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THE PRIMAL-DUAL POTENTIAL REDUCTION ALGORITHM FOR POSITIVE SEMI-DEFINITE PROGRAMMING*[1)]

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Abstract

In this paper we introduce a primal-dual potential reduction algorithm for positive semi-definite programming. Using the symetric preserving scalings for both primal and dual interior matrices, we can construct an algorithm which is very similar to the primal-dual potential reduction algorithm of Huang and Kortanek [6] for linear programming. The complexity of the algorithm is either $O(n\log(X^0\bullet S^0/\epsilon)$ or $O(\sqrt{n}\log(X^0\bullet S^0/\epsilon)$ depends on the value of ρ in the primal-dual potential function, where X^0 and S^0 is the initial interior matrices of the positive semi-definite programming.

Key words: Positive semi-definite programming, Potential reduction algorithms, Complexity.

1. Introduction

In this paper, we consider the following standard form of positive semi-definite programming:

(PSP) Minimize $C\bullet X$

Subject to $A_iX=b_i,\quad i=1,...,m,\quad X\succeq 0,$

where $C,X\in\mathcal{M}^n$, $A_i\in\mathcal{M}^n$, $i=1,...,m$, and $b\in R^m$. Here $\mathcal{M}^n$ denotes the set of symetric matrices in $R^{n\times n}$. Let $\mathcal{M}^n_+$ denotes the set of positive semi-definite matrices in $\mathcal{M}^n$ and $\mathcal{M}^n_{++}$ denotes the set of positive definite matrices in $\mathcal{M}^n$. We call $\mathcal{M}^n_{++}$ the interior of $\mathcal{M}^n$. The notation $X\succeq 0$ means that $X\in\mathcal{M}^n_+$, and $X\succ 0$ means that $X\in\mathcal{M}^n_{++}$. If $X\succ 0$ satisfies all equations in (PSP), it is called a primal interior feasible solution. The $\bullet$ operation is the matrix inner product

$$A\bullet B := trA^TB=\sum_{i,j}A_{ij}B_{ij}.$$

The dual problem to (PSP) can be written as:

(PSD) Maximize b^Ty

Subject to $S=C-\sum_{i=1}^m y_iA_i,\qquad S\succeq 0,$

where $S\in\mathcal{M}^n$, $y\in R^m$. If a point $(y,S\succ 0)$ satisfies all equations in (PSD), it is called a dual interior feasible solution.

Define the Frobenius norm, or the l_2 norm, of the matrix $X\in\mathcal{M}^n$ by

$$\|X\| := \|X\|_f=\sqrt{X\bullet X}=\sqrt{\sum_{j=1}^n(\lambda_j(X))^2},$$

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where $\lambda_j(X)$ is the jth eigenvalue of X, and the l_∞ norm of X by

$$\|X\|_\infty := \max_{j\in\{1,\ldots,n\}}\{|\lambda_j(X)|\}.$$

Since semi-definite programming has many applications in combinatorial optimization, control theory, statistics, etc., it becomes a hot research topic in optimization over the last decade. Many interior point algorithms have been developed to solve the semi-definite programming. The primal potential reduction algorithms were developed by Alizadeh [1], Nesterov and Nemirovskii [7], Ye [13], etc.; the primal-dual potential reduction algorithms using symetric matrix scaling were proposed by Nesterov and Todd [8], Kojima, Shindoh and Hara [4], among others. In this paper, we introduce a primal-dual potential reduction algorithm, which uses seperate matrices scaling, for above positive semi-definite programming. This kind of scaling has been used extensively in interior point algorithms for linear programming (e.g.,Kojima et la' [4], Huang and Kortanek [7],[8], Gonzaga and Todd [6]). To the best of our knowledge we have not seen a paper on interior-point algorithms for semi-definite programming which uses such seperate matrices scaling.

To measure the progress of the algorithm, we will use the following primal-dual potential function

$$\phi(X,S) = \rho \log X \bullet S - \log det XS. \tag{1}$$

The reduction in potential function is controlled by the length of projection of the search directions. In this paper we show that the length of projection is bounded below by 1/4 if $\rho = n + \sqrt{n}$. Furthermore, we prove that the length is greater than or equal to one if $\rho \geq 2n + \sqrt{2n}$. These results are the extentions of the results in Huang and Kortanek [8] for linear programming to semi-definite programming.

2. The Search Directions

The gradient matrices of (1) are

$$\nabla\phi_X(X,S) = \frac{\rho}{X\bullet S}S - X^{-1}, \tag{2}$$

$$\nabla\phi_S(X,S) = \frac{\rho}{X\bullet S}X - S^{-1}. \tag{3}$$

Let $A = (a_1,\cdots,a_n)$ be any $n\times n$ matrix, where a_j $(j = 1,\cdots,n)$ are columns of A, we define the vector of A as follows:

$$vec(A) = (a_1^T,\cdots,a_n^T)^T.$$

Then define

$$\mathcal{A} = \begin{pmatrix} vec(A_1)^T \\ vec(A_2)^T \\ \vdots \\ vec(A_m)^T \end{pmatrix}.$$

Also define the operator $\mathcal{A} : \mathcal{M}^n \to R^m$ as follows:

$$\mathcal{A}X = \mathcal{A}vec(X).$$

Furthermore

$$\mathcal{A}^T y = \sum_{i=1}^m y_i A_i.$$

Given a primal-dual interior feasible solution (X^0, y^0, S^0) such that $\mathcal{A}X^0 = b$ and $S^0 = C - \mathcal{A}^T y^0$, and a $\beta \in (0,1)$, we consider the following homogeneous minimization problem:

$$\begin{array}{lll}
\text{(HPSD)} & \min & \nabla\phi_X(X^0,S^0)\bullet\triangle X + \nabla\phi_S(X^0,S^0)\bullet\triangle S \\
 & \text{s.t.} & \mathcal{A}\triangle X = 0 \\
 & & \mathcal{A}^T\triangle y + \triangle S = 0 \\
 & & \|(X^0)^{-.5}\triangle X(X^0)^{-.5}\|^2 + \|(S^0)^{-.5}\triangle S(S^0)^{-.5}\|^2 \leq \beta^2 < 1.
\end{array}$$

Let $\bar{X} = (X^0)^{-.5}X(X^0)^{-.5}$, $\bar{S} = (S^0)^{-.5}S(S^0)^{-.5}$, $\triangle\bar{X} = (X^0)^{-.5}\triangle X(X^0)^{-.5}$, $\triangle\bar{S} = (S^0)^{-.5}\triangle S(S^0)^{-.5}$, $\nabla\bar{\phi}_X(X^0,S^0) = (X^0)^{-.5}\nabla\phi_X(X^0,S^0)(X^0)^{-.5}$, $\nabla\bar{\phi}_S(X^0,S^0) = (S^0)^{-.5}\nabla\phi_X(X^0,S^0)\ (X^0)^{-.5}$, $A_i{}' = (X^0)^{-.5}A_i(X^0)^{-.5}$ and $A_i" = (S^0)^{-.5}A_i(S^0)^{-.5}$ for $i = 1, ..., m$, and

$$\mathcal{A}' = \begin{pmatrix} vec(A'_1)^T \\ vec(A'_2)^T \\ \vdots \\ vec(A'_m)^T \end{pmatrix}, \qquad \mathcal{A}" = \begin{pmatrix} vec(A''_1)^T \\ vec(A''_2)^T \\ \vdots \\ vec(A''_m)^T \end{pmatrix}.$$

Note that for any symmetric matrices $A, B \in \mathcal{M}^n_+$ and $X \in \mathcal{M}^n_{++}$,

$$A \bullet X^{.5}BX^{.5} = X^{.5}AX^{.5} \bullet B$$

and

$$\|XA\|_l = \|AX\|_l = \|X^{.5}AX^{.5}\|_l,$$

where $l = 2, \infty$. Then (HPSD) becomes:

$$\begin{array}{lll} \text{(HPSD')} & \min & \nabla\bar{\phi}_X(X^0,S^0) \bullet \triangle\bar{X} + \nabla\bar{\phi}_S(X^0,S^0) \bullet \triangle\bar{S} \\ & \text{s.t.} & \mathcal{A}'\triangle\bar{X} = 0 \\ & & \mathcal{A}"^T\triangle y + \triangle\bar{S} = 0 \\ & & \|\triangle\bar{X}\|^2 + \|\triangle\bar{S}\|^2 \le \beta^2 < 1. \end{array}$$

Let $u \in R^m$, $V \in R^{n\times n}$ and $\lambda \in R$ denote Lagrangians for constriants of (HPSD') respectively. Then the KKT conditions for (HPSD') are:

$$\nabla\bar{\phi}_X + \mathcal{A}'^Tu + 2\lambda\triangle\bar{X} = 0, \tag{4}$$

$$\nabla\bar{\phi}_S + V + 2\lambda\triangle\bar{S} = 0, \tag{5}$$

$$\mathcal{A}"V = 0,. \tag{6}$$

Let operator $\mathcal{A}'$ acts on (4) and $\mathcal{A}"$ acts on (5), and using the constriants in (HPSD') we obtain the solution for (HPSD'):

$$\triangle\bar{X} = -\beta\ \frac{P_{\mathcal{A}'}\nabla\bar{\phi}_X(X^0,S^0)}{\|p(X^0,S^0)\|}, \tag{7}$$

$$\triangle\bar{S} = -\beta\ \frac{Q_{\mathcal{A}"}\nabla\bar{\phi}_S(X^0,S^0)}{\|p(X^0,S^0)\|}, \tag{8}$$

where

$$P_{\mathcal{A}'} = I - \mathcal{A}'^T(\mathcal{A}'\mathcal{A}'^T)^{-1}\mathcal{A}',$$

$$Q_{\mathcal{A}"} = \mathcal{A}"^T(\mathcal{A}"\mathcal{A}"^T)^{-1}\mathcal{A}",$$

$$p(X^0,S^0) = \begin{pmatrix} p_X(X^0,S^0) & \\ & p_S(X^0,S^0) \end{pmatrix} = \begin{pmatrix} P_{\mathcal{A}'}\nabla\bar{\phi}_X & \\ & Q_{\mathcal{A}"}\nabla\bar{\phi}_S \end{pmatrix}. \tag{9}$$

Having found the search directions $(\triangle X, \triangle S)$ of (7) and (8), we define the new iterates $X^1 = X^0 + \triangle X$, $S^1 = S^0 + \triangle S$. Constraints in $(HPSD)$ guarantee that $\{X^1, S^1\}$ is a feasible primal-dual interior solution. To show the relationship between the reduction in potential function and the projection $p(X^0,S^0)$, we state the following lemma. Its proof is similar to the proof of lemma 2.2.3 in Huang [8].

Lemma 2.1. *Let X^1, S^1 be defined as above, then*

$$\phi(X^1,S^1) - \phi(X^0,S^0) \le -\beta\|p(X^0,S^0)\| + \frac{\beta^2}{2(1-\beta)} \tag{10}$$

From lemma 2.1 we can see that if $\|p(X^0,S^0)\| \ge \alpha$ for some $\alpha \in (0,1)$, then

$$\phi(X^1,S^1) - \phi(X^0,S^0) \le -\beta\alpha + \frac{\beta^2}{2(1-\beta)}, \tag{11}$$

hence the potential function will decrease by a constant.

3. The Length of the Primal-Dual Projection with $\rho = n + \sqrt{n}$

From (8) we can see that the length of $p(X^0, S^0)$ is a key measurement of the reduction in potential function $\phi(X,S)$. Obviously, it would be nice if $\|p(X,S)\|$ is bounded away from zero. In this section we show that $\|p(X,S)\|$ of (9) is always great than or equal to 1/4 for $\rho = n + \sqrt{n}$. The analysis uses the similar approach as in [8] except that we are dealing with the semi-definite programming now. We introduce some notations first.

Let $F(P)$ and $F(D)$ be the feasible regions of (PSP) and (PSD) respectively, and $F_+(P)$ and $F_+(D)$ be the interiors of $F(P)$ and $F(D)$ respectively. Define

$$F_+ = \{W = \begin{pmatrix} X & \\ & S \end{pmatrix} | X \in F_+(P), S \in F_+(D)\}.$$

For $W \in F_+$, we define a primal-dual penalized function by

$$f_\mu(W) = \mu X \bullet S - \log det W = \mu X \bullet S - \log det XS. \tag{12}$$

Note that, if we let $D = \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}b$, then $\mathcal{A}D = b$, hence $b^Ty = b^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}(C - S) = D \bullet C - D \bullet S$ for $y \in F(D)$. Therefore,

$$X \bullet S = C \bullet X - b^Ty = C \bullet X + D \bullet S - D \bullet C.$$

So (12) becomes

$$f_\mu(W) = \mu(C \bullet X + D \bullet S) - \mu D \bullet C - \log det W = \mu \hat{C} \bullet W - \mu D \bullet C - \log det W, \tag{13}$$

where $\hat{C} = \begin{pmatrix} C & \\ & D \end{pmatrix}$. We also define a class of primal-dual potential functions on F_+ by

$$\phi(W) = \rho \log(\hat{C} \bullet W - D \bullet C) - \log det W \tag{14}$$

for $\rho = n + \sqrt{n}$.

Our analysis is based on the relationship between $\phi(W)$ and $f_\mu(W)$.

The minimizer of $f_\mu(W)$ over F_+ is called the μ-center of F_+ and is denoted by $W(\mu) = \begin{pmatrix} X(\mu) & \\ & S(\mu) \end{pmatrix}$. Note that if $W(\mu)$ is the minimizer of $f_\mu(W)$, then

$$X(\mu) \bullet S(\mu) = \frac{n}{\mu}. \tag{15}$$

The path $\{W(\mu))|\mu \geq 0\}$ is called the central path.

Let $p(X,S)$ be the primal-dual projection of (6), and $p = \|p(X,S)\|$. Then we have the following lemma.

Lemma 3.1. *Let $W \in F_+$ and $W(\mu) \in F_+$ be a μ center of F_+, $\bar{W}(\mu) = W^{-.5}W(\mu)W^{-.5}$ and let*

$$\gamma = \frac{\|W^{-.5}(W(\mu) - W)W^{-.5}\|_2}{\|W^{-.5}(W(\mu) - W)W^{-.5}\|_\infty} = \frac{\|\bar{W}(\mu) - I_{2n}\|_2}{\|\bar{W}(\mu) - I_{2n}\|_\infty}$$

where $I_{2n} \in R^{2n \times 2n}$ is the unit matrix. If $p < \frac{1}{4}$, then

$$q = \|\bar{W}(\mu) - I_{2n}\|_2 \leq 2p, \tag{16}$$

and

$$|\hat{C} \bullet W(\mu) - \hat{C} \bullet W| < \frac{\sqrt{n}}{\mu}. \tag{17}$$

Proof. Let $P = \begin{pmatrix} P_{\mathcal{A}'} & \\ & Q_{\mathcal{A}"} \end{pmatrix}$, and

$$\bar{G} = \frac{\bar{W}(\mu) - I_{2n}}{\|\bar{W}(\mu) - I_{2n}\|_2},$$

then $\|\bar{G}\|_2 = 1$, and $\bar{W}(\mu) = I_{2n} + q\bar{G}$. Since scaled primal-dual penalized function

$$\bar{f}_\mu = \mu \bar{C} \bullet \bar{W} - \mu D \bullet C - \log det W det \bar{W}$$

is convex and minimized at $\bar{W}(\mu)$, where $\bar{C} = W^{.5}\hat{C}W^{.5}$,

$$\bar{G} \bullet \nabla \bar{f}_\mu(I_{2n} + \lambda \bar{G}) < 0$$

for $0 \le \lambda < p$. But,

$$\begin{aligned}\nabla \bar{f}_\mu(I_{2n} + \lambda \bar{G}) &= \mu \bar{C} - (I_{2n} + \lambda \bar{G})^{-1} \\ &= \mu \bar{C} - I_{2n} + \lambda \bar{G}(I_{2n} + \lambda \bar{G})^{-1} \\ &= \nabla \bar{f}_\mu(I_{2n}) + \lambda \bar{G} - \lambda^2 \bar{G}^2 (I_{2n} + \lambda \bar{G})^{-1}.\end{aligned}$$

Also,

$$\begin{aligned}P\bar{G} &= \frac{1}{q}\begin{pmatrix} P_{\mathcal{A}'}(\bar{X}(\mu) - I_n) & \\ & Q_{\mathcal{A}"}(\bar{S}(\mu) - I_n)\end{pmatrix} \\ &= \frac{1}{q}\begin{pmatrix} \bar{X}(\mu) - I_n & \\ & \bar{S}(\mu) - I_n\end{pmatrix} \\ &= \frac{1}{q}(\bar{W}(\mu) - I_{2n}) \\ &= \bar{G}.\end{aligned}$$

Since $(BA) \bullet C = A \bullet (BC)$ for symmetric matrices A and B, and

$$|\bar{G} \bullet \nabla \bar{f}_\mu(I_{2n})| = |P\bar{G} \bullet \nabla \bar{f}_\mu(I_{2n})| \le \|\bar{G}\| \|P \nabla \bar{f}_\mu(I_{2n})\| \le p.$$

Therefore,

$$\begin{aligned}0 &> \bar{G} \bullet \nabla \bar{f}_\mu(I_{2n} + \lambda \bar{G}) \\ &= \bar{G} \bullet \nabla \bar{f}_\mu(I_{2n}) + \|\lambda \bar{G}\|^2 - \lambda^2 \bar{G} \bullet \bar{G}^2 (I_{2n} + \lambda \bar{G})^{-1} \\ &\ge -p + \lambda - \lambda^2 \|\bar{G}\|_\infty \|\bar{G}\|^2 \\ &= -p + \lambda - \frac{\lambda^2}{\gamma},\end{aligned} \tag{18}$$

where the last equality uses the facts $\|\bar{G}\|_\infty = \frac{1}{\gamma}$, and $\|\bar{G}\| = 1$. Thus,

$$\begin{aligned}0 &< p - \lambda + \frac{\lambda^2}{\gamma} \\ &\le \frac{\lambda^2}{\gamma} - \lambda + \frac{\gamma}{4} \\ &= \frac{(\lambda - \frac{\gamma}{2})^2}{\gamma}\end{aligned}$$

for $0 \le \lambda < q$, so $q \le \frac{\gamma}{2}$. This implies that

$$\frac{\lambda^2}{\gamma} \le \frac{q\lambda}{\gamma} \le \frac{\lambda}{2}$$

for $0 \le \lambda < q$, so (18) yields

$$-p + \lambda - \frac{\lambda}{2} < 0 \quad \text{for} \quad 0 \le \lambda < q,$$

from which $q \le 2p$ follows.

To prove (17), we have

$$\begin{aligned}&|\bar{C} \bullet (\bar{W}(\mu) - I_{2n})| \\ &= \frac{|\mu \bar{C} \bullet (q\bar{G})|}{\mu} = \frac{|(\mu \bar{C} - I_{2n}) \bullet (q\bar{G}) + I_{2n} \bullet q\bar{G}|}{\mu} \\ &= \frac{|q \nabla \bar{f}_\mu(I_{2n}) \bullet \bar{G} + q I_{2n} \bullet \bar{G}|}{\mu} \le \frac{q}{\mu}(\nabla \bar{f}_\mu(I_{2n}) \bullet \bar{G} + |I_{2n} \bullet \bar{G}|) \\ &\le \frac{q}{\mu}(p + \|I_{2n}\| \|\bar{G}\|) \le \frac{2p}{\mu}(p + \sqrt{2n}) \le \frac{2p}{\mu}(\frac{1}{4} + \sqrt{2n}) \\ &\le \frac{5\sqrt{2n}}{2}\frac{p}{\mu} < \frac{4p\sqrt{n}}{\mu} < \frac{\sqrt{n}}{\mu},\end{aligned}$$

where the last inequality uses the assumption $p < \frac{1}{4}$.

Using lemma 3.1 we can prove the following main result of this section immediately.

Theorem 3.2. *Let $W \in F_+$, and let $\mu = \frac{\rho}{X \bullet S}$, where $\rho = n + \sqrt{n}$. Then*

$$p = \|p(x,s)\| \geq \frac{1}{4}.$$

Proof. Assume $p < \frac{1}{4}$, then lemma 3.1 applies since $\gamma \geq 1$, and we have

$$|\hat{C} \bullet W(\mu) - \hat{C} \bullet W| < \frac{\sqrt{n}}{\mu}.$$

Since

$$\hat{C} \bullet W(\mu) - \hat{C} \bullet W = X(\mu) \bullet S(\mu) - X \bullet S,$$

therefore by (15) and the fact that $\mu = \frac{\rho}{X \bullet S}$ we have

$$\begin{aligned} \frac{n}{\rho} X \bullet S = \frac{n}{\mu} &= X(\mu) \bullet S(\mu) \\ &> X \bullet S - \frac{\sqrt{n}}{\mu} \\ &= \frac{(\rho - \sqrt{n}) X \bullet S}{\rho} \\ &= \frac{n X \bullet S}{\rho}, \end{aligned}$$

a contradiction.

Theorem 3.2 shows that $p(X,S)$ can be bounded away from zero. Therefore, we can give a primal-dual potential reduction algorithm as follows.

Primal-Dual Potential Reduction Algorithm

Given $X^0, y^0, S^0 \in F_+$ set $\alpha = \frac{1}{4}$, $\beta = 0.5$, $\epsilon > 0$ and $k = 0$;
while $(X^k) \bullet S^k \geq \epsilon$ do
begin
Compute $p(X^k, S^k)$ of (9), let

$$X^{k+1} = X^k - \beta (X^k)^{.5} p_X(X^k, S^k)(X^k)^{.5} / \|p(X^k, S^k)\|;$$
$$S^{k+1} = S^k - \beta (S^k)^{.5} p_S(X^k, S^k)(S^k)^{.5} / \|p(X^k, S^k)\|.$$

end
$k = k + 1$
end.

Of course the iteration complexity of the above algorithm is $O(\sqrt{n} \log X^0 \bullet S^0 / \epsilon)$. In the next section we try to increase the lower bound of $\|p(x,s)\|$ by choosing a larger ρ in the primal-dual potential function of (1).

4. The Length of Primal-Dual Projection with $\rho = 2n + \sqrt{2n}$

In the last section we have shown that if $\rho = n + \sqrt{n}$, then $\|p(X,S)\| \geq 1/4$. Obviously, one can see from (2), (3), and (9) that the length of $p(X,S)$ increases as the value of ρ increases. Naturally one would ask whether it is possible to improve the lower bound of $\|p(X,S)\|$ if we choose a larger ρ in the potential function. We prove, in this section, that this is indeed possible. That is $\|p(X,S)\| \geq 1$ if $\rho = 2n + \sqrt{2n}$. We prove the following lemma first.

Lemma 4.1. *Let $\{X, y, S\} \in F_+$ and $p_X(X,S)$, $p_S(X,S)$ as given in (6), then for any $X' \in F(PSP)$ and $(S', y') \in F(PSD)$, we have*

$$p_X(X,S) \bullet X^{-.5}(X' - X)X^{-.5} = \nabla \phi_X(X,S) \bullet (X' - X), \tag{19}$$

$$p_S(X,S) \bullet S^{-.5}(S' - S)S^{-.5} = \nabla \phi_S(X,S) \bullet (S' - S). \tag{20}$$

Proof. By (6) we have

$$\begin{aligned} & p_X(X,S)\bullet X^{-.5}(X'-X)X^{-.5} \\ &= P_{\mathcal{A}'}X^{.5}\nabla\phi_X(X,S)X^{.5}\bullet \quad X^{-.5}(X'-X)X^{-.5} \\ &= X^{.5}\nabla\phi_X(X,S)X^{.5}\bullet \quad P_{\mathcal{A}'}X^{-.5}(X'-X)X^{-.5} \\ &= X^{.5}\nabla\phi_X(X,S)X^{.5}\bullet X^{-.5}(X'-X)X^{-.5} \\ &= \nabla\phi_X(X,S)\bullet(X'-X) \end{aligned}$$

since $\mathcal{A}(X'-X)=0$. Similarly,

$$\begin{aligned} & p_S(X,S)\bullet S^{-.5}(S'-S)S^{-.5} \\ &= Q_{\mathcal{A}"}S^{.5}\nabla\phi_S(X,S)S^{.5}\bullet \quad S^{-.5}(S'-S)S^{-.5} \\ &= S^{.5}\nabla\phi_S(X,S)S^{.5}\bullet \quad Q_{\mathcal{A}"}S^{-.5}(-\mathcal{A}^T(y'-y))S^{-.5} \\ &= S^{.5}\nabla\phi_S(X,S)S^{.5}\bullet \quad Q_{\mathcal{A}"}(-\mathcal{A}"^T(y'-y)) \\ &= S^{.5}\nabla\phi_S^T(X,S)S^{.5}\bullet \quad (-\mathcal{A}"^T(y'-y)) \\ &= \nabla\phi_S(X,S)\bullet(S'-S). \end{aligned}$$

since $Q_{\mathcal{A}"}\mathcal{A}"=\mathcal{A}"$.

Now we can prove the following theorem.

Theorem 4.2. *Let $\rho\geq 2n+\sqrt{2n}$ in (1), and X^*,S^* be the optimal solution for the $(PSP)-(PSD)$, then*

$$\|p(x,s)\|\geq 1. \tag{18}$$

Proof. Since $X^*\bullet S^*=0$ and $0=(X-X^*)\bullet(S-S^*)=X^*\bullet S+S^*\bullet X-X\bullet S$ implies $X\bullet S=X^*\bullet S+S^*\bullet X$. By Lemma 4.1 we have

$$\begin{aligned} & p(X,S)\bullet\begin{pmatrix} X^{-.5}(X-X^*)X^{-.5} & \\ & S^{-.5}(S-S^*)S^{-.5}\end{pmatrix} \\ &= p_X(X,S)\bullet X^{-.5}(X-X^*)X^{-.5}+p_S(X,S)\bullet S^{-.5}(S-S^*)S^{-.5} \\ &= \nabla\phi_X(X,S)\bullet(X-X^*)+\nabla\phi_S(X,S)\bullet(S-S^*) \\ &= (\frac{\rho}{X\bullet S}S-X^{-1})\bullet(X-X^*)+(\frac{\rho}{X\bullet S}X-S^{-1})\bullet(S-S^*) \\ &= \frac{\rho}{X\bullet S}(2X\bullet S-X^*\bullet S-S^*\bullet X)-2n+I\bullet\bar{X}^*+I\bullet\bar{S}^* \\ &= \rho-2n+I\bullet\bar{X}^*+I\bullet\bar{S}^* \\ &= \rho-2n+I_{2n}\bullet\bar{W}^*, \end{aligned}$$

where $\bar{X}^*=X^{-.5}X^*X^{-.5}$, $\bar{S}^*=S^{-.5}S^*S^{-.5}$, and $\bar{W}^*=W^{-.5}W^*W^{-.5}$. Hence,

$$\begin{aligned} \|p(x,s)\|\|(I_{2n}-\bar{W}^*)\| &\geq \rho-2n+I_{2n}\bullet\bar{W}^* \\ &\geq \rho-2n+\|\bar{W}^*\| \end{aligned}$$

Therefore,

$$\|p(x,s)\|\geq\frac{\rho-2n+\|\bar{W}^*\|}{\|I_{2n}-\bar{W}^*\|}\geq\frac{\rho-2n+\|\bar{W}^*\|}{\sqrt{2n}+\|\bar{W}^*\|}, \tag{19}$$

because

$$\|I_{2n}-\bar{W}^*\|\leq\|I_{2n}\|+\|\bar{W}^*\|=\sqrt{2n}+\|\bar{W}^*\|.$$

Now if $\rho-2n-\sqrt{2n}\geq 0$, then from (19) we obtain

$$\|p(X,S)\|\geq 1+\frac{\rho-2n-\sqrt{2n}}{\sqrt{2n}+\|\bar{W}^*\|}\geq 1.$$

Combining Lemma 1.1 and Theorem 4.2 we have the following potential reduction corollary.

Corollary 4.3. *Let $\rho\geq 2n+\sqrt{2n}$, and $(\triangle X,\triangle S)$ be defined as in (7), (8). Let $X^1=X+\triangle X$, $S^1=S+\triangle S$, then*

$$\phi(X^1,S^1)-\phi(X^0,S^0)\leq-\beta+\frac{\beta^2}{2(1-\beta).}$$

Corollary 4.3 shows that the potential function achieves a larger reduction if we choose a larger $\rho \geq 2n+\sqrt{2n}$, and we can take a larger step since the step length β can be larger. One drawback of choosing a larger ρ is that the iteration complexity will increase to $O(n \log X^0 \bullet S^0/\epsilon)$ since the iteration complexity for potential reduction algorithms is $O((\rho - n) \log X^0 \bullet S^0/\epsilon)$.

5. Conclusions

In this paper, we introduced a primal-dual potential reduction algorithm for solving the semi-definite programming. The value of $\|p(X,S)\|$ takes an important role in analyzing the complexity of the potential reduction algorithm, and of course is related to ρ parameter in potential function. We have shown that $\|p(x,s)\| \geq 1/4$ for $\rho = n + \sqrt{n}$ and $\|p(x,s)\| \geq 1$ if $\rho \geq 2n + \sqrt{2n}$. Therefore the algorithm does not need to take the centering steps as some primal potential reduction algorithms (e.g., [1], [13]) do. Hence the algorithm is much simpler.

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