Journal of Computational Mathematics, Vol.21, No.3, 2002, 305–310.

A NUMERICAL METHOD FOR DETERMINING THE OPTIMAL EXERCISE PRICE TO AMERICAN OPTIONS*[1)]

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Abstract

American options can be exercised prior to the date of expiration, the valuation of American options then constitutes a free boundary value problem. How to determine the free boundary, i.e. the optimal exercise price, is a key problem. In this paper, a nonlinear equation is given. The free boundary can be obtained by solving the nonlinear equation and the numerical results are better.

Key words: American options, Free boundary, Optimal exercise price, Nonlinear equation.

1. Introduction

In the early 1970s, Fischer Black and Myron Scholes made a major differential equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock under some assumptions. The Black-Scholes analysis is of great importance in today's derivative pricing. How to solve the partial differential equation faster and more accurately is one of the important contents in today's computational financial field.

For an American option, the holder can exercise it prior to the date of expiration, the valuation of it then constitutes a free-boundary problem. To solve the free-boundary problem, there are several methods for the equation developed in the past two decades. For example, Jaillet, Lamberton and Lapeyre ([1]) turn the free-boundary problem into variational inequalities, then construct a numerical scheme to obtain the solution. A numerical method of the integral equation that is based on an analytic approximation is described in [2].

A new method for determining the optimal exercise price is provided in this paper. The numerical results from our method agree with those from the numerical method of the integral equation.

This paper is organized as follows. In section 2, we describe the model of American options on a continuous dividend yield. In section 3, we describe our method. In section 4, an example and its numerical results are given.

2. The American Option Pricing Model

The Black-Scholes model for American call options with continuous dividend yield is the following:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - q)S\dfrac{\partial V}{\partial S} - rV = 0 & 0 \le t \le T,\ 0 \le S \le S_f(t); \\ V(S,T) = \max(S - E, 0) & 0 \le S \le S_f(T) = \max\left(E, \dfrac{r}{q}E\right); \\ V(S_f(t),t) = S_f(t) - E & 0 \le t \le T; \\ \dfrac{\partial V}{\partial S}(S_f(t),t) = 1 & 0 \le t \le T; \\ V(0,t) = 0 & 0 \le t \le T; \end{cases} \tag{1}$$

* Received September 21, 1999; final revised June 12, 2000.
1) Supported by National Science Foundation of China.

where $V-$ the value of the option; $S-$ the price of the underlying asset;
$\sigma-$ the volatility of the underlying asset; $E-$ the exercise price;
$r-$ the risk-free interest rate; $q-$ the dividend yield; $T-$ the expiry;
$S_f(t)-$ the optimal exercise price at a given time t.
This equation is based on these assumptions:
There are no transaction costs and taxes.
The risk-free interest rate r and the asset volatility σ are constants.
There are no arbitrage possibilities.

According to the no-arbitrage principle, the price of American call options should satisfies the inequality $V(S,t) \geq (S-E)^+$, the details can be found in [2], [3].

If V, S and $S_f(t)$ are divided by E, and for the dimensionless quantities we use the same notation, then the dimensionless V still satisfies (1), but$E$should be replaced by 1. Problem (1) with $E=1$ will be referred to as the standardized American call option problem. For different$E$'s, we need just to solve the standardized problem and get the final answer by multiplying the results of the standardized problem by E.

3. Numerical Methods

For the standardized problem, let $\tau = T - t$, then (1) becomes

$$\begin{cases} \dfrac{\partial \mathrm{V}}{\partial \tau} - \dfrac{1}{2}\sigma^2 \mathrm{S}^2 \dfrac{\partial^2 \mathrm{V}}{\partial \mathrm{S}^2} - (\mathrm{r - q})\mathrm{S}\dfrac{\partial \mathrm{V}}{\partial \mathrm{S}} + \mathrm{rV} = 0 & 0 \leq \tau \leq T,\ 0 \leq S \leq S_{\mathrm{f}}(\tau); \\ V(S,0) = \max(S-1,0) & 0 \leq S \leq S_{\mathrm{f}}(0) = \max\left(1, \dfrac{r}{q}\right); \\ V(S_{\mathrm{f}}(\tau),\tau) = S_{\mathrm{f}}(\tau) - 1 & 0 \leq \tau \leq T; \\ \dfrac{\partial V}{\partial S}(S_{\mathrm{f}}(\tau),\tau) = 1 & 0 \leq \tau \leq T; \\ V(0,\tau) = 0 & 0 \leq \tau \leq T; \end{cases} \tag{2}$$

For (2), let $\xi = S/S_f(\tau)$, this transformation turns $S \in [0, S_f(\tau)]$ into $\xi \in [0,1]$, then $S = \xi\ S_f(\tau)$,

$$v(\xi,\tau) = v(S/S_f(\tau),\tau) = V(S,\tau) = V(\xi\ S_f(\tau),\tau).$$

Since

$$\frac{\partial V}{\partial S} = \frac{1}{S_f(\tau)}\frac{\partial v}{\partial \xi},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{S_f(\tau)^2}\frac{\partial^2 v}{\partial \xi^2},$$

$$\frac{\partial V}{\partial \tau} = -\frac{dS_f(\tau)}{d\tau}\frac{\xi}{S_f(\tau)}\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \tau},$$

from (2), we have

$$\frac{\partial v}{\partial \tau} - \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 \mathrm{v}}{\partial \xi^2} - \left(r - q + \frac{dS_f(\tau)}{d\tau}\frac{1}{S_f(\tau)}\right)\xi\frac{\partial v}{\partial \xi} + \mathrm{rv} = 0\ . \tag{3}$$

Suppose $\Delta\tau = T/K$, where Ki s a given integer, $\frac{dS_f(\tau)}{d\tau}$, $\frac{\partial v}{\partial \tau}$ can be discretized by $\frac{dS_f(\tau)}{d\tau} = \frac{S_f - S_f^k}{\Delta\tau}$, $\frac{\partial v}{\partial \tau} = \frac{v - v^k}{\Delta\tau}$, where $v = v(\tau), v^k = v(\tau - \Delta\tau), S_f = S_f(\tau), S_f^k = S_f(\tau - \Delta\tau)$.

Therefore, (3) can be written as

$$\xi^2\frac{\partial^2 \mathrm{v}}{\partial \xi^2} + (2b+1)\xi\frac{\partial v}{\partial \xi} + c\mathrm{v} = g\ , \tag{4}$$

where

$$b = \frac{1}{\sigma^2}\Big(r - q - \frac{\sigma^2}{2} + \frac{S_f - S_f^k}{\Delta\tau}\frac{1}{S_f}\Big), \quad c = -\frac{2}{\sigma^2}\Big(r + \frac{1}{\Delta\tau}\Big), \quad g = -\frac{2}{\sigma^2\Delta\tau}v^k,$$

Let $x = \ln\xi$, (4) can be written as

$$\frac{\partial^2 \mathrm{v}}{\partial x^2} + 2b\frac{\partial v}{\partial x} + \mathrm{cv} = g, \tag{5}$$

The boundary and initial conditions of (2) can be written as

$$v(x,0) = (\xi\ S_f(0) - 1)^+ = (e^x S_f(0) - 1)^+,$$

$$v(0,\tau) = S_f - 1,$$

$$v_x(0,\tau) = v_\xi \xi_x = S_f v_s \xi\,|_{\xi=1} = S_f,$$

$$v(-\infty) = 0.$$

Let $y_1 = \frac{dv}{dx}, \quad y_2 = v, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, for (5), we have

$$\frac{dy_1}{dx} = -2by_1 - cy_2 + g, \quad \frac{dy_2}{dx} = y_1,$$

then (5) can be written as

$$\frac{dY}{dx} = AY + F, \tag{6}$$

where $A = \begin{bmatrix} -2b & -c \\ 1 & 0 \end{bmatrix}$, $F = \begin{bmatrix} g \\ 0 \end{bmatrix}$,

$$x \in (-\infty, 0), y_1(0) = S_f, y_2(0) = S_f - 1, y_2(-\infty) = 0.$$

We know, the two characteristic values of A are

$$\lambda_1 = -b + \sqrt{b^2 - c}, \quad \lambda_2 = -b - \sqrt{b^2 - c},$$

the two characteristic vectors of A are $p_1,\ p_2$, let $P = [p_1, p_2] = \frac{1}{2\sqrt{b^2-c}}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, then $P^{-1} = \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. Let $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then we have $e^{Ax} = Pe^{\Lambda x}P^{-1}$. The solution of the system (6) can be written as

$$Y(x) = e^{A(x-x_0)}Y(x_0) + e^{Ax}\int_{x_0}^{x} e^{-Aw}F\,dw,$$

or

$$P^{-1}Y(x) = e^{\Lambda(x-x_0)}P^{-1}Y(x_0) + e^{\Lambda x}\int_{x_0}^{x} e^{-\Lambda w}P^{-1}F\,dw,$$

or

$$\begin{bmatrix} v_x(x) - \lambda_2 v(x) \\ -v_x(x) + \lambda_1 v(x) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1(x-x_0)}(v_x(x_0) - \lambda_2 v(x_0)) \\ e^{\lambda_2(x-x_0)}(-v_x(x_0) + \lambda_1 v(x_0)) \end{bmatrix} + \begin{bmatrix} e^{\lambda_1 x} & \\ & e^{\lambda_2 x} \end{bmatrix} \bullet \int_{x_0}^{x} g \begin{bmatrix} e^{-\lambda_1 w} \\ -e^{-\lambda_2 w} \end{bmatrix} dw. \tag{7}$$

If we take the second term of (7), we have

$$-v_x(x) + \lambda_1 v(x) = e^{\lambda_2(x-x_0)}(-v_x(x_0) + \lambda_1 v(x_0)) - e^{\lambda_2 x}\int_{x_0}^{x} g e^{-\lambda_2 w} dw. \tag{8}$$

Let $x_0 \to -\infty, x = 0$, because$\lambda_2 < 0$, (8) can be written as

$$-v_x(0) + \lambda_1 v(0) + \int_{-\infty}^{0} g e^{-\lambda_2 w} dw = 0, \tag{9}$$

In the integration, let $w = \ln \xi$, then (9) becomes

$$-S_f + \lambda_1(S_f - 1) + \int_0^1 g \xi^{-\lambda_2 - 1} d\xi = 0. \tag{10}$$

For (10), we use numerical integration,S_f satisfies the following nonlinear equation

$$f(y) = -y + \lambda_1(y-1) - \frac{2}{\sigma^2 \Delta\tau N}\left(\sum_{i=1}^{N-1} v_i^k \xi_i^{-\lambda_2-1} + \frac{1}{2}v_N^k\right) = 0 \tag{11}$$

where λ_1, λ_2 are the functions of b and c, and $b = \frac{1}{\sigma^2}$(r - q - $\frac{\sigma^2}{2} + \frac{y - S_f^k}{\Delta\tau}\frac{1}{y}$)is a nonlinear function of y. when $y \in [1, +\infty)$, the solution of the equation (11) exists, because when $y = 1$, $f(1) = -1 - \frac{2}{\sigma^2\Delta\tau N}(\sum_{i=1}^{N-1} v_i^k \xi_i^{-\lambda_2-1} + \frac{1}{2}v_N^k) < 0$, when $y \to +\infty$,$b \to \frac{1}{\sigma^2}(r - q + \frac{1}{\Delta\tau}) - \frac{1}{2}$,

$$\begin{aligned}
\lambda_1 - 1 &= -b - 1 + \sqrt{b^2 - c} \\
&= -[\frac{1}{\sigma^2}(r - q + \frac{1}{\Delta\tau}) + \frac{1}{2}] + \sqrt{[\frac{1}{\sigma^2}(r - q + \frac{1}{\Delta\tau}) - \frac{1}{2}]^2 + \frac{2}{\sigma^2}(r + \frac{1}{\Delta\tau})} \\
&= -[\frac{1}{\sigma^2}(r - q + \frac{1}{\Delta\tau}) + \frac{1}{2}] + \sqrt{[\frac{1}{\sigma^2}(r - q + \frac{1}{\Delta\tau}) + \frac{1}{2}]^2 + \frac{2q}{\sigma^2}} > 0,
\end{aligned}$$

then $\lim_{y\to+\infty}(\lambda_1 - 1)y \to +\infty$, when $y \to +\infty$, $-\lambda_1 - \frac{2}{\sigma^2\Delta\tau N}(\sum_{i=1}^{N-1} v_i^k \xi_i^{-\lambda_2-1} + \frac{1}{2}v_N^k)$ is a constant, then $\lim_{y\to+\infty} f(y) \to +\infty$.

Because $f(y)$ is a continuous function of y the solution of the equation (11) exists in $[1, +\infty)$.

Because S_f is close to S_f^k and $S_f^k$is a good initial value of S_f, the equation (11) can be solved by the iteration method. Once $S_f$is known, the equation (4) can be easily solved by a difference method in $\xi \in [0, 1]$, then v can be obtained. This means we can obtain $S_\mathrm{f}(\tau), v(\tau)$ from $S_\mathrm{f}(\tau - \Delta\tau), v(\tau - \Delta\tau)$.

A numerical method of the integral equation that is based on an analytic approximation for American put options is shown in [2]. We describe it briefly in the following:

The American put value $P(S, \tau)$ satisfies

$$P(S,\tau) = Ee^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1) + \int_0^\tau [rEe^{-r\xi}N(-d_{\xi,2}) - qSe^{-q\xi}N(-d_{\xi,1})]d\xi,$$

where

$$d_1 = \frac{\ln(S/E) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

$$d_{\xi,1} = \frac{\ln(S/S_f(T-\xi)) + (r - q + \sigma^2/2)\xi}{\sigma\sqrt{\xi}}, \quad d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi},$$

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt.$$

The boundary condition is $P(S_f(\tau),\tau)=E-S_f(\tau)$, then

$$E-S_f(\tau)=Ee^{-r\tau}N(-\hat{d}_2)-S_f(\tau)e^{-q\tau}N(-\hat{d}_1)+\int_0^{\tau}[rEe^{-r\xi}N(-\hat{d}_{\xi,2})-qS_f(\tau)e^{-q\xi}N(-\hat{d}_{\xi,1})]d\xi$$

where

$$\hat{d}_1=\frac{\ln(S_f(\tau)/E)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\hat{d}_2=\hat{d}_1-\sigma\sqrt{\tau},$$

$$\hat{d}_{\xi,1}=\frac{\ln(S_f(\tau)/S_f(T-\xi))+(r-q+\sigma^2/2)\xi}{\sigma\sqrt{\xi}},\quad \hat{d}_{\xi,2}=\hat{d}_{\xi,1}-\sigma\sqrt{\xi}.$$

The integral equation for a given value of τ can be solved by the following numerical algorithm.

We divided τinto n equally spaced subintervals with end points $\tau_k(k=0,1\cdots n)$, where $\tau_0=0,\tau_n=\tau,\Delta\tau=\frac{\tau}{n}$. For convenience, we denote the integrand function by

$$f(S_f(\tau),S_f(\tau-\xi),\tau,\xi)=rEe^{-r\xi}N(-\hat{d}_{\xi,2})-qS_f(\tau)e^{-q\xi}N(-\hat{d}_{\xi,1}),$$

Let S_k^* denote the numerical approximation to $S_f(\tau_k),(k=0,1\cdots n)$. The solution S_k^* can be obtained by solving the following equation

$$E-S_k^*=p(S_k^*,\tau_k)+\frac{\Delta\tau}{2}[f(S_k^*,S_k^*,\tau_k,\tau_0)+f(S_k^*,S_0^*,\tau_k,\tau_k)+2\sum_{i=1}^{k-1}f(S_k^*,S_{k-i}^*,\tau_k,\tau_i)]$$
$$(k=2,3\cdots n),$$

where $p(S,\tau)=Ee^{-r\tau}N(-d_2)-Se^{-q\tau}N(-d_1)$ denotes the European put value.

The American call value and the optimal exercise prices can be easily obtained after knowing the corresponding put values.

4. Numerical Results

Some numerical experiments have been made both using our method and the method of the integral equation. Here, we give an example with the following parameters:

$$r=0.12,\quad q=0.08,\quad \sigma=0.2,\quad T=1,E=1.$$

The free boundaries obtained by using our method are shown in Table 1.

Table 1

S_f $\Delta\tau$	0.25	0.125	0.0625	0.03125
$S_f(0.25)$	1.5741	1.5834	1.5901	1.5957
$S_f(0.50)$	1.6178	1.6254	1.6303	1.6343
$S_f(0.75)$	1.6519	1.6586	1.6627	1.6662
$S_f(1.00)$	1.6813	1.6877	1.6915	1.6947

The free boundaries obtained by using the method of the integral equation are shown in Table 2.

Table 2

S_f $\Delta\tau$	0.25	0.125	0.0625	0.03125
$S_f(0.25)$	1.6186	1.6062	1.6012	1.5980
$S_f(0.50)$	1.6502	1.6432	1.6384	1.6355
$S_f(0.75)$	1.6827	1.6742	1.6696	1.6670
$S_f(1.00)$	1.7099	1.7023	1.6979	1.6954

When we take $\Delta\tau = 0.03125$, the difference between two methods at $\tau = 1.00$ is less then 10^{-3}.

Our method is very simple. The free boundaries can be easily obtained step by step and only a few time steps are enough. The numerical results are better.

Acknowledgments. The authors are grateful to Prof. Lishang Jiang and Prof. Youlan Zhu for providing some useful papers and helpful talks.

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