solution u (case A), or these three points cover a discontinuity (case B).

For case A, we must have, for $h \to 0$, $f_{j-1} = f_{j+1} = f_j$ so that

$$a_{-1} f^{(nu)}_{j-\frac{3}{2}} + (a_0 - a_{-1}) f^{(nu)}_{j-\frac{1}{2}} + (a_1 - a_0) f^{(nu)}_{j+\frac{1}{2}} - a_1 f^{(nu)}_{j+\frac{3}{2}} = 0 \tag{3.11}$$

since $b_{-1} + b_0 + b_1 = 0$.

For case B, we may for convenience consider the discontinuty to lie between $j-1$ and j, so that

$$a_{-1} f^{(nu)}_{j-\frac{3}{2}} + (a_0 - a_{-1}) f^{(nu)}_{j-\frac{1}{2}} + (a_1 - a_0) f^{(nu)}_{j+\frac{1}{2}} - a_1 f^{(nu)}_{j+\frac{3}{2}} = -b_{-1} \langle f \rangle_j \tag{3.12}$$

where $\langle f \rangle_j = f_j - f_{j-1}$ denotes the jump at the discontinuity.

Due to the special structure of the linear system defined by (3.11) and (3.12), there is no difficulty to see that for each j lying inside two discontinuities, we have

$$f^{(nu)}_{j+\frac{1}{2}} = f(u_j)$$

In summary, the result in the following lemma holds

Lemma 3.2. *The numerical flux determined by (3.6) satisfies the constraints (2.8)-(2.9).*

By Lemmas 3.1 and 3.2, the conditions required by Theorem 2.2 are all met. This leads to the following theorem.

Theorem 3.3. *The compact scheme defined by (3.1)-(3.2) can be made equivalent to the usual conservative form (3.4) with the numerical flux satisfying the consistency constraints (2.8)-(2.9), so that if the numerical solution converges for $h \to 0$, it converges to a weak solution of the exact problem.*

Remark 3.4. The key point for the compact scheme is that, when made equivalent to a scheme in the usual conservative form, it has a numerical flux involving an infinite number of grid points. This is the new feature in establishing the convergence result, slightly different from the classical result of Lax-Wendroff.

3.3. Conservation for the Initial-Boundary-Value Problem

3.3.1. Conservation For the initial-boundary-value problem, the compact scheme (3.2), now defined for the interior points $j = 1, 2, \cdots, J$, must be supplied with suitable boundary conditions at $j = 0$ and $j = J+1$. The boundary conditions should fulfill the requirement of accuracy and stability. This has been rigorously studied in [2]. Here we do not require the exact form of the boundary conditions. The compact scheme along with suitable boundary conditions implies the following relations

$$F_j = \sum_{l=0}^{J+1} a_l^{(j)} f_l, 1 \leqslant j \leqslant J \tag{3.13}$$

where the coefficients $a_l^{(j)}$ depend on j and on the boundary condition, and g_j depends on the data on the boundary for incoming waves. Furthermore, consistency assumption leads to

$$\sum_{l=0}^{J+1} a_l^{(j)} = r_j, \quad 1 \leqslant j \leqslant J \tag{3.14}$$

where each r_j simulates an rounding error. The reason to introduce an rounding error is that, it is the scheme (3.2) to be solved by the computer which always has rounding errors. Since the scheme (3.2) is solved with rounding errors, the computed $a_l^{(j)}$ involves rounding errors so that there is a difference between the exact consistency relation $\sum_{l=0}^{J+1} a_l^{(j)} = 0$ and the numerical one.

Similarly as for the initial-value problem, we look for a numerical flux of the form

$$f_{j+\frac{1}{2}}^{(nu)} = \sum_{l=0}^{J+1} y_l^{(j)} f_l, 0 \leqslant j \leqslant J \tag{3.15}$$

so that

$$F_j = -\frac{1}{h}(f_{j+\frac{1}{2}}^{(nu)} - f_{j-\frac{1}{2}}^{(nu)}), 1 \leqslant j \leqslant J. \tag{3.16}$$

Here $y_l^{(j)}$ must depend on j. Now we write F_j as

$$\begin{aligned} F_j &= -\frac{1}{h}(f_{j+\frac{1}{2}}^{(nu)} - f_{j-\frac{1}{2}}^{(nu)}) \\ &= -\frac{1}{h}(\sum_{l=0}^{J+1} y_l^{(j)} f_l - \sum_{l=0}^{J+1} y_l^{(j-1)} f_l) \\ &= -\frac{1}{h}\sum_{l=0}^{J+1}(y_l^{(j)} - y_l^{(j-1)}) f_l \end{aligned}$$

which, on using (3.13), yields

$$-\frac{y_l^{(j)} - y_l^{(j-1)}}{h} = a_l^{(j)}, 0 \leqslant l \leqslant J+1, 1 \leqslant, j \leqslant J \tag{3.17}$$

Moreover, the numerical flux should be consistent with the exact flux, so that the following consistency relations must hold

$$\sum_{l=0}^{J+1} y_l^{(j)} = 1, 0 \leqslant j \leqslant J \tag{3.18}$$

Hence (3.17) and (3.18) define $(J+1)^2 + J$ relations for the $(J+1) \times (J+1)^2 + J + 1$ unknowns $y_l^{(j)}$ $(0 \leqslant j \leqslant J, 0 \leqslant l \leqslant J+1)$.

Lemma 3.5. *If the rounding errors r_j in (3.14) do not vanish, the system (3.17)-(3.18) has no solution; if the rounding errors r_j vanish, the system (3.17)-(3.18) is underdetermined and has nontrivial solutions.*

Proof. The use of (3.14) eliminates J relations. Hence the relations (3.17)-(3.18) only contain $(J+1)^2$ linearly independent relations for the $(J+1) \times (J+1)^2 + J + 1$ free parameters

$y_l^{(j)}$. In consequence, the system (3.17)-(3.18) is underdetermined. Summing (3.17) over l for each j yields

$$-\frac{\sum_{l=0}^{J+1} y_l^{(j)} - \sum_{l=0}^{J+1} y_l^{(j-1)}}{h} = \sum_{l=1}^{J} a_l^{(j)}, 1 \leqslant j \leqslant J$$

which, on using (3.14) and (3.18), yields

$$0 = r_j, 1 \leqslant j \leqslant J$$

Hence the underdetermined system (3.17)-(3.18) has nontrivial solutions if and only if the rounding errors vanish.

From Lemma 3.5, we see that if the rounding error vanishes, then we can indeed find a numerical flux ensuring the condition (3.16) to be satisfied.

Theorem 3.6. *For the initial-boundary-value problem, the scheme defined by (3.1)-(3.2) is conservative if and only if the rounding errors r_j are negligible.*

3.3.2. Rounding Errors Generated by a Shock

We have shown that the conservation of the compact scheme depends on the magnitude of the rounding errors. The rounding error should be small enough for the conservation error to be neglected. Obviously, rounding errors could be large near discontinuous solutions such as shock waves in gas dynamics.

Denote $e_j = F_j - F_j^{(e)}$ where F_j is the computer solution of (3.2) and $F_j^{(e)}$ is the exact solution of (3.2). The equation for e_j is given by

$$a_{-1}e_{j-1} + a_0 e_j + a_1 e_{j+1} = r_j \quad \forall j \tag{3.19}$$

where r_j is the rounding error.

Let $r_j \equiv 0$ and $e_j = \tau^j$ in (3.19), we obtain the characteristic equation

$$a_{-1} + a_0\tau + a_1\tau^2 = 0.$$

The following result is well-known.

Lemma 3.7. *Let the compact scheme be invertible. Then the roots τ_1, τ_2 are separted by the unit circle.*

For convenience, we let $|\tau_1| < 1$, $|\tau_2| > 1$. By a straightforward calculation using Table 1, we obtain

Lemma 3.8. *The roots τ_1, τ_2 are are real (a) for the second-order scheme if $\alpha\beta > \frac{1}{4}$, (b) for the third-order scheme $\forall\alpha \in \mathbf{R}$ and (c) for the fourth-order scheme.*

We consider both mid-point shock (MPS) and cell-centered shock (CCS). The mid-point shock locates at $j = \frac{1}{2}$ while the cell-centered shock locates at $j = 0$. The rounding error satsifies the relation $|r_j| \ll |r_0| = |r_1|$ for the mid-point shock and $|r_j| \ll |r_0|$ for the cell-centered shock. We therefore consider the solution of (3.19) with

$$\text{(MPS)} \;\; r_j = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ 1 & j = 1 \\ 0 & j > 1 \end{cases} ; \quad \text{(CCS)} \;\; r_j = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ 0 & j > 0 \end{cases} \tag{3.20}$$

Proposition 3.9. *The solution of (3.19) with r_j defined by (3.20) is*

$$e_j = c^- \tau_2^j, \; j \leqslant 0; \quad e_j = c^+ \tau_1^j, \; j \geq 1 \tag{3.21}$$

where

$$c^- = \frac{a_{-1} + \tau_1 a_1}{a_{-1}a_1(\tau_1 - \tau_2)}, \quad c^+ = \frac{a_{-1} + \tau_2 a_1}{a_{-1}a_1(\tau_1 - \tau_2)}$$

for the mid-point shock and

$$c^- = c^+ = \frac{\tau_2}{2a_{-1} + \tau_2 a_0}$$

for the cell-centerd shock.

Proof. First consider the mid-point shock case. The general solutions of (3.19)-(3.20) for $j \neq 0$ and $j \neq 1$ are given by

$$\begin{aligned} e_j &= c_1^- \tau_1^j + c_2^- \tau_2^j \qquad j < 0 \\ e_j &= c_1^+ \tau_1^j + c_2^+ \tau_2^j \qquad j > 1 \end{aligned}$$

Introducing these solutions in (3.19) for $j = 0$ and $j = 1$ yield the relations

$$c_1^+ - c_1^- = \frac{a_{-1} + \tau_2 a_1}{a_{-1}a_1(\tau_1 - \tau_2)}, \quad c_2^+ - c_2^- = \frac{a_{-1} + \tau_1 a_1}{a_{-1}a_1(\tau_2 - \tau_1)} \tag{3.22}$$

Since the solution far away from the shock is not perturbed, we must have $c_2^+ = 0$ and $c_1^- = 0$. This leads to (3.21).

For the cell-centerd shock, the proof is similar.

Remark. The key in this proof is the domain decomposition technique. Such a technique was already used in [22] to study the solution behaviour near a shock on overlapping grids.

Example 1. Consider the fourth-order compact scheme for the mid-point shock model. In this case, we have $\tau_1 = \sqrt{3} - 2$ and $\tau_2 = -\sqrt{3} - 2$. The solution has the following simple form

$$e_j = \frac{\sqrt{3} - 1}{2\sqrt{3}}(-\sqrt{3} - 2)^j, \;\; j \leqslant 0; \quad e_j = -\frac{\sqrt{3} + 1}{2\sqrt{3}}(\sqrt{3} - 2)^j, \;\; j \geq 1$$

The maximum occurs at $j = 0$ and $j = 1$. Precisely, $e_0 = e_1 \approx 0.21$. Away from $j = 0$ and $j = 1$, the error e_j rapidely decays.

Example 2. Consider the fourth-order compact scheme for the cell-centerd shock model. The solution has the following simple form

$$e_j = \frac{\sqrt{3}}{6}(-\sqrt{3} - 2)^j, \;\; j \leqslant 0; \quad e_j = \frac{\sqrt{3}}{6}(\sqrt{3} - 2)^j, \;\; j \geq 0$$

The maximum occurs at $j = 0$ and we have $e_0 \approx 0.29$. Away from $j = 0$, the error e_j rapidely decays.

4. Compact Scheme on a Multiblock Patched Grid

Little effort has been done in the past for domain decomposition with a compact scheme. Domain decomposition is useful for treating complex geometries and for doing parallel computation. Stability, uniqueness, convergence, conservation, accuracy, and parallization of the interface treatment are important issues. This has been much studied in the past for patched grid and overlapping grid methods using traditional difference schemes other than the compact scheme, see for instances [1, 16, 15, 22, 23, 24]. Here we address conservation, accuracy and parallelization for multiblock patched grids. Precisely, we look for the most accurate interface conditions that ensure both conservation and parallization. Parallization means that the interface treatment allows for independent solution of the implicit schemes in each subdomain.

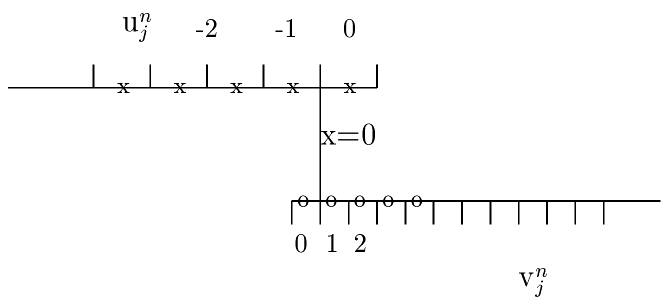

Fig 4.1. Patched grid in one dimension

This is important for using parallel computers. Conservation is important for correctly capturing shock waves. The construction is basically for one dimension, but its extension to high dimension is straightforward.

4.1. Conservative Treatment at Grid Interfaces

Consider two uniform grids separated by an interface located at $x = 0$. The configuration is shown in Fig. 4.1.

In the left subdomain D_u, the mesh size is h_u. Denoting u_j, f_j and F_j, with $j = 0, -1, -2, \dots$, the solutions, the fluxes and the flux derivatives, respectively, the compact scheme can be written as:

$$a_{-1}h_uF_{j-1} + a_0h_uF_j + a_1h_uF_{j+1} = b_{-1}f_{j-1} + b_0f_j + b_1f_{j+1}),\ \ j \leqslant -1 \tag{4.1}$$

In the right subdomain D_v, the mesh size is h_v. Denoting v_j, g_j and G_j, with $j = 0, 1, 2, \dots$, the solutions, the fluxes and the flux derivatives, respectively, the compact scheme can be written as:

$$a_{-1}h_vG_{j-1} + a_0h_vG_j + a_1h_vG_{j+1} = b_{-1}g_{j-1} + b_0g_j + b_1g_{j+1},\ \ j \geqslant 1 \tag{4.2}$$

The solutions of the two subdomains should be coupled at the interface through some interface condition. Obviously, in order to solve (4.1) for $j = -1$, we require the interface value F_0. Similarly, we need the interface value G_0 for (4.2) with $j = 1$. These are to be defined by interface conditions.

It is quite convenient to use the conservative criterion of Berger [1] to derive conservative interface conditions.

The Berger's conservation criterion can be described as follows. In the case of a Cauchy problem without interface, conservation of the difference scheme can be expressed by stating that the following quantity is conserved in time:

$$S = \sum_{j=-\infty}^{\infty} hu_j. \tag{4.3}$$

When there is an interface, a similar quantity can be defined which can be split into three parts accounting for the contributions from the left and right subdomains and the interface:

$$S = S_{D_u} + S_{D_v} + S_{interface} \tag{4.4}$$

If

$$\frac{dS}{dt} = 0 \tag{4.5}$$

for any time, the interface treatment is conservative. In this case, if the numerical solution of the interface problem is convergent, it converges to a weak solution of the exact problem, *i.e.*, it allows for a correct shock capturing.

In any case, the quantity S should be a consistent approximation to the integral $\int_{-\infty}^{\infty} u dx$ of the exact solution u. It is sufficient enough to approximate this integral by the mid-point formula.

4.2. Basic Requirement for Conservation

Using (3.1) for each subgrid, and (4.5), the Berger's conservative criterion can be written as

$$\sum_{j\leqslant -1} F_j h_u + \sum_{j\geq 1} G_j h_v = 0 \tag{4.6}$$

Proposition 4.1. *If F_0 and G_0 satisfy the relation*

$$h_u(a_1 F_0 - a_{-1}F_{-1}) + h_v(a_{-1}G_0 - a_1 G_1) = b_1 f_0 - b_{-1} f_{-1} + b_{-1} g_0 - b_1 g_1 \tag{4.7}$$

then the problem is conservative.

Proof. Summing (4.1) over $j = -1, -2, \ldots$, we obtain the following equation

$$a_{-1}\sum_{j=-1}^{-\infty} F_{j-1} + a_0 \sum_{j=-1}^{-\infty} F_j + a_1 \sum_{j=-1}^{-\infty} F_{j+1} = \frac{1}{h_u}(b_{-1}\sum_{j=-1}^{-\infty} f_{j-1} + b_0 \sum_{j=-1}^{-\infty} f_j + b_1 \sum_{j=-1}^{-\infty} f_{j+1}) \tag{4.8}$$

Noting that

$$\begin{aligned}\sum_{j=-1}^{-\infty} F_{j-1} &= \sum_{j=-1}^{-\infty} F_j - F_{-1}, \ \sum_{j=-1}^{-\infty} F_{j+1} = \sum_{j=-1}^{-\infty} F_j + F_0,\\ \sum_{j=-1}^{-\infty} f_{j-1} &= \sum_{j=-1}^{-\infty} f_j - f_{-1}, \ \sum_{j=-1}^{-\infty} f_{j+1} = \sum_{j=-1}^{-\infty} f_j + f_0,\end{aligned}$$

we can rewrite (4.8) as:

$$(a_{-1} + a_0 + a_1)\sum_{j=-1}^{-\infty} F_j - a_{-1}F_{-1} + a_1 F_0 = \frac{1}{h_u}[(b_{-1} + b_0 + b_1)\sum_{j=-1}^{-\infty} f_j - b_{-1} f_{-1} + b_1 f_0].$$

Making use of the consistency relations in (3.3), we finally obtain

$$\sum_{j=-1}^{-\infty} F_j + a_1 F_0 - a_{-1}F_{-1} = \frac{1}{h_u}(b_1 f_0 - b_{-1} f_{-1}). \tag{4.9}$$

Summing (4.2) over $j = 1, 2, \ldots$, we obtain the following equation

$$a_{-1}\sum_{j=1}^{\infty} G_{j-1} + a_0 \sum_{j=1}^{\infty} G_j + a_1 \sum_{j=1}^{\infty} G_{j+1} = \frac{1}{h_v}(b_{-1}\sum_{j=1}^{\infty} g_{j-1} + b_0 \sum_{j=1}^{\infty} g_j + b_1 \sum_{j=1}^{\infty} g_{j+1}) \tag{4.10}$$

Noting that

$$\begin{aligned}\sum_{j=1}^{\infty} G_{j-1} &= \sum_{j=1}^{\infty} G_j + G_0, \ \sum_{j=1}^{\infty} G_{j+1} = \sum_{j=1}^{\infty} G_j - G_1,\\ \sum_{j=1}^{\infty} g_{j-1} &= \sum_{j=1}^{\infty} g_j + g_0, \ \sum_{j=1}^{\infty} g_{j+1} = \sum_{j=1}^{\infty} g_j - g_1,\end{aligned}$$

we can rewrite (4.8) as:

$$(a_{-1}+a_0+a_1)\sum_{j=1}^{\infty}G_j + a_{-1}G_0 - a_1G_1 = \frac{1}{h_v}[(b_{-1}+b_0+b_1)\sum_{j=-1}^{-\infty}f_j + b_{-1}g_0 - b_1g_1].$$

Making use of the consistency relations in (3.3), we finally obtain

$$\sum_{j=1}^{\infty}G_j + a_{-1}G_0 - a_1G_1 = \frac{1}{h_v}(b_{-1}g_0 - b_1g_1). \tag{4.11}$$

Multiplying the relation (4.9) by h_u and the relation (4.11) by h_v, summing the resulting equations, and making use of (4.6), we obtain (4.7).

Proposition 4.2. *The interface condition (4.7) has locally a (p+1)-th order of accuracy for all the three-point compact schemes if* $h_u - h_v = O[h_u^p, h_v^p]$ *with some integer* $p > 0$.

Proof. Using Taylor expansion, we have

$$\begin{aligned}
F_0 &= f_x(0) + \frac{h_u}{2}f_{xx}(0) + \frac{1}{2}\left(\frac{h_u}{2}\right)^2 f_{xxx}(0) + O[h_u^3]\\
f_0 &= f(0) + \frac{h_u}{2}f_x(0) + \frac{1}{2}\left(\frac{h_u}{2}\right)^2 f_{xx}(0) + \frac{1}{6}\left(\frac{h_u}{2}\right)^3 f_{xxx}(0) + O[h_u^4]\\
F_{-1} &= f_x(0) - \frac{h_u}{2}f_{xx}(0) + \frac{1}{2}\left(\frac{h_u}{2}\right)^2 f_{xxx}(0) + O[h_u^3]\\
f_{-1} &= f(0) - \frac{h_u}{2}f_x(0) + \frac{1}{2}\left(\frac{h_u}{2}\right)^2 f_{xx}(0) - \frac{1}{6}\left(\frac{h_u}{2}\right)^3 f_{xxx}(0) + O[h_u^4]\\
G_0 &= f_x(0) - \frac{h_v}{2}f_{xx}(0) + \frac{1}{2}\left(\frac{h_v}{2}\right)^2 f_{xxx}(0) + O[h_v^3]\\
g_0 &= f(0) - \frac{h_v}{2}f_x(0) + \frac{1}{2}\left(\frac{h_v}{2}\right)^2 f_{xx}(0) - \frac{1}{6}\left(\frac{h_v}{2}\right)^3 f_{xxx}(0) + O[h_v^4]\\
G_1 &= f_x(0) + \frac{h_v}{2}f_{xx}(0) + \frac{1}{2}\left(\frac{h_v}{2}\right)^2 f_{xxx}(0) + O[h_v^3]\\
g_1 &= f(0) + \frac{h_v}{2}f_x(0) + \frac{1}{2}\left(\frac{h_v}{2}\right)^2 f_{xx}(0) + \frac{1}{6}\left(\frac{h_v}{2}\right)^3 f_{xxx}(0) + O[h_v^4]
\end{aligned}$$

Hence one can expand the left-hand side of (4.7) as

$$\begin{aligned}
& h_u(a_1F_0 - a_{-1}F_{-1}) + h_v(a_{-1}G_0 - a_1G_1) - [b_1f_0 - b_{-1}f_{-1} + b_{-1}g_0 - b_1g_1]\\
= \; & K_0f(0) + K_1f_x(0) + K_2f_{xx}(0) + K_3f_{xxx}(0) + O[h_u^4, h_v^4]
\end{aligned}$$

with

$$\begin{aligned}
K_0 &= 0\\
K_1 &= [a_1 - a_{-1} - \frac{1}{2}(b_1 + b_{-1})](h_u - h_v)\\
K_2 &= \frac{1}{2}[a_1 + a_{-1} - \frac{1}{4}(b_1 - b_{-1})](h_u^2 - h_v^2)\\
K_3 &= \frac{1}{8}[a_1 - a_{-1} - \frac{1}{6}(b_1 + b_{-1})](h_u^3 - h_v^3)
\end{aligned}$$

Using Table 3.1, we find that $K_1 = 0$ for all the second to fourth order schemes, while $K_2 = \frac{1}{12}$ for both of the third and fourth order schemes and $K_2 = \frac{\alpha+\beta}{1+\alpha+\beta} - \frac{1}{4}$ for the second order scheme. This proves the proposition.

According to the above proposition, if the interface involves an abrupt refinement so that $h_u - h_v = O[h_u, h_v]$, then the local accuracy of a conservative interface treatment for the compact scheme drops to second order. If there is no refinement (as would occur in parallel computations), then this interface treatment is as accurate as the interior scheme.

4.3. Full Interface Condition

The interface condition (4.7) is incomplete in the following two senses

1) it has only one relation while we need to determine two unknowns F_0 and G_0,

2) it couples the unknowns F_{-1} and G_1 belonging to different subdomains so that the difference equation in each subdomain can not be solved independently.

Thus we have to add an additional interface condition. This is done by requiring a suitable order of accuracy and, most desirably, by requiring the difference equation in each subdomain to be solved in an independent way (parallelization requirement). This is only possible if the additional interface condition has the form

$$h_u(a_1 F_0 - a_{-1} F_{-1}) = A \tag{4.12}$$

or

$$h_v(a_{-1} G_0 - a_1 G_1) = B \tag{4.13}$$

where A and B are linearly functions of their arguments to be determined by accuracy consideration.

Let us concentrate on (4.12). In order the problem to be parallelizable, A must be independent of G_0 and G_1. Hence we rewrite (4.12) as

$$h_u(a_1 F_0 - a_{-1} F_{-1}) = a_u f_0 + b_u f_{-1} + c_u g_0 + d_u g_1 \tag{4.14}$$

For convenience, let $r = h_v/h_u$. Performing a Taylor expansion up to fourth order, we obtain the following relations from (4.14)

$$\begin{cases} a_u + b_u + c_u + d_u = 0 \\ \frac{1}{2}(a_u - b_u) - \frac{r}{2}(c_u - d_u) = a_1 - a_{-1} \\ \frac{1}{8}(a_u + b_u) + \frac{r^2}{8}(c_u + d_u) = \frac{1}{2}(a_1 + a_{-1}) \\ \frac{1}{48}(a_u - b_u) - \frac{r^3}{48}(c_u - d_u) = \frac{1}{8}(a_1 - a_{-1}) \end{cases} \tag{4.15}$$

which, when using the exact solution, ensures that

$$a_1 F_0 - a_{-1} F_{-1} - \frac{1}{h_u}(a_u f_0 + b_u f_{-1} + c_u g_0 + d_u g_1) = O(h_u^3, r h_v^3)$$

that is, if the coefficients a_u, b_u, c_u, and d_u are determined by (4.15), then the additional condition (4.14) has a local fourth order of accuracy (globally third-order accurate) for any given set of a_1 and a_{-1}.

Unluckily, the system (4.15) has no nontrivial solution and is thus unusable.

Now let us assume

$$h_u(a_1 F_0 - a_{-1} F_{-1}) = a_u f_0 + b_u f_{-1} + c_u f_{-2} + d_u f_{-3} \tag{4.16}$$

which leads to the following relations for third order accuracy

$$\begin{cases} a_u + b_u + c_u + d_u = 0 \\ a_u - b_u - 3c_u - 5d_u = 2(a_1 - a_{-1}) \\ a_u + b_u + 9c_u + 25d_u = 4(a_1 + a_{-1}) \\ a_u - b_u - 27c_u - 125d_u = 6(a_1 - a_{-1}) \end{cases} \tag{4.17}$$

Now the system (4.17) has a unique set of solution and is given by

$$\begin{aligned} a_u &= \frac{11}{6}a_1 - \frac{1}{3}a_{-1} \\ b_u &= -3a_1 - \frac{1}{2}a_{-1} \\ c_u &= \frac{3}{2}a_1 + a_{-1} \\ d_u &= -\frac{1}{3}a_1 - \frac{1}{6}a_{-1} \end{aligned}$$

Similarly, if we write (4.13) as

$$h_v(a_{-1}G_0 - a_1G_1) = a_v g_0 + b_v g_1 + c_v g_2 + d_v g_3 \tag{4.18}$$

then the following relations hold for third order accuracy requirement

$$\begin{cases} a_v + b_v + c_v + d_v = 0 \\ -a_v + b_v + 3c_v + 5d_v = -2(a_1 - a_{-1}) \\ a_v + b_v + 9c_v + 25d_v = -4(a_1 + a_{-1}) \\ -a_v + b_v + 27c_v + 125d_v = -6(a_1 - a_{-1}) \end{cases} \tag{4.19}$$

which yields uniquely

$$\begin{aligned} a_v &= \frac{1}{3}a_1 - \frac{11}{6}a_{-1} \\ b_v &= \frac{1}{2}a_1 + 3a_{-1} \\ c_v &= -a_1 - \frac{3}{2}a_{-1} \\ d_v &= \frac{1}{6}a_1 + \frac{1}{3}a_{-1} \end{aligned}$$

One can repeat the above analysis to derive lower order additional treatment by setting $d_u = 0$ or $d_v = 0$.

Proposition 4.3. *The most accurate full set of conservative interface conditions, allowing independent or parallel computations, can be written as*

$$\begin{cases} h_u(a_1F_0 - a_{-1}F_{-1}) = A \\ h_v(a_{-1}G_0 - a_1G_1) = b_1f_0 - b_{-1}f_{-1} + b_{-1}g_0 - b_1g_1 - A \end{cases} \tag{4.20}$$

or

$$\begin{cases} h_v(a_{-1}G_0 - a_1G_1) = B \\ h_u(a_1F_0 - a_{-1}F_{-1}) = b_1f_0 - b_{-1}f_{-1} + b_{-1}g_0 - b_1g_1 - B \end{cases} \tag{4.21}$$

where

$$\begin{aligned} A &= a_u f_0 + b_u f_{-1} + c_u f_{-2} + d_u f_{-3} \\ B &= a_v g_0 + b_v g_1 + c_v g_2 + d_v g_3 \end{aligned}$$

with

$$\begin{aligned} a_u &= \frac{11}{6}a_1 - \frac{1}{3}a_{-1},\ a_v = \frac{1}{3}a_1 - \frac{11}{6}a_{-1} \\ b_u &= -3a_1 - \frac{1}{2}a_{-1},\ b_u = -3a_1 - \frac{1}{2}a_{-1} \\ c_u &= \frac{3}{2}a_1 + a_{-1},\ c_u = \frac{3}{2}a_1 + a_{-1} \\ d_u &= -\frac{1}{3}a_1 - \frac{1}{6}a_{-1},\ d_u = -\frac{1}{3}a_1 - \frac{1}{6}a_{-1} \end{aligned}$$

4.4. Further Consideration and Extension to High Dimensions

The choice among (4.20) and (4.21) may depend on stability consideration. If the wave is right-going, the interface condition (4.20) is more stable since its first relation involves upstream upwinding. Similarly, if the wave is left-going, the interface condition (4.21) is preferred.

The extension of the interface treatment to high dimensions with grid lines from the adjacent subdomains not matching exactly at the interface is not difficult. Consider for instance the interface condition (4.20), the first relation in (4.20) can be implemented as in one dimension since it uses information only from the left subdomain. The second relation must be adapted to account for interpolation in the plane tangent to the interface and can be roughly written as

$$\begin{aligned} h_v(a_{-1}G_0(\overrightarrow{x}) - a_1 G_1(\overrightarrow{x})) &= b_1 I(f_0, \overrightarrow{x}) - b_{-1} I(f_{-1}, \overrightarrow{x}) \\ &\quad + b_{-1} g_0(\overrightarrow{x}) - b_1 g_1(\overrightarrow{x}) - A(\overrightarrow{x}) \end{aligned}$$

where $I(f, \overrightarrow{x})$ is some interpolation to the interface grid point $\overrightarrow{x}$ belonging to the right subdomain.

Numerical experiments will be conducted in subsequent studies.

Appendix A. Remark on Conservation for Compact Schemes with Non-constant Coefficients

In thisn appendix, we use the conservation criterion of M. Berger[1] to analyze the conservation for compact schemes with non-constant coefficients, here restricted to the case of a Cauchy problem.

For the hyperbolic equation (2.1), the semidiscrete compact scheme with three points in space can be written as

$$\frac{du_j}{dt} = F_j \qquad (A.1)$$

Here $F = f_x(u)$, in the case of a non-constant coefficients, is given by the implicit formula

$$a_{-1}^{(j)} hF_{j-1} + a_0^{(j)} hF_j + a_1^{(j)} hF_{j+1} = b_{-1}^{(j)} f_{j-1} + b_0^{(j)} f_j + b_1^{(j)} f_{j+1} \qquad (A.2)$$

For the scheme (A.2) to be consistent with the equation $F = f_x$, the coefficients $a_i^{(j)}$, $b_i^{(j)}$ with $i = -, 0, +$, after properly normalized, must satisfy the relations

$$a_{-1}^{(j)} + a_0^{(j)} + a_1^{(j)} = 1, \quad b_{-1}^{(j)} + b_0^{(j)} + b_1^{(j)} = 0. \qquad (A.3)$$

If the conservation criterion (4.3) is satisfied, then the scheme is conservative. Using (A.1), the condition (4.3) reduces to

$$\sum_{j=-\infty}^{\infty} hF_j = 0 \qquad (A.4)$$

Using (A.3), we obtain

$$a_{-1}^{(j)}h(F_{j-1}-F_j)+hF_j+a_1^{(j)}h(F_{j+1}-F_j)=b_{-1}^{(j)}(f_{j-1}-f_j)+b_1^{(j)}(f_{j+1}-f_j) \qquad (A.5)$$

Summing (A.5) over all j, we obtain

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} a_{-1}^{(j)}h(F_{j-1}-F_j)+\sum_{j=-\infty}^{\infty} hF_j+\sum_{j=-\infty}^{\infty} a_1^{(j)}h(F_{j+1}-F_j) \\ =&\sum_{j=-\infty}^{\infty} b_{-1}^{(j)}(f_{j-1}-f_j)+\sum_{j=-\infty}^{\infty} b_1^{(j)}(f_{j+1}-f_j) \end{aligned}$$

which can also be written as

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} (a_{-1}^{(j+1)}-a_{-1}^{(j)})hF_j+\sum_{j=-\infty}^{\infty} hF_j+\sum_{j=-\infty}^{\infty} (a_1^{(j-1)}-a_1^{(j)})hF_j \\ =&\sum_{j=-\infty}^{\infty} (b_{-1}^{(j+1)}-b_{-1}^{(j)})f_j+\sum_{j=-\infty}^{\infty} (b_1^{(j-1)}-b_1^{(j)})f_j \end{aligned}$$

or

$$\sum_{j=-\infty}^{\infty} hF_j=\sum_{j=-\infty}^{\infty} (b_{-1}^{(j+1)}-(b_{-1}^{(j)}+b_1^{(j)})+b_{-1}^{(j-1)})f_j-\sum_{j=-\infty}^{\infty} (a_1^{(j-1)}-(a_{-1}^{(j)}+a_1^{(j)})+a_{-1}^{(j+1)})hF_j \quad (A.6)$$

From (A.6), it is clear that for the conservation criterion (A.4) to be satisfied for all f_j, we must have

$$b_{-1}^{(j+1)}-(b_{-1}^{(j)}+b_1^{(j)})+b_{-1}^{(j-1)}=0 \quad \forall j \qquad (A.7)$$

$$a_1^{(j-1)}-(a_{-1}^{(j)}+a_1^{(j)})+a_{-1}^{(j+1)}=0 \quad \forall j \qquad (A.8)$$

Hence a compact scheme, with non-constant coefficients, must satisfy the constraints (A.7)-(A.8) to have conservation. It is not sure that the compact schemes with non-constant coefficients all satisfy these constraints. This point need be addressed in a separate paper.

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