Journal of Computational Mathematics, Vol.21, No.3, 2003, 321–324.

ON KORN'S INEQUALITY*1)

Lie-heng Wang
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This paper is devoted to give a new proof of Korn's inequality in L^r-norm $(1 < r < \infty)$.

Key words: Korn's inequality.

1. Introduction

Korn's inequality is fundamental in the theory and the numerical analysis for the elasticity. There have been many nice proofs of Korn's inequality in the literatures(see [4] and the references therein). The work [5] proposed an intuitive exposition and heuristic proof of Korn's inequality. And the works [2] and [7] give an interesting result, which is useful tool in the proof of Korn's inequality, as for example in the works [3], [6].

In this paper, we intend to show a new proof of Korn's inequality in L^r-norm $(1 < r < \infty)$, in the plane, with the help of the heurisitc work [4] and the result of [2], [7].

2. Notation and Preliminaries

We begin with some notation. Let $\Omega \subset R^n (n = 2, 3)$ denote the bounded domain with smoothly boundary $\partial\Omega$ or the polygon. Let $\vec{v}$ be the n-dimensional vector valued function defined in Ω, and

$$\epsilon_{ij}(\vec{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j), \quad \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad 1 \le i,\ j \le n. \tag{2.1}$$

And in this paper, the notation in Sobolev spaces [1] will be used.

Korn's inequality, in L^2 version, can be stated as follows: There exists C=Const. > 0, such that

$$\sum_{i,j} \|\epsilon_{ij}(\vec{v})\|^2_{0,\Omega} + \|\vec{v}\|^2_{0,\Omega} \ge C\|\vec{v}\|^2_{1,\Omega} \quad \forall \mathbf{v} \in E, \tag{2.2}$$

where

$$E = \{\vec{w} \in (L^2(\Omega))^2 : \epsilon_{ij}(\vec{w}) \in L^2(\Omega) \forall i, j\}. \tag{2.3}$$

Korn's inequality (2.2) means that the following cotaining relationship holds:

$$E \subset (H^1(\Omega))^n. \tag{2.4}$$

The relation (2.4) seems to be unexpected at the first glance, because, for the case $n = 3$, only six independent linear combinations of partial derivatives of $\vec{v} \in (H^1(\Omega))^3$ belong to $L^2(\Omega)$. However when we consider it in depth, as in [5], we find that all second order partial derivatives of $\vec{v}$ can be presented by the partial derivatives of $\epsilon_{ij}(\vec{v})$:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\epsilon_{ik}(\vec{v}) + \frac{\partial}{\partial x_k}\epsilon_{ij}(\vec{v}) - \frac{\partial}{\partial x_i}\epsilon_{jk}(\vec{v}). \tag{2.5}$$

---

* Received July 10, 2000, final evised October 20, 2000.
1) The propject supported by Natural Science Foundation of China.

Thus if $\vec{v} \in E$, then

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial v_i}{\partial x_j}\Big) \in H^{-1}(\Omega)\ \forall\ i,j,k, \tag{2.6}$$

which, roughly speaking, can be seen (the rigorous proof can be found in [5], added by $\partial v_i/\partial x_j \in H^{-1}(\Omega)\forall i,j$) as

$$\frac{\partial v_i}{\partial x_j} \in L^2(\Omega)\ \forall\ i,j. \tag{2.7}$$

This means that $\mathbf{v} \in (H^1(\Omega))^n$.

3. The Proof of Korn's Inequality

In this section, we present a new proof of Korn's inequality in L^r version, $1 < r < \infty$, in the plane ($n = 2$), which can be stated in the following:

Theorem 1 (Korn's Inequality). *There exists a positive constant α, such that*

$$\sum_{i,j} \|\epsilon_{ij}(\vec{v})\|_{0,r,\Omega} + \|\vec{v}\|_{0,r,\Omega} \geq \alpha\|\vec{v}\|_{1,r,\Omega}\ \ \forall\ \vec{v} \in (W^{1,r}(\Omega))^2. \tag{3.1}$$

In order to prove Theorem 1, we need some lemmas.

Lemma 1. *For all $w \in L^r(\Omega)$,*

$$\begin{cases} \|w\|_{-1,r,\Omega} \leq \|w\|_{0,r,\Omega}, \\ \|\nabla w\|_{-1,r,\Omega} \leq \|w\|_{0,r,\Omega}. \end{cases} \tag{3.2}$$

Lemma 1 can be proved easily by the definition of the $W^{-1,r}(\Omega)-$norm.

Lemma 2 (c.f.[2],[6]). *Assume that $\Omega \subset R^2$ be a bounded smoothly domain or polygon. Let*

$$L_0^r(\Omega) = \{p \in L^r(\Omega) : \int_\Omega pdx = 0\}. \tag{3.3}$$

Then for any given $p \in L_0^{r'}(\Omega),\ 1 < r' = r/(r-1) < \infty$, r' the conjugate number of r, there exists $\vec{\phi}_0 \in (W_0^{1,r'}(\Omega))^2$, such that

$$div\vec{\phi}_0 = p\ in\ \Omega, \quad \|\vec{\phi}_0\|_{1,r',\Omega} \leq C\|p\|_{0,r',\Omega}, \tag{3.4}$$

with a constant C independent of $\vec{\phi}_0$ and p.

Lemma 3. *For every function $v \in L^r(\Omega),\ 1 < r < \infty$,*

$$\|v\|_{0,r,\Omega} \leq \frac{1}{2C}\|\nabla v\|_{-1,r,\Omega} + \frac{1}{|\Omega|^{1/r'}}\Big|\int_\Omega vdx\Big|, \tag{3.5}$$

with the same $C = Const.$ as in (3.4) and $|\Omega| = \int_\Omega 1dx$.

Proof. For any given $v \in L^r(\Omega)$ let

$$\hat{v} = v - \frac{1}{|\Omega|}\int_\Omega vdx,$$

then

$$\|v\|_{0,r,\Omega} \leq \|\hat{v}\|_{0,r,\Omega} + \frac{1}{|\Omega|^{1/r'}}\Big|\int_\Omega vdx\Big|. \tag{3.6}$$

And

$$\|\hat{v}\|_{0,r,\Omega} = \sup_{w\in L^{r'}(\Omega)} \frac{\int_\Omega \hat{v}\cdot wdx}{\|w\|_{0,r',\Omega}} = \sup_{w\in L^{r'}(\Omega)} \frac{\int_\Omega \hat{v}(\hat{w} + \frac{1}{\Omega}\int_\Omega wdy)dx}{\|w\|_{0,r',\Omega}} \leq \frac{1}{2}\sup_{\hat{w}\in L_0^{r'}(\Omega)} \frac{\int_\Omega \hat{v}\cdot \hat{w}dx}{\|\hat{w}\|_{0,r',\Omega}},$$

since $\hat{w} = w - \frac{1}{|\Omega|}\int_\Omega w dx$, and $\|\hat{w}\|_{0,r',\Omega} \le 2\|w\|_{0,r',\Omega}$. And $\hat{w} \in L_0^{r'}(\Omega)$, then by Lemma 2, it can be seen that

$$\begin{aligned}\|\hat{v}\|_{0,r,\Omega} &\le \frac{1}{2C} \sup_{\vec{\phi}_0 \in (W_0^{1,r'}(\Omega))^2} \frac{\int_\Omega \hat{v} \cdot div\vec{\phi}_0 dx}{\|\vec{\phi}_0\|_{1,r',\Omega}} \\ &= \frac{1}{2C} \sup_{\vec{\phi}_0 \in (W_0^{1,r'}(\Omega))^2} \frac{\int_\Omega \nabla\hat{v} \cdot \vec{\phi}_0 dx}{\|\vec{\phi}_0\|_{1,r',\Omega}} \\ &= \frac{1}{2C} \sup_{\vec{\phi}_0 \in (W_0^{1,r'}(\Omega))^2} \frac{\int_\Omega \nabla v \cdot \vec{\phi}_0 dx}{\|\vec{\phi}_0\|_{1,r',\Omega}} = \frac{1}{2C}\|\nabla v\|_{-1,r,\Omega}. \end{aligned} \tag{3.7}$$

By the inequalities (3.6) and (3.7), the Lemma 3 is proved.

Lemma 4 (c.f.[5]).

$$\|v\|_{0,r,\Omega} \le C_1\|\nabla v\|_{-1,r,\Omega} + C_2\|v\|_{-1,r,\Omega} \quad \forall\ v \in L^r(\Omega), \tag{3.8}$$

with the constants C_1, and C_2 independent of $\mathbf{v}$.

Proof. There exists a function $\rho \in W_0^{1,r'}(\Omega)$, such that

$$\|1-\rho\|_{0,r',\Omega} < |\Omega|^{1/r'}, \tag{3.9}$$

since the space $C_0^\infty(\Omega)$ is dense in $L^{r'}(\Omega)$. Then

$$\Big|\int_\Omega v dx\Big| \le \Big|\int_\Omega v\rho dx\Big| + \Big|\int_\Omega (1-\rho) v dx\Big| \le \|v\|_{-1,r,\Omega}\|\rho\|_{1,r',\Omega} + \|1-\rho\|_{0,r',\Omega}\|v\|_{0,r,\Omega}$$

from which and Lemma 3, it can be seen that

$$\begin{aligned}\|v\|_{0,r,\Omega} \le \frac{1}{2C}\|\nabla v\|_{-1,r,\Omega} &+ \frac{1}{|\Omega|^{1/r'}}\|\rho\|_{1,r',\Omega}\|v\|_{-1,r,\Omega} \\ &+ \frac{1}{|\Omega|^{1/r'}}\|1-\rho\|_{0,r',\Omega}\|v\|_{0,r,\Omega}. \end{aligned} \tag{3.10}$$

Thus, since the function ρ is found with (3.9) independent of $\mathbf{v}$, then from (3.9) and (3.10), the proof is completed.

Proof of Theorem 1. Let $v = \partial v_i/\partial x_j$ in (3.8), taking into account Lemma 1 and the relation (2.5), then

$$\begin{aligned}\|\partial v_i/\partial x_j\|_{0,r,\Omega} &\le C_1\|\nabla(\partial v_i/\partial x_j)\|_{-1,r,\Omega} + C_2\|\partial v_i/\partial x_j\|_{-1,r,\Omega} \\ &\le C_1\|\nabla(\partial v_i/\partial x_j)\|_{-1,r,\Omega} + C_2\|\nabla v_i\|_{-1,r,\Omega} \\ &\le C_1\sum_k \|\partial\epsilon_{ik}(\vec{v})/\partial x_j + \partial\epsilon_{ij}(\vec{v})/\partial x_k - \partial\epsilon_{jk}(\vec{v})/\partial x_i\|_{-1,r,\Omega} + C_2\|v_i\|_{0,r,\Omega}. \end{aligned}$$

Thus

$$\begin{aligned}|\vec{v}|_{1,r,\Omega} &\le C_1''\sum_{i,j}\|\nabla\epsilon_{ij}(\vec{v})\|_{-1,r,\Omega} + C_2\|\vec{v}\|_{0,r,\Omega} \\ &\le C_1'\sum_{i,j}\|\epsilon_{ij}(\vec{v})\|_{0,r,\Omega} + C_2\|\vec{v}\|_{0,r,\Omega}, \end{aligned}$$

from which, the proof is completed.

Remark 1. Let $r = 2$, our result (3.1) is weaker than the result (2.2) of [5].

As a corollary of Theorem 1, we have the following second Korn's inequality.

Firstly we introduce some spaces. Let Γ_0 be a part of the bondary $\partial\Omega$ of Ω, and

$$(W_{\Gamma_0}^{1,r}(\Omega))^2 = \{\vec{v} \in (W^{1,r}(\Omega))^2 : \vec{v} = 0 \ \ on \ \ \Gamma_0\}. \tag{3.11}$$

And

$$(\hat{W}^{1,r}(\Omega))^2 = \{\vec{v} \in (W^{1,r}(\Omega))^2 : \int_\Omega \vec{v}dx = \vec{0}, \quad \int_\Omega rot\vec{v}dx = 0\}, \tag{3.12}$$

where

$$rot\vec{v} = -\partial v_1/\partial x_2 + \partial v_2/\partial x_1. \tag{3.13}$$

Then we have

Theorem 2 (Second Korn's Inequality) (c.f.[8], [9] and [10]). *There exists a positive constant β, such that*

$$\sum_{i,j} \|\epsilon_{ij}(\vec{v})\|_{0,r,\Omega} \geq \beta \|\vec{v}\|_{1,r,\Omega}, \tag{3.14}$$

for both cases: Foe all $\vec{v} \in (W^{1,r}_{\Gamma_0}(\Omega))^2$ with masure $(\Gamma_0) > 0$, or $\vec{v} \in (\hat{W}^{1,r}(\Omega))^2$.

Proof. Assume that inequality (3.14) is not true. Then there is a sequence $(\vec{v}_n)$ such that

$$\|\vec{v}_n\|_{1,r,\Omega} = 1, \quad \sum_{i,j} \|\epsilon_{ij}(\vec{v}_n)\|_{1,r,\Omega} < \frac{1}{n} \ \ \forall\, n > 1.$$

Hence, there is a subsequence, again denoted by $(\vec{v}_n)$, and $\vec{v} \in (W^{1,r}(\Omega))^2$, such that

$$\vec{v}_n \longrightarrow \vec{v} \quad \textit{weakly in} \ \ (W^{1,r}(\Omega))^2, \quad \textit{strongly in} \ L^r(\Omega)^2,$$

and

$$\|\epsilon_{ij}(\vec{v}_n)\|_{0,r,\Omega} \longrightarrow 0,$$

from which and Theorem 1 (Korn's inequality), it can be seen that the subsequence $(\vec{v}_n)$ is Cauchy sequence in $(W^{1,r}(\Omega))^2$, and then

$$\vec{v}_n \longrightarrow \vec{v} \quad \textit{strongly in} \ (W^{1,r}(\Omega))^2.$$

Thus $\epsilon_{ij}(\vec{v}) = 0$, which means that $\vec{v}$ is infinitesimal rigid motion:

$$\vec{v} = \vec{a} + b(x_2, -x_1)^t,$$

and $\vec{a}, b$ are constants. Then due to $\vec{v} \in (W^{1,r}_{\Gamma_0}(\Omega))^2$, meas $(\Gamma_0) > 0$, or $\vec{v} \in (\hat{W}^{1,r}(\Omega))^2$, we have $\vec{v} = 0$, which is a contradiction with $1 = \|\vec{v}_n\|_{1,r,\Omega} \longrightarrow \|\vec{v}\|_{1,r,\Omega}$. This completes the proof.

Remark 2. In fact, let

$$\mathbf{RM} = \{\vec{v} : \epsilon_{ij}(\vec{v}) = 0 \ \ \forall\, i,j\} = \{\vec{v} : \vec{v} = \vec{a} + b(x_2, -x_1)^t, \ \vec{a} \in R^2, \ b \in R\}, \tag{3.15}$$

then second Korn's inequality (3.14) holds for all $\vec{v} \in \hat{W}$, if

$$\hat{W} \cap \mathbf{RM} = \vec{0}. \tag{3.16}$$

References

[1] Adams, R.A., Sobolev Spaces., Academic Press, New York, 1975.
[2] Anold, D.N., Scott, L.R., Vogelius, M., Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon, *Ann. Scula Norm. Sup. Pisa CI. Sci.,* **IV XV** (1988), 169–192.
[3] Brenner, S.C., Scott, L.R., The Mathematical Theory of Finite Element Methods, Spring-Verlage, 1994.
[4] P.G. Ciarlet, Mathematical Elasticity, I. Three-Dimensional Elasticity, North-Holland, Amsterdam, 1988.
[5] Duvaut, G., Lions, J.L., Inequalities in Mechanics and Physics, Springer-Verlag, 1976.
[6] Falk, R.S., Nonconforming finite element methods of the equations of linear elasticity, *Math. Comp.,* **57** (1991), 529–550.
[7] Girault, V., Raviart, P.-A., Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms, Springer-Verlag, 1984.
[8] Kikuchi, N., Oden, J.T., Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM Philadelphia, 1988.
[9] Miyoshi, T., Foundations of the Numerical Analysis of Plasticity, North-Holland, 1985.
[10] Nitsche, J.A., On Korn's second inequality, *RAIRO. Analyse Numerique,* **15**:3 (1981), 237–248.