Journal of Computational Mathematics, Vol.21, No.3, 2003, 347–356.

A COMBINED HYBRID FINITE ELEMENT METHOD FOR PLATE BENDING PROBLEMS*[1)]

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Abstract

In this paper, a combined hybrid method is applied to finite element discretization of plate bending problems. It is shown that the resultant schemes are stabilized, i.e., the convergence of the schemes is independent of inf-sup conditions and any other patch test. Based on this, two new series of plate elements are proposed.

Key words: Combined hybrid finite element, Weakly compatible.

1. Introduction

The success of finite element methods for the numerical solution of boundary value problems for elliptic partial differential equations is, to a large extent, due to the variational principles upon which these methods are built. The assumed stress hybrid methods pioneered by Pian and Tong (see[8],[10],etc.) are based on modified complementary energy principles and proved to be very successful in a number of applications; see, e.g.,[3],[7],[8],[10],[11],[19], etc.. For 4th-order problems, the hybrid methods can relax the C^1-continuity for deflection elements so that sufficient flexibility in the finite element solution can be gained. However, because of the "saddle-point" nature of the hybrid models, some strict stability conditions such as inf-sup conditions or LBB conditions must be satisfied by deflection and bending moments (e.g.,[3]), and then the formulations of hybrid elements can not yet be simplified to a degree comparable to the use of shape function routine in the conventional displacement methods. And due to the complicated self-equilibrium equations, application of assumed stress hybrid elements to shell analysis is not convenient.

To avoid the inf-sup difficulties, the least-squares method (see,e.g.,[1] and the references therein) having developed in the past decade seems to be an efficient way. But as pointed out in [1], for 4th-order problems, some conforming shape functions are required and the resulting least-squares finite element method also fails to be practical because the condition numbers of the corresponding discrete problems are $O(h^{-4})$ compared with the $O(h^{-2})$ condition numbers that result from standard Galerkin methods for the same problem.

Recently, a new hybrid finite element method for linear elasticity problems named as combined hybrid method was suggested by Zhou (see [23],[26]). This approch is based on a so-called combined variational principle, i.e., a homotopy family of optimization conditions of two dual systems of saddle point problem–one is the domain-decomposed Hellinger-Reissner principle, the other is the primal hybrid variational principle, a dual to the former. Theoritical analyses[23] and numerical tests[26] both showed that the combined hybrid method possesses not only the features of hybrid methods, but also almost all the significant and valuable properties of the least-squres methods, such as: it can circumvent the inf-sup conditions, and then the

* Received December 14, 2000.
1) This work was supported by the National Science Foundation of China.

weak problems are in general coercive; the resulting algebraic problems are symmetric and positive definite; essential boundary conditions can be imposed in a weak sense; and finite element spaces for displacement and stress can be chosen independently, etc..

In this paper, the combined variational principle, as a rational approach to incompatible displacement schemes, is applied to finite element discretization of plate bending problems. It is shown that the resultant schemes, named as combined hybrid finite element schemes, is stabilized, i.e., the convergence of the new incompatible element schemes is independent of inf-sup conditions and any other patch test. Then the deflection and the bending moments subspace can be chosen independently. The C^1-continuity for deflection interpolations is relaxed. The self-equilibrium constraint on the bending moments subspace $\mathbf{V}^h$ is not required.Two new series of plate elements to are given to show another feature of the combined hybrid method, i.e., in building the plate bending finite elements there exists great possibility in the choice of stress/strain-enriched interpolations to enhance accurary of the schemes.

The paper is arranged as follows. In section 2 the combined variational principle is derived and then a mathematical foundation of the stabilized hybrid method is established. Section 3 is devoted to the discussion of stabilized hybrid schemes and convergence. The error estimates are deduced. Finally, two new series of plate elements are given in section 4.

In what follows the letter C will represent different constant independent of the mesh size h at its each occurence.

2. Combined Variational Principle

We consider the following plate bending problem:

$$\begin{cases} \mathbf{divdiv}\sigma = f, & in\ \Omega, \\ \sigma = m(\mathbf{D}_2 u), & in\ \Omega, \\ u = \nabla u \cdot n = 0, & on\ \Gamma = \partial\Omega. \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded open set, u represents vertical deflection, σ the bending moments, and n the outer normal unit vector along Γ. The operators $\mathbf{divdiv}, \mathbf{D}_2$ and m are defined respectively as follows:

$$\mathbf{divdiv}\tau = \partial_{11}\tau_{11} + 2\partial_{12}\tau_{12} + \partial_{22}\tau_{22},$$

$$\mathbf{D}_2 v = \begin{pmatrix} \partial_{11}v & \partial_{12}v \\ \partial_{12}v & \partial_{22}v \end{pmatrix},$$

$$m(\tau) = \begin{pmatrix} \tau_{11} + \nu\tau_{22} & (1-\nu)\tau_{12} \\ (1-\nu)\tau_{12} & \nu\tau_{11} + \tau_{22} \end{pmatrix}$$

for any symmetric tensor τ, and $\nu \in (0, 0.5)$ denotes the Poisson's coefficient, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $i, j = 1, 2$.

We know that for this problem the two basic solution spaces are the deflection space $H_0^2(\Omega)$ and the bending moments space $H(\mathbf{divdiv}; \Omega) := \{\tau \in (L^2(\Omega))_s^4; \mathbf{divdiv}\tau \in L^2(\Omega)\}$, where $(L^2(\Omega))_s^4$ is the space of square integrable 2×2 symmetric tensors.

To relax continuity, we introduce the following two piecewise Sobolev spaces to replace $H_0^2(\Omega)$ and $H(divdiv; \Omega)$:

$$\mathbf{V} := \prod_{K \in T_h} H(\mathbf{divdiv}; K),$$

$$U := \{v \in \prod_{K \in T_h} H^2(K); u = \nabla u \cdot n = 0, on\ \Gamma\},$$

where $T_h = \{K\}$ denotes a regular subdivision of Ω, with mesh diameter h_K for any $K \in T_h$. We also need the following Lagrange multiplier space as

$$U_c := H_0^2(\Omega) / \prod_{K \in T_h} H_0^2(K).$$

We equip $\mathbf{V}$ and $U \times U_c$ with the norms

$$||\tau||_{\mathbf{V}} := [\int_\Omega m^{-1}(\tau) : \tau d\mathbf{x} + \sum_K h_K^4 |divdiv\tau|_{0,K}^2]^{\frac{1}{2}}$$

$$||(v, v_c)||_{U \times U_c} := [\sum (\int_K m(\mathbf{D}_2 v) : \mathbf{D}_2 v d\mathbf{x} + ||v - v_c||_{P,K}^2)]^{\frac{1}{2}}$$

where

$$||v - v_c||_{P,K}^2 = \inf_{w \in H_0^2(K)} [h_K^{-4} ||v - v_c - w||_{0,K}^2 + \int_K m(\mathbf{D}_2(v - v_c - w)) : \mathbf{D}_2(v - v_c - w) d\mathbf{x}].$$

As to the validity of the second norm, we only need to check that $||(v, v_c)||_{U \times U_c} = 0$ implies $v = v_c = 0$. In fact, $\sum ||v - v_c||_{P,K} = 0$ yields $v \in H_0^2(\Omega)$, which, together with $\sum \int_K m(\mathbf{D}_2 v) : \mathbf{D}_2 v d\mathbf{x} = 0$, implies $v = 0$. And then $v_c = 0$.

Next we will show that corresponding to the deflection/bending moments space $\mathbf{V} \times (U \times U_c)$, there are two variational principles related to the problem (2.1). In other words, we have the following equivalence theorem:

Theorem 2.1. *Assume that $f \in L^2(\Omega)$. Then problem (2.1) is equivalent to either of the following two saddle point problems:*

$$\inf_{\tau \in \mathbf{V}} \sup_{(v, v_c) \in U \times U_c} \{1/2 a(\tau, \tau) - b_2(\tau, v) + b_1(\tau, v - v_c) + f(v)\} \tag{2.2}$$

and

$$\inf_{(v, v_c) \in U \times U_c} \sup_{\tau \in \mathbf{V}} \{1/2 d(v, v) - b_1(\tau, v - v_c) - f(v)\} \tag{2.3}$$

where

$$a(\sigma, \tau) = \int_\Omega m^{-1}(\sigma) : \tau d\mathbf{x},$$
$$b_2(\tau, v) = \sum \int_K \tau : \mathbf{D}_2 v d\mathbf{x},$$
$$b_1(\tau, v - v_c) = \sum \oint_{\partial K} [M_{nn}(\tau) \nabla (v - v_c) \cdot \mathbf{n} + M_{ns}(\tau) \nabla (v - v_c) \cdot \mathbf{s} - Q_n(\tau)(v - v_c)] ds,$$
$$d(u, v) = \sum \int_K m(\mathbf{D}_2 u) : \mathbf{D}_2 v d\mathbf{x},$$
$$f(v) = \int_\Omega f v d\mathbf{x},$$
$$M_{nn}(\tau) = (\tau \mathbf{n}) \cdot \mathbf{n},\ M_{ns}(\tau) = (\tau \mathbf{n}) \cdot \mathbf{s},\ Q_n(\tau) = \nabla (tr(\tau)) \cdot \mathbf{n},$$

$\mathbf{n} =$ *unit outer normal vector along* ∂K,
$\mathbf{s} =$ *unit tangent vector along* ∂K.

Proof. Firstly we prove problem (2.1) is equivalent to (2.2).

We know from [2] that the condition $\mathbf{divdiv}\tau = f$ in Ω is equivalent

$$\begin{cases} \mathbf{divdiv}\tau = f, & in\ K, \\ b_1(\tau, v_c) = 0, & \forall v_c \in U_c. \end{cases} \tag{2.4}$$

Thus the principle of minimum complementary energy can be written as follows:

$$\inf_{\substack{\tau \in (L^2(\Omega))_s^4 \\ \mathbf{divdiv}\tau = f\ in\ \Omega}} 1/2\ a(\tau, \tau) = \inf_{\substack{\tau \in \mathbf{V},\ \mathbf{divdiv}\tau = f\ in\ K \\ b_1(\tau, v_c) = 0,\ \forall v_c \in U_c}} 1/2\ a(\tau, \tau) \tag{2.5}$$

By using the same technique as in [2] (see also [22],[23]), we rewrite (2.5) as the following unconstrained problem:

$$\inf_{\tau\in\mathbf{V}}\{1/2a(\tau,\tau)+\sup_{v\in U}\sum\int_K v(-\mathbf{divdiv}\tau+f)d\mathbf{x}+\sup_{v_c\in U_c}b_1(\tau,-v_c)\}$$

$$=\inf_{\tau\in\mathbf{V}}\sup_{(v,v_c)\in U\times U_c}\{1/2a(\tau,\tau)+\sum\int_K v(-\mathbf{divdiv}\tau+f)d\mathbf{x}+b_1(\tau,-v_c)\}$$

Applying Green's formula

$$\int_K \tau:\mathbf{D}_2(v)d\mathbf{x}-\int_K \mathbf{divdiv}\tau\ v d\mathbf{x}=\oint_{\partial K}[M_{nn}(\tau)\nabla v\cdot\mathbf{n}+M_{ns}(\tau)\nabla v\cdot\mathbf{s}-Q_n(\tau)v]ds$$

for all $\tau\in(H^2(K))^4_s$ and $v\in H^2(K)$, We then get the saddle point problem (2.2).

Next, we have

$$H_0^2(\Omega)=\{v\in U;\exists v_c\in U_c,s.t.b_1(\tau,v-v_c)=0,\forall\tau\in\mathbf{V}\}=:U_0 \tag{2.6}$$

In fact, it's trivial that $H_0^2(\Omega)\subset U_0$. Thus it is sufficient to prove the following conclusion: If $(v,v_c)\in U\times U_c$ is such that for $\forall K\in T_h,\forall\tau\in\mathbf{V}$,

$$\oint_{\partial K}[M_{nn}(\tau)\nabla(v-v_c)\cdot\mathbf{n}+M_{ns}(\tau)\nabla(v-v_c)\cdot\mathbf{s}-Q_n(\tau)(v-v_c)]ds=0, \tag{2.7}$$

we have $v=v_c$ on ∂K, that is, $v\in H_0^2(\Omega)$.

For a given couple $(v,v_c)\in U\times U_c$, by virtue of Lax–Milgram theorem, there exists a unique solution $w_K\in H_0^2(K)$ such that

$$\int_K m(\mathbf{D}_2w):\mathbf{D}_2(v-v_c-w_K)d\mathbf{x}=0,\ \ \forall w\in H_0^2(K). \tag{2.8}$$

Setting $\tau=m(\mathbf{D}_2w)$, by Green's formula, (2.7) and (2.8) we get

$$\begin{aligned}0&=\oint_{\partial K}[M_{nn}(\tau)\nabla(v-v_c)\cdot\mathbf{n}+M_{ns}(\tau)\nabla(v-v_c)\cdot\mathbf{s}-Q_n(\tau)(v-v_c)]ds\\ &=\int_K[\tau:\mathbf{D}_2(v-v_c-w_K)-\mathbf{divdiv}\tau:(v-v_c-w_K)]d\mathbf{x}\\ &=\int_K[m(\mathbf{D}_2w):\mathbf{D}_2(v-v_c-w_K)-\mathbf{divdiv}(m(\mathbf{D}_2w))(v-v_c-w_K)]d\mathbf{x}\\ &=-\int_K\triangle^2w\cdot(v-v_c-w_K)d\mathbf{x},\ \forall w\in H_0^2(K)\end{aligned}$$

which implies that $(v-v_c-w_K)|_K=0,\forall K$. Therefor (2.6) holds.

So we can write the principle of minimum potential energy as follows:

$$\inf_{v\in H_0^2(\Omega)}[1/2\ d(v,v)-f(v)]=\inf_{\substack{(v,v_c)\in U\times U_c\\ b_1(\tau,v-v_c)=0,\ \forall\tau\in\mathbf{V}}}[1/2\ d(v,v)-f(v)]$$

which can as well be written as

$$\inf_{(v,v_c)\in U\times U_c}[1/2\ \ d(v,v)-f(v)+\sup_{\tau\in\mathbf{V}}b_1(\tau,v_c-v)],$$

just the saddle point problem (2.3). The theorem is proven.

Remark 2.1. If $v \in U \bigcap H_0^1(\Omega)$, the term $b_1(\tau, v - v_c)$ in (2.6) can be reduced to

$$b_1(\tau, v - v_c) = \sum \oint_{\partial K} M_{nn}(\tau) \nabla(v - v_c) \cdot \mathbf{n} ds. \tag{2.9}$$

According to the optimality conditions of saddle point problems, the problems (2.2) and (2.3) can be changed as:

Find $(\sigma, u, u_c) \in \mathbf{V} \times U \times U_c$ such that

$$a(\sigma, \tau) - b_2(\tau, u) + b_1(\tau, u - u_c) = 0, \quad \forall \tau \in \mathbf{V} \tag{2.10}$$

$$b_2(\sigma, v) - b_1(\sigma, v - v_c) = f(v), \quad \forall (v, v_c) \in U \times U_c \tag{2.11}$$

Find $(\sigma, u, u_c) \in \mathbf{V} \times U \times U_c$ such that

$$b_1(\tau, u - u_c) = 0, \quad \forall \tau \in \mathbf{V} \tag{2.12}$$

$$-b_1(\sigma, v - v_c) + d(u, v) = f(v), \quad \forall (v, v_c) \in U \times U_c \tag{2.13}$$

These are the desired variational formulations. However, in this paper we do not directly discretize the two saddle point problems which require LBB condition. We will use the same stability-enhanced technique—combined stabilization as in the papers [22][23][26] to circumvent inf-sup conditions.

The combined hybrid variational principle reads as:

Find $(\sigma, u, u_c) \in \mathbf{V} \times U \times U_c$ such that

$$\alpha a(\sigma, \tau) - \alpha\, b_2(\tau, u) + b_1(\tau, u - u_c) = 0, \ \forall \tau \in \mathbf{V} \tag{2.14}$$

$$\alpha b_2(\sigma, v) - b_1(\sigma, v - v_c) + (1 - \alpha) d(u, v) = f(v), \quad \forall (v, v_c) \in U \times U_c \tag{2.15}$$

where the weight factor $\alpha \in (0, 1)$.

Assume that (σ, u) is the solution of the plate bending problem (2.1) , then it is not hard to see that $(\sigma, (u, u_c))$ is the solution of the problem (2.14)(2.15), where $u_c|_{\partial K} = u|_{\partial K}$ and $\nabla u_c \cdot n|_{\partial K} = \nabla u \cdot n|_{\partial K}$ for $\forall K \in T_h$. In fact, we have

Theorem 2.2. *$(\sigma, (u, u_c))$ is the unique solution of the combined hybrid problem (2.14)(2.15).*

Proof. For the uniqueness of the solution of (2.14)(2.15), we only need to show that the following problem

$$\alpha a(\bar{\sigma}, \tau) - \alpha\, b_2(\tau, \bar{u}) + b_1(\tau, \bar{u} - \bar{u}_c) = 0, \quad \forall \tau \in \mathbf{V} \tag{2.16}$$

$$\alpha b_2(\bar{\sigma}, v) - b_1(\bar{\sigma}, v - v_c) + (1 - \alpha) d(\bar{u}, v) = 0, \quad \forall (v, v_c) \in U \times U_c \tag{2.17}$$

only has a zero solution. Actually, take $\tau = \bar{\sigma}$ in (2.16) and $(v, v_c) = (\bar{u}, \bar{u}_c)$ in (2.17), and combine the two relations, then we have

$$\alpha a(\bar{\sigma}, \bar{\sigma}) + (1 - \alpha) d(\bar{u}, \bar{u}) = 0$$

which implies $\bar{\sigma} = 0$ and $\bar{u}|_K$ is a linear polynomial. Thus by (2.16) we get

$$b_1(\tau, \bar{u} - \bar{u}_c) = 0, \quad \forall \tau \in \mathbf{V},$$

which yields $\bar{u} \in H_0^2(\Omega)$ and $(\bar{u} - \bar{u}_c)|_{\partial K} = 0$. So $\bar{u} = 0$ and $\bar{u}_c = 0$.

3. Stabilized Hybrid Schemes

In this section, we will discuss the construction of stabilized hybrid schemes and the convergence. For the sake of simplicity, we assume that Ω is a polygonal domain. In what follows $P_t(K)$ denotes the set of polynomials of degree $\leq t$ for an integer $t \geq 0$.

Firstly we introduce three finite dimentional subspaces of piecewise polynomials $\mathbf{V}^h, U^h$ and U_c^h such that $\mathbf{V}^h \subset \mathbf{V}, U^h \subset U$ and $U_c^h \subset U_c$.

We discretize the problem (2.14)(2.15) as follows:

Find $(\sigma_h, u_h, u_{c,h}) \in \mathbf{V}^h \times U^h \times U_c^h$ such that

$$\alpha a(\sigma_h, \tau) - \alpha b_2(\tau, u_h) + b_1(\tau, u_h - u_{c,h}) = 0, \quad \forall \tau \in \mathbf{V}^h \tag{3.1}$$

$$\alpha b_2(\sigma_h, v) - b_1(\sigma_h, v - v_c) + (1-\alpha) d(u_h, v) = f(v), \quad \forall (v, v_c) \in U^h \times U_c^h \tag{3.2}$$

For this problem , the interelement deflection subspace U_h and the interelement boundary deflection subspace $U_{c,h}$ may be chosen independently, but in application the two are usually coupled so that the two displacement subspaces U_h and $U_{c,h}$ can be reduced into one (see also [23,26,27]). For this we give:

Definition 3.1. *A nonconforming space U^h is weakly compatible if, for $K \in T_h$, there exists a set S_K of C^1-continuous nodal points on ∂K such that :*

(D1) $d(v,v) = 0$ *implies* $v = 0$*;*

(D2) A linear mapping $T_c : v \in U^h \to v_c = T_c(v) \in U_c$ can be established, i.e., $v \in U^h$ has a corresponding element-boundary conforming component $T_c(v) \in U_c^h$. In other words, $T_c(v)$ is determined by the nodal parameters of $v \in U^h$.

Remark 3.1. A nonconforming element which is C^1-continuous at all vertices of element is weakly compatible, and an element which is not C^1-continuous at all the vertices but is interpolated indirectly by these C^1-continuous nodal parameters $C^1(p_v, S_K) = \{p_v(a_i), \partial_1 p_v(a_i), \partial_2 p_v(a_i)\}$, is also weakly compatible. Thus, all the nonconforming plate elements with C^1-continuous vertices mentioned in Ciarlet's book [4] and [15-18] can be considered as weakly compatible elements.

As to the construction of the operator T_c, we have

Proposition 3.1. *(i) Assume that the set of nodal parameters of $v \in U^h$ on each side K' of element K (a triangle or a quadrilateral) is*

$$\Sigma_{K'}(p_v) = \{p_v(a_i), \partial_1 p_v(a_i), \partial_2 p_v(a_i), i = 1,2\},$$

where a_1 and a_2 are the endpoints of K'. Then T_c can be constructed as

$$\forall v \in U^h, T_c(v)|_{K'} \in P_3(K'), \nabla T_c(v) \cdot \mathbf{n}|_{K'} \in P_1(K') \tag{3.3}$$

such that for $i = 1,2$,

$$T_c(v)(a_i) = p_v(a_i), \nabla T_c(v)(a_i) \cdot \mathbf{s} = \nabla p_v(a_i) \cdot \mathbf{s}, \tag{3.4}$$

$$\nabla T_c(v)(a_i) \cdot \mathbf{n} = \nabla p_v(a_i) \cdot \mathbf{n}, \tag{3.5}$$

and then the following invariance

$$\begin{cases} T_c(p_{k+2})|_{\partial K} = p_{k+2}|_{\partial K}, \\ \nabla T_c(p_{k+2}) \cdot \mathbf{n}|_{\partial K} = \nabla p_{k+2} \cdot \mathbf{n}|_{\partial K} \end{cases} \tag{3.6}$$

holds for $\forall p_{k+2} \in P_{k+2}(K)$ with $k = 0$;

(ii) Assume that $\sum_{K'} = \{p_v(a_i), \partial_1 p_v(a_i), \partial_2 p_v(a_i), i = 1,2; \nabla p_v(a_3) \cdot \mathbf{n}\}$, where $a_3 = a_{12}$ is the midpoint of K'. Then T_c can be constructed as

$$\forall v \in U^h, T_c(v)|_{K'} \in P_3(K'), \nabla T_c(v) \cdot \mathbf{n}|_{K'} \in P_2(K') \tag{3.7}$$

such that for $i = 1,2$,

$$T_c(v)(a_i) = p_v(a_i), \nabla T_c(v)(a_i) \cdot \mathbf{s} = \nabla p_v(a_i) \cdot \mathbf{s}, \tag{3.8}$$

and for $i = 1,2,3$,

$$\nabla T_c(v)(a_i) \cdot \mathbf{n} = \nabla p_v(a_i) \cdot \mathbf{n}, \tag{3.9}$$

and then the invariance (3.6) holds with $k = 1$.

The proof is trivial.

Now we replace U_c^h with $T_c(U^h) := \{T_c(v);\ \forall v \in U^h\}$ to couple U_c^h with U^h. The problem (3.1)(3.2) reduces to:

Find $(\sigma_h, u_h) \in \mathbf{V}^h \times U^h$ such that

$$\alpha a(\sigma_h, \tau) - \alpha b_2(\tau, u_h) + b_1(\tau, u_h - T_c(u_h)) = 0, \quad \forall \tau \in \mathbf{V}^h \tag{3.10}$$

$$\alpha b_2(\sigma_h, v) - b_1(\sigma_h, v - T_c(v)) + (1-\alpha) d(u_h, v) = f(v), \quad \forall v \in U^h \tag{3.11}$$

Remark 3.2. If the weakly compatible space $U^h \subset C^0(\bar{\Omega})$, then by Proposition 3.1, $T_c(v)|_{\partial K} = v|_{\partial K}$, and the following relation holds:

$$b_1(\tau, v - T_c(v)) = \sum \oint_{\partial K} M_{nn}(\tau) \nabla (v - T_c(v)) \cdot \mathbf{n} ds$$

Moreover, if $U^h \subset C^1(\bar{\Omega})$, then $T_c(v)$ is such that $b_1(\tau_h, v - T_c(v)) = 0$, and the combined hybrid scheme (3.10)(3.11) is reduced to a usual conforming form.

It is easy to see that the scheme (3.10)(3.11) is stabilized for a weakly compatible subspace U^h, i.e., the existence and uniqueness of the discrete solution (σ_h, u_h) is independent of inf-sup conditions and any other patch test. In fact, since the inequality

$$\alpha a(\tau, \tau) + (1-\alpha) d(v, v) \geq C(||\tau||^2_{0,\Omega} + ||v||^2_U)$$

holds for $\forall (\tau, v) \in \mathbf{V}^h \times U^h$, where $||v||_U := (\sum \int_K m(\mathbf{D}_2 v) : \mathbf{D}_2 v)^{\frac{1}{2}}$, by Lax-Milgram Theorem we then conclude that there is a unique solution for the problem (3.10)(3.11). Furthermore, we have

Theorem 3.1. *Assume that U^h is weakly compatible and that (σ, u) is the exact solution to problem (2.1), (σ_h, u_h) the discrete solution of the problem (3.10)(3.11). Assume that*

$$|b_1(\tau, v - T_c(v))| \leq C ||\tau||_{\mathbf{V}} ||v||_U, \forall (\tau, v) \in \mathbf{V} \times U^h. \tag{3.12}$$

Then there holds the following estimate

$$\begin{aligned} &||\sigma - \sigma_h||_{0,\Omega} + ||u - u_h||_U \\ &\leq C\{\inf_{\tau \in \mathbf{V}^h} ||\sigma - \tau||_{\mathbf{V}} + \inf_{v \in U_h} [||u - v||_U + \sup_{\tau \in \mathbf{V}^h,\, \tau \neq 0} \frac{b_1(\tau, v - T_c(v))}{||\tau||_{\mathbf{V}}}]\} \end{aligned} \tag{3.13}$$

Proof. Assuming that $(\prod_1 \sigma, \prod_0 u) \in \mathbf{V}^h \times U^h$ is any given approximation of (σ, u).

Substracting the equations (3.10)(3.11) respectively from (2.14)(2.15) and recalling that $(u - u_c)|_{\partial K} = 0, \nabla (u - u_c) \cdot \mathbf{n}|_{\partial K} = 0$, we have

$$\begin{aligned} &\alpha a(\textstyle\prod_1 \sigma - \sigma_h, \tau) - \alpha b_2(\tau, \prod_0 u - u_h) + b_1(\tau, \prod_0 u - u_h - T_c(\prod_0 u - u_h)) \\ &= \alpha a(\textstyle\prod_1 \sigma - \sigma, \tau) - \alpha b_2(\tau, \prod_0 u - u) + b_1(\tau, \prod_0 u - T_c(\prod_0 u)), \forall \tau \in \mathbf{V}^h, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\alpha b_2(\textstyle\prod_1 \sigma - \sigma_h, v) - b_1(\prod_1 \sigma - \sigma_h, v - T_c(v)) + (1-\alpha) d(\prod_0 u - u_h, v) \\ &= \alpha b_2(\textstyle\prod_1 \sigma - \sigma, v) - b_1(\prod_1 \sigma - \sigma, v - T_c(v)) + (1-\alpha) d(\prod_0 u - u, v), \forall v \in U^h. \end{aligned} \tag{3.15}$$

Setting up $\tau = \delta\sigma_h := \prod_1 \sigma - \sigma_h$ and $v = \delta u_h := \prod_0 u - u_h$ in the above equations and then adding both, we get

$$\begin{aligned} &\alpha a(\delta\sigma_h, \delta\sigma_h) + (1-\alpha) d(\delta u_h, \delta u_h) \\ &= \{\alpha a(\textstyle\prod_1 \sigma - \sigma, \delta\sigma_h) - \alpha b_2(\delta\sigma_h, \prod_0 u - u_h) \\ &\quad + \alpha b_2(\textstyle\prod_1 \sigma - \sigma, \delta u_h)\} + \{-b_1(\prod_1 \sigma - \sigma, \delta u_h - T_c(\delta u_h)) \\ &\quad + (1-\alpha) d(\textstyle\prod_0 u - u, \delta u_h)\} + b_1(\delta\sigma_h, \prod_0 u - T_c(\prod_0 u)) \\ &=: \textstyle\sum(\sigma, u) + b_1(\delta\sigma_h, \prod_0 u - T_c(\prod_0 u)). \end{aligned} \tag{3.16}$$

For the algebraic sum $\sum(\sigma,u)$, by virtue of Cauchy-Schwarz inequality and $b_2(\tau,v) \leq (a(\tau,\tau))^{1/2}(d(v,v))^{1/2}$, we have

$$\begin{aligned}\sum(\sigma,u) \leq\ & \alpha(a(\prod_1 \sigma-\sigma,\prod_1 \sigma-\sigma))^{1/2}(a(\delta\sigma_h,\delta\sigma_h))^{1/2}\\ & + (1-\alpha)(d(\prod_0 u-u,\prod_0 u-u))^{1/2}(d(\delta u_h,\delta u_h))^{1/2}\\ & + |\alpha b_2(\delta\sigma_h,\prod_0 u-u_h) - \ +\alpha b_2(\prod_1 \sigma-\sigma,\delta u_h)\\ & + b_1(\prod_1 \sigma-\sigma,\delta u_h - T_c(\delta u_h))|\\ \leq\ & \alpha(a(\delta\sigma_h,\delta\sigma_h))^{1/2}[(a(\prod_1 \sigma-\sigma,\prod_1 \sigma-\sigma))^{1/2} + (d(\prod_0 u-u,\prod_0 u-u))^{1/2}]\\ & + (1-\alpha)(d(\delta u_h,\delta u_h))^{1/2}[(d(\prod_0 u-u,\prod_0 u-u))^{1/2}\\ & + \tfrac{\alpha}{1-\alpha}(a(\prod_1 \sigma-\sigma,\prod_1 \sigma-\sigma))^{1/2}] + C||\prod_1 \sigma-\sigma||_{\mathbf{V}}||\delta u_h||_U.\end{aligned}$$

Noting that $||v||_U \leq C(d(v,v))^{1/2} \leq C||v||_U$ and $(a(\tau,\tau))^{1/2} \leq C||\tau||_{\mathbf{V}}$, and using Cauch-Schwarz inequality again, we have

$$\begin{aligned}|\sum(\sigma,u)| \leq\ & C[\alpha a(\delta\sigma_h,\delta\sigma_h) + (1-\alpha)d(\delta u_h,\delta u_h)]^{1/2}\\ & \cdot [a(\prod_1 \sigma-\sigma,\prod_1 \sigma-\sigma) + d(\prod_0 u-u,\prod_0 u-u) + ||\prod_1 \sigma-\sigma||_{\mathbf{V}}^2]^{1/2}\\ \leq\ & C[\alpha a(\delta\sigma_h,\delta\sigma_h) + (1-\alpha)d(\delta u_h,\delta u_h)]^{1/2}[||\prod_0 u-u||_U^2 + ||\prod_1 \sigma-\sigma||_{\mathbf{V}}^2]^{1/2}.\end{aligned}$$

From this estimate and (3.16) we obtain

$$\begin{aligned}& [\alpha a(\delta\sigma_h,\delta\sigma_h) + (1-\alpha)d(\delta u_h,\delta u_h)]^{1/2}\\ \leq\ & C\{[||\prod_0 u-u||_U^2 + ||\prod_1 \sigma-\sigma||_{\mathbf{V}}^2]^{1/2} + \sup_{\delta\sigma_h}\frac{b_1(\delta\sigma_h,\prod_0 u - T_c(\prod_0 u))}{||\delta\sigma_h||_{\mathbf{V}}}\}.\end{aligned}$$

Thus by triangle inequalities we have

$$\begin{aligned}& ||\sigma-\sigma_h||_{0,\Omega} + ||u-u_h||_U\\ \leq\ & C[\alpha a(\prod_1 \sigma-\sigma,\prod_1 \sigma-\sigma) + (1-\alpha)d(\prod_0 u-u,\prod_0 u-u)]^{1/2}\\ & + \alpha a(\delta\sigma_h,\delta\sigma_h) + (1-\alpha)d(\delta u_h,\delta u_h)]^{1/2}\\ \leq\ & C[||\prod_0 u-u||_U + ||\prod_1 \sigma-\sigma||_{\mathbf{V}}^2 + \sup_{\tau}\frac{b_1(\tau_h,\prod_0 u - T_c(\prod_0 u))}{||\tau||_{\mathbf{V}}}].\end{aligned}$$

Because this estimate holds for any $(\prod_1 \sigma, \prod_0 u) \in \mathbf{V}^h \times U^h$, thus the theorem is proven.

By Definition 3.1, Proposition 3.1 and Theorem 3.1, we obtain the following result:

Theorem 3.2. *Assume that the exact solution (σ,u) to problem (2.1) is such that $(\sigma,u) \in (H^{k+1}(\Omega))_s^4 \times (H_0^2(\Omega)\bigcap H^{k+3}(\Omega))$ with $k \geq 0$. Let T_c be defined as in Proposition 3.1. Assume that the deflection displacement subspace U^h and the bending moments subspace $\mathbf{V}^h$ are such that:*

(A1) $\mathbf{V}^h \supset \{\tau \in \mathbf{V}; \tau_{ij}|_K \in P_k(K),\ i,j=1,2,\ \forall K \in T_h\}$;

(A2) *U^h is interpolated directly or indirectly by nodal parameters $\bigcup C^1(p_v,S_K)$ such that*

$$U^h \supset \{v \in U;\ v|_K \in P_{k+2}(K),\ \forall K \in T_h\}$$

and

$$d(v,v) = 0 \Rightarrow C^1(p_v,S_K) = \{0\}, \forall K \in T_h,$$

where $C^1(p_v,S_K)$ consists of all $\sum_{K'}(p_v)$ on element K introduced in Proposition 3.1. Then the combined hybrid scheme (3.10)(3.11) is convergent and there holds

$$||\sigma-\sigma_h||_{0,\Omega} + ||u-u_h||_U \leq Ch^{k+1}[|\sigma|_{k+1,\Omega} + |u|_{k+3,\Omega}] \qquad (3.17)$$

4. Plate Bending Element Series

Based on Theorem 3.1, we have two series of combined hybrid (abbr. CH) elements which are useful in practice. The CH-triangles with 9 DOF and CH-quadrilaterals with 12 DOF will be denoted respectively by $PB-9-CH(B,Z_3)$ and $PB-12-CH(B,A_3)$, where the DOF is as same as in Proposition 3.1 (i), Z_3 and A_3 denote respectively Zienkiewicz's interpolations and Adini's interpolations (see [4]) for the incompatible deflection displacements, and B denotes interpolations for the bending moments. Since any complete or incomplete polynomial $p_k \in P_k$ or $Q_k = \{p(x_1,x_2); p = \sum\limits_{0\le i,j\le k} a_{ij}x_1^i x_2^j\}$ with $k \ge 0$ can be chosen as the approximated bending moments, we then obtain two series of CH-plate bending elements.

By Theorem 3.2, we immediately obtain the following conclusion:

Corollary 4.1. *The combined hybrid plate elements $PB-9-CH(B,Z_3)$ and $PB-12-CH(B,A_3)$ are all convergent with accurary at least $O(h)$.*

Proof. For the above two series of CH-elements, it is obvious that the two conditions (A1) and (A2) in Theorem 3.2 are satisfied with at least $k=0$.

The combined hybrid plate elements corresponding to the case of $k=1$ in Theorem 3.1 can also be constructed in a similar way.

Conclusion. In this paper, a combined hybrid approach based on a combined variational principle is proposed for the plate bending problem. Compared with the saddle-point-type hybrid methods, it possess two main features: 1) The inf-sup difficulty is avoided so that the deflection subspace U^h and the bending moments subspace $\mathbf{V}^h$ can be chosen independently and arbitrarily; 2) The complicated equilibrium equations that should be satisfied by $\mathbf{V}^h$ is also circumvented. Then application of this method to shell analysis will be convenient. Two new series of elements $PB-9-CH(B,Z_3)$ and $PB-12-CH(B,A_3)$ are given to point out great possibility in building the plate bending elements of higher accurary according to energy compatibility condition due to many different choices of stress/strain-enriched interpolations, as we will discuss in a forthcoming paper.

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