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DELAY-DEPENDENT TREATMENT OF LINEAR MULTISTEP METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS [*1)]

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Abstract

This paper deals with a delay-dependent treatment of linear multistep methods for neutral delay differential equations $y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau),\ t > 0,\ y(t) = g(t),\ -\tau \le t \le 0,\ a, b$ and$c \in \mathbb{R}$. The necessary condition for linear multistep methods to be $N\tau(0)$-stable is given. It is shown that the trapezoidal rule is $N\tau(0)$-compatible. Figures of stability region for some linear multistep methods are depicted.

Key words: Delay-dependent stability, Linear multistep methods, Neutral delay differential equations.

1. Introduction

The stability analysis for delay differential equations can be classified into two different categories, i.e. delay-independent and delay-dependent. In the former criterion the stability analysis is carried out for all delay, but in the delay-dependent criterion stability analysis is carried out for arbitrary but fixed delay. The delay-independent analysis was studied by many researches (see e.g. [2, 3, 5]).

Consider the following neutral delay differential equations (NDDEs).

$$\begin{aligned} y'(t) &= ay(t) + by(t-\tau) + cy'(t-\tau), \quad t > 0, \\ y(t) &= g(t), t \in [-\tau, 0]. \end{aligned} \tag{1.1}$$

For $a,\ b$ and $c \in \mathbb{C}$, Bellen *et al.* [2] proved that if:

$$|a\overline{c} - \overline{b}| + |ac + b| < -2\Re[a]. \tag{1.2}$$

Then every solution of (1.1) tends to zero as $t \to \infty$ for all delay. If $a,\ b$ and $c \in \mathbb{R}$ then the condition (1.2) is equivalent to the following condition:

$$|b| < -a \text{ and } |c| < 1, \tag{1.3}$$

which is given by Brayton *et.al.* [5] and is shown in Fig.1.

In the delay-dependent case the delay term also plays a role in the stability analysis. The delay-dependent analysis was first carried out by Al-Mutib [1], but his analysis was based on some numerical experiments only. Recently Guglielmi and Hairer [9, 10, 11] did some work on the delay dependent stability analysis of $\Theta-$ and Runge-Kutta methods for DDEs. In [10] it was proved that linear Θ-methods are $\tau(0)$-stable if and only if they are A-stable. In [13] it was proved that BDF method of second order is $\tau(0)$-stable. Sidibe and Liu [15] proved that all Gauss methods are $N\tau(0)$-stable. In this work we address the delay-dependent stability analysis of linear multistep methods when they are applied to the neutral delay differential equation with real coefficients.

For the sake of simplicity and without losing generality, we consider (1.1) with $\tau = 1$,

$$\begin{aligned} y'(t) &= ay(t) + by(t-1) + cy'(t-1), \quad t > 0, \\ y(t) &= g(t), \quad t \in [-1, 0], \end{aligned} \tag{1.4}$$

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where $a,\ b$ and $c \in \mathbb{R}$.

In this paper we study the stability region of linear multistep methods applied to (1.4) with an arbitrary but fixed value of τ.

The organization of this paper is as follows. In the section 2, the analytical stability region is studied. In section 3, an introduction of linear multistep methods is given and then they are applied to linear test equation (1.4). In section 4, a necessary condition for $N\tau(0)$ -stability is provided and applied to some well-known implicit linear multistep methods for $N\tau(0)$-stability. The results are presented in tabular form (See Table 1). In the section 4, conclusions are presented.

2. Analytical Stability Region

Classically the analytic solution of (1.4) can be expressed by a power series.

$$y(t) = \sum_k \left(A_k \exp(\lambda_k t) + B_k t\ \exp(\lambda_k t)\right),$$

where the coefficients $A_k, B_k \in \mathbb{C}$ are determined by the provided initial function and $\{\lambda_k\}_{k=0}^{\infty}$ are the roots of the quasi-polynomial characteristic equation:

$$\lambda = a + be^{-\lambda} + c\lambda e^{-\lambda}. \tag{2.1}$$

It is well known that the sufficient condition for the asymptotic stability of $y(t)$, independent of the initial function $g(t)$ is,

$$\Re[\lambda_k] < 0, \tag{2.2}$$

for all k.

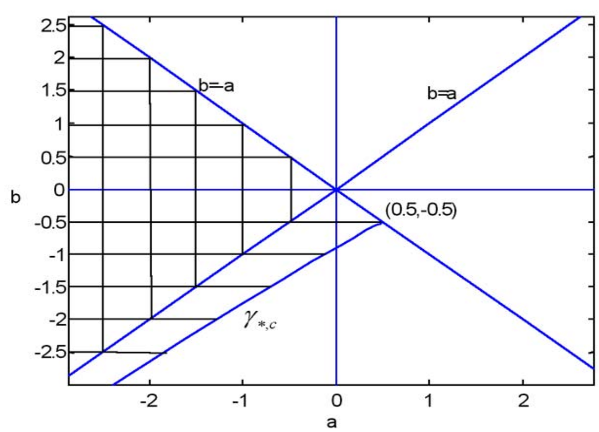

Figure 1: Analytical stability region of (1.4) for $c = 0.5$.

For the case $a,\ b$ and $c \in\ \mathbb{R}$, the stability region Σ_* of (1.4) is given by the connected domain included in the half-plane $a < 1 - c$, and bounded by the planes $|c| = 1$, the straight half-plane l_* and the transcendental surface γ_*. Denoting $\partial\Sigma_*$ as the boundary of the region Σ_*, then

$$\partial\Sigma_* = l_* \cup \gamma_*, \tag{2.3}$$

where

$$l_* = \left\{(a,b,c) \in \mathbb{R}^3 |\ a = -b, a \in (-\infty, 1-c] \text{ and } |c| < 1\right\},$$

$$\gamma_* = \left\{(a,b,c) \in \mathbb{R}^3 |\ c^2\theta^2 + b^2 = a^2 + \theta^2, \theta = \operatorname{arc\,cot} \frac{ab - c\theta^2}{\theta\,(b + ac)},\ \theta \in \mathbb{R}\right\}.$$

We define the intersection of the plane l_* and the surface γ_* is the segment $\mathbf{P} = \{(1-c, c-1, c),\ |c| < 1\}$, and all points $P = (a,b,c) = (1-c, c-1, c)$ of the segment $\mathbf{P}$, are the double points of the boundary $\partial\Sigma_*$ which is shown in the Fig.1.

3. Linear Multistep Methods for NDDEs

Linear k-step methods [12, 14] for ODEs of the form $y'(t) = f(t,y),\ t \geq 0,\ y(0) = y_0$ is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{3.1}$$

for some fixed numbers α_j, β_j, $j = 0, 1, 2, \dots, k$ subject to the condition $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0$. Methods (3.1) is called explicit if $\beta_k = 0$, otherwise implicit. In this paper, we shall consider constant stepsize,

$$h = \frac{1}{m} \quad m \in \mathbb{Z}^+. \tag{3.2}$$

Applying method (3.1) to (1.4), we get

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j (a y_{n+j} + b y_{n-m+j}) + c \sum_{j=0}^{k} \alpha_j y_{n-m+j}.$$

or

$$\sum_{j=0}^{k} a_j y_{n+j} = \sum_{j=0}^{k} b_j y_{n+j-m}, \tag{3.3}$$

where

$$a_j = \alpha_j - ha\beta_j \quad \text{and} \quad b_j = hb\beta_j + c\alpha_j, \quad j = 0, 1, ..., k.$$

Characteristic polynomial for (3.3) is given by

$$C_m(a,b,c;\zeta) = (1 - c\zeta^{-m})\varrho(\zeta) - \frac{1}{m}(a + b\zeta^{-m})\sigma(\zeta), \tag{3.4}$$

where $\varrho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ and $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$ are the usual *generating polynomials* of the multistep methods (3.1) [12].

It is well known that the numerical solution of (3.3) is asymptotically stable if and only if all zeros of characteristic polynomial (3.4) lie within the open unit disk in the complex plane for all m. Let

$$\Sigma_m = \{(a,b,c)\ ;\ \text{ all zeros } \zeta \text{ of } (3.4) \text{ satisfy } |\zeta| < 1\}, \tag{3.5}$$

then the numerical stability region is $\Sigma = \bigcap_{m=1}^{\infty} \Sigma_m$.

Definition 3.1. *A numerical method for NDDEs is $N\tau(0)$-stable if* $\Sigma_* \subseteq \bigcap_{m=1}^{\infty} \Sigma_m$.

3.1. Taking c as the Fixed Parameter

In order to study $N\tau(0)$-stability of numerical methods, we use the Boundary Locus Technique (BLT)[4] which means to determine whether $\Sigma_* \subset \Sigma_m$ for all $m \in Z^*$ and any, but fixed c, where

$$\Sigma_{*,c} = \{(a,b)\ |\ \text{ all roots of (2.1) satisfying } \Re(\lambda) < 0\}. \tag{3.6}$$

$$\Sigma_{m,c} = \{(a,b)\ |\ \text{all roots } \zeta \text{ of (3.4)satisfying } |\zeta| < 1\}. \tag{3.7}$$

Let

$$\gamma_{*,c} = \left\{(a,b)\ |\ a = \frac{\theta}{\sin\theta}(\cos\theta - c), b = \frac{\theta}{\sin\theta}(c\cos\theta - 1), \theta \in (0,\pi)\right\} \tag{3.8}$$

and $\Gamma_{*,c}(a)$ is a smooth function for $a \in (-\infty, 1-c)$, which is implicitly defined by the curve $\gamma_{*,c}$.

We denote the intersection of the line $a = -b$ and the transcendental curve $\gamma_{*,c}$ by $P^c = (1-c, c-1)$. Using L'Hospital's rule, we get the first and second derivative of $\Gamma_{*,c}(a)$ at P^c,

$$\begin{aligned} \Gamma'_{*,c}(1-c)\mid_{\theta=0} &= \frac{1+2c}{2+c}. \\ \Gamma''_{*,c}(1-c)|_{\theta=0} &= -\tfrac{6}{5}\cdot\frac{1-c^2}{c^3+6c^2+12c+8}. \end{aligned} \tag{3.9}$$

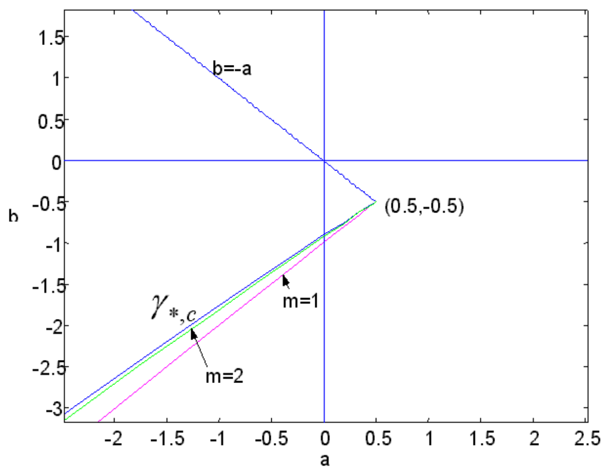

Figure 2: Stability region for trapezoidal rule for c=0.5

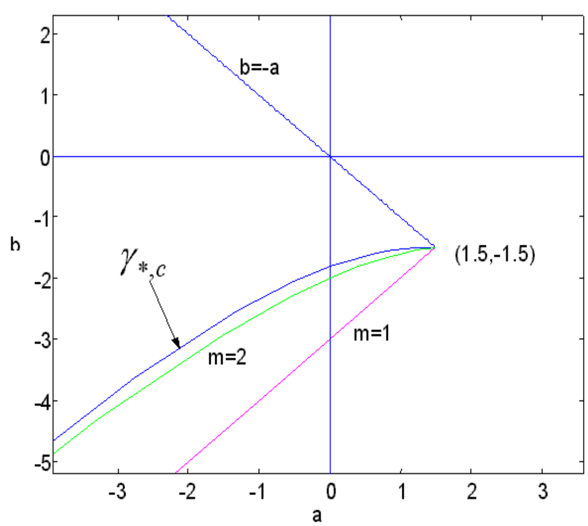

Figure 3: Stability region for trapezoidal rule for c=-0.5

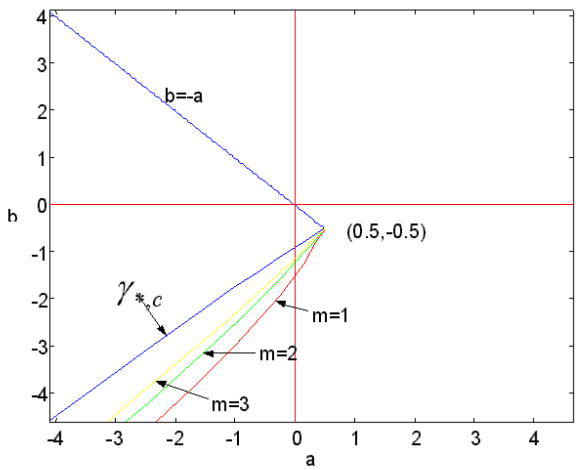

Figure 4: Stability region for BDF of first order for c=0.5

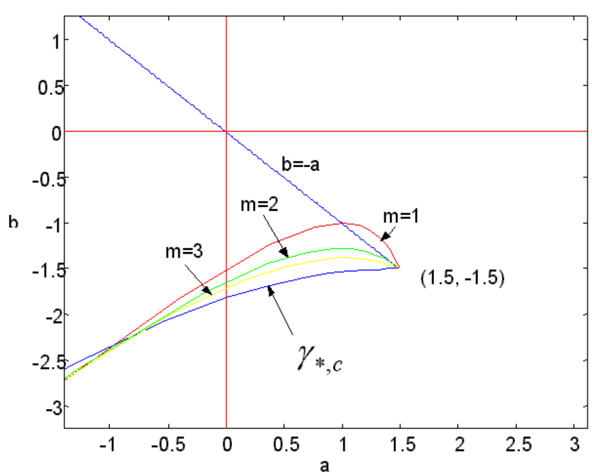

Figure 5: Stability region for BDF of first order for c=-0.5

4. A Necessary Condition for $N\tau(0)$-Stability

Let

$$A_n = \sum_{j=0}^{k} j^n \alpha_j,\ B_n = \sum_{j=0}^{k} j^n \beta_j,\ n = 0, 1, 2, \ldots, l_q = A_q - qB_{q-1},\ \forall\ q \in \mathbb{Z}^+.$$

From the order condition we know that for linear multistep methods of order $p \geq 1$, we have

$$A_0 = 0, A_q = qB_{q-1}, q = 1, 2, \ldots, p.$$

We define the set

$$U_m^c = \{(a, b) \in \mathbb{R}^2 |\ \exists\ \zeta = \exp(\mathrm{i}\theta)\ \theta \in (-\pi, \pi)\ :\ C_m(a, b, c; \zeta) = 0\}. \tag{4.1}$$

Obviously $\partial\Sigma_{m,c} \subseteq U_m^c$. For the determination of the stability region of a numerical method, the set U_m^c plays a fundamental role.

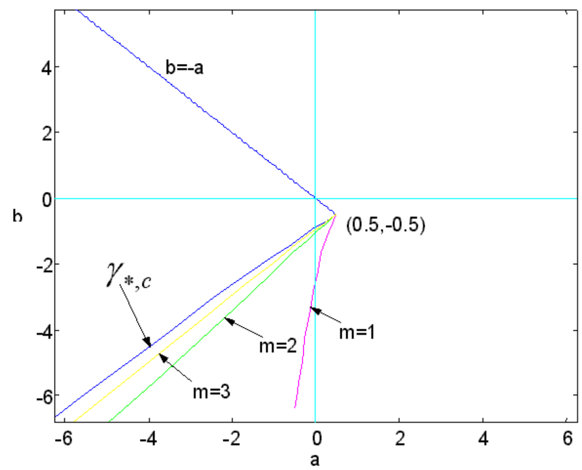

Figure 6: Stability region for BDF of second order for c=0.5

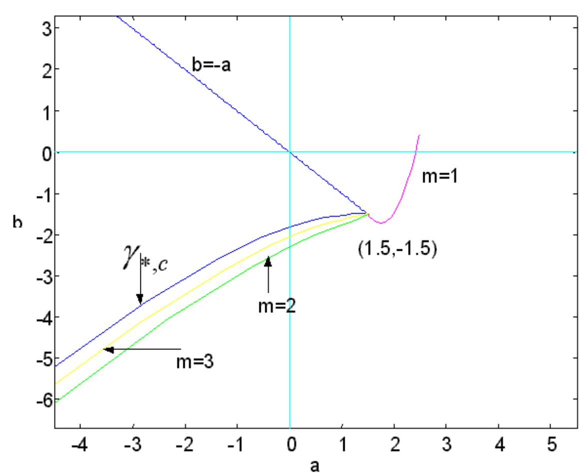

Figure 7: Stability region for BDF of second order for c=-0.5

Lemma 4.1. *For any linear multistep method for NDDEs applied to the test equation (1.4) the following relation holds*

$$U^\circ = \{(a,b) \in \mathbb{R}^2 |\ b = -a,\ |c| < 1\} \subset U_m^c. \tag{4.2}$$

Proof. Substituting $a = -b$ in the characteristic equation (3.4), yields

$$C_m(a,b,c;\zeta) = (1 - c\zeta^{-m})\varrho(\zeta) - \frac{a}{m}(1 - \zeta^{-m})\sigma(\zeta).$$

For $\zeta = 1$.

$$C_m(a,-a,c;\zeta) = (1-c)\varrho(1) = 0.$$

Which is identically fulfilled for $\zeta = 1$, independent of a.

As we shall observe that P^c is the only point of U° to which a multiple zero corresponds. The following theorem provides the information that P^c is still a double point in the numerical case, that is it belongs to two different branches of the locus U_m^c.

Theorem 4.2. *For any linear multistep method for NDDEs applied to the test equation (1.4), the point $P^c = (1-c, c-1)$ is a double point of U_m^c.*

Proof. Let us define a function $F : \mathbb{R}^2 \times (-\pi, \pi] \times \mathbb{Z}^+ \to \mathbb{C}$ as:

$$F(a, b, \theta; m) = C_m(a, b, c; \exp(\mathrm{i}\theta)).$$

Since $P^c \in U^\circ$, we now apply Dini's implicit function theorem to the equation $F(a, b, \theta; m) = 0$ at the point $(a, b, \theta) = (1-c, c-1, 0)$.

Define

$$F_\rho = 0 \quad \text{and} \quad F_\mathrm{i} = 0, \tag{4.3}$$

where

$$F_\rho = \Re[F] \quad F_\mathrm{i} = \Im[F].$$

Consider the Jacobian of F at $(a, b, \theta) = (1-c, c-1, 0)$

$$J_F(1-c, c-1, 0) = B_0 \begin{bmatrix} -h & -h & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

It can be observed that at the point $P^c = (1-c, c-1)$, there do not exist full rank principal minors of J_F. So implicit function theorem cannot be applied, consequently P^c is a potential bifurcation point of the boundary locus U_m^c . Now we study the structure of U_m^c, in the neighborhood of the point $(a, b) = (1-c, c-1)$.

Let us consider the following equivalent form of the system of algebraic equation (4.3).

$$\begin{aligned} F_\rho(a, b, \theta; m) &= 0, \\ \theta \cdot \tfrac{F_\mathrm{i}(a,b,\theta;m)}{\theta} &= 0. \end{aligned} \tag{4.5}$$

The system (4.5) is equivalent to the following couple of algebraic system:

$$F_1 : \begin{cases} F_\rho(a, b, \theta; m) = 0 \\ \theta = 0 \end{cases} \qquad \text{and} \quad F_2 : \begin{cases} F_\rho(a, b, \theta; m) = 0 \\ \dfrac{F_\iota(a, b, \theta; m)}{\theta} = 0 \end{cases} \tag{4.6}$$

The solution of the system F_1 leads to the set U°. Let us consider

$$G(a, b, \theta; m) = G_\rho(a, b, \theta; m) + \mathrm{i}G_\mathrm{i}(a, b, \theta; m),$$

where

$$\begin{aligned} G_\rho(a, b, \theta; m) &= F_\rho(a, b, \theta; m), \\ G_\mathrm{i}(a, b, \theta; m) &= \frac{F_\mathrm{i}(a, b, \theta; m)}{\theta}. \end{aligned} \tag{4.7}$$

Since $F(a, b, \theta; m)$ is $\mathbf{C}^\infty$ in its arguments, the considered function G is also $\mathbf{C}^\infty$. Applying the implicit function theorem to the system (4.7). Let J_G denotes the Jacobian of the function G at $(1-c, c-1)$, then we obtain

$$J_G(1-c, c-1, 0) = -\frac{1}{m} \begin{bmatrix} B_0 & B_0 & 0 \\ B_1 & B_1 - mB_0 & 0 \end{bmatrix}. \tag{4.8}$$

The first principal minor of the matrix J_G

$$J_1 = -\frac{1}{m} \begin{bmatrix} B_0 & B_0 \\ B_1 & B_1 - mB_0 \end{bmatrix}. \tag{4.9}$$

is nonsingular, which is independent of B_1. It means there exist an open neighborhood $\Theta \subset \mathbb{R}$ containing zero and a neighborhood $\Omega \subset \mathbb{R}^2$ including the double point $P^c = (1-c, c-1)$ such that for all $\theta \in \Theta$ there is an unique curve $\gamma_{m,c}(\theta)(\Theta \to \Omega)$, parameterizing a smooth function $\Gamma_{m,c}(a)$ defined as follows:

$$\gamma_{m,c}(\theta) = \left\{(a_m(\theta), b_m(\theta)) \mid \; G(a_m(\theta), b_m(\theta), \theta) = 0 \; \forall \; \theta \in \Theta\right\}. \tag{4.10}$$

Since G is of class $\mathbf{C}^\infty$ so as $\gamma_{m,c}$. The characteristic polynomial (3.4) is an algebraic polynomial of real coefficients in the complex variable ζ, so every complex zero has its complex conjugate too, which makes the smooth function $\gamma_{m,c}(\theta)$ an even function. Therefore all the odd entries of Taylor's expansion of $a_m(\theta),\ b_m(\theta)$ in the neighborhood of $\theta=0$ are zero. Hence we obtain

$$\begin{aligned} a_m(\theta) &= 1-c+N_1\tfrac{\theta^2}{2}+N_2\tfrac{\theta^4}{4!}+N_3\tfrac{\theta^6}{6!}+\mathcal{O}(\theta^8),\\ b_m(\theta) &= c-1+D_1\tfrac{\theta^2}{2}+D_2\tfrac{\theta^4}{4!}+D_3\tfrac{\theta^6}{6!}+\mathcal{O}(\theta^8), \end{aligned} \tag{4.11}$$

where

$$N_i=\frac{d^{2i}a_m}{d\theta^{2i}} \quad \text{and} \quad D_i=\frac{d^{2i}b_m}{d\theta^{2i}}, \quad i=1,2,3,\dots .$$

By applying the second step of the implicit function theorem to G, we obtain following matrix equation

$$\begin{bmatrix} \dfrac{d^2a_m}{d\theta^2}|_{\theta=0} \\ \dfrac{d^2b_m}{d\theta^2}|_{\theta=0} \end{bmatrix} = -J_1^{-1}\begin{bmatrix} \dfrac{\partial^2 G_\rho}{\partial\theta^2} \\ \dfrac{\partial^2 G_{\rm i}}{\partial\theta^2} \end{bmatrix}. \tag{4.12}$$

A straightforward manipulation in (4.12), yields

$$\begin{aligned} \left.\frac{d^2a_m}{d\theta^2}\right|_{\theta=0} &= \tfrac{1}{3B_0^2}\left[N_{12}m^2+N_{11}m+N_{10}\right],\\ \left.\frac{d^2b_m}{d\theta^2}\right|_{\theta=0} &= \tfrac{1}{3B_0^2}\left[D_{12}m^2+D_{11}m+D_{10}\right], \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} N_{12} &= -2(1+2c)B_0^2+3cB_0B_1,\\ N_{11} &= -3l_2B_0,\\ N_{10} &= (1-c)K_1,\\ D_{12} &= -(1-c)B_0^2-3cB_0B_1,\\ D_{11} &= 3cl_2B_0,\\ D_{10} &= -N_{10},\\ K_1 &= 3l_2B_1-l_3B_0. \end{aligned}$$

Using (4.11), we obtain the first derivative of $\Gamma_m(a)$ at $a=1-c$

$$\Gamma'_m(1-c)=\frac{D_1}{N_1}=\frac{D_{12}m^2+D_{11}m+D_{10}}{N_{12}m^2+N_{11}m+N_{10}}, \tag{4.14}$$

which implies that slope is not equal to -1 at $(1-c,c-1)$. So it necessarily corresponds to a second branch, different from U°, crossing the point P^c. Hence P^c is a double point of U_m^c in the numerical case too. This proves the theorem.

Now we investigate the local behavior of the algebraic approximation $\gamma_{m,c}$ to the transcendental curve $\gamma_{*,c}$, and also the stability properties of the numerical methods.

Let $\Gamma^{(k)}_{m,c}(1-c)$ and $\Gamma^{(k)}_{*,c}(1-c)$ denote the kth derivative of $\Gamma_{m,c}(a)$ and $\Gamma_{*,c}(a)$ at $a=1-c$ respectively. Obviously $\Gamma^{(0)}_{m,c}(1-c)=\Gamma_{m,c}(1-c)=c-1$ and $\Gamma^{(0)}_{*,c}(1-c)=\Gamma_{*,c}(1-c)=c-1$. Now we give the necessary condition for $N\tau(0)$-stability.

Lemma 4.3.[13] *Assume* $\Gamma^{(k)}_{m,c}(1-c)\equiv\Gamma^{(k)}_{*,c}(1-c),\ k=0,1,2,\dots,j-1$ *and* $\Gamma^{(j)}_{m,c}(1-c)\not\equiv\Gamma^{(j)}_{*,c}(1-c)$. *Then, a necessary condition for $N\tau(0)$-stability is given by*

$$\begin{aligned} &\Gamma^{(j)}_{m,c}(1-c)\geq\Gamma^{(j)}_{*,c}(1-c) \quad \textit{if } j \textit{ is odd}\\ &\Gamma^{(j)}_{m,c}(1-c)\leq\Gamma^{(j)}_{*,c}(1-c) \quad \textit{if } j \textit{ is even } \forall m\in\mathbb{Z}^+ \end{aligned} \tag{4.15}$$

for all c with $|c|<1$ and all $m\in Z^+$.

Definition 4.4. *A numerical method for NDDEs is $N\tau(0)$-compatible if it satisfies the conditions of Lemma 4.3.*

By applying implicit function theorem to $G(a,b,\theta;m)=0$, it is possible to calculate derivatives of any order. To calculate the higher derivatives of a and b w.r.t. θ, we proceed as follows.

By applying the implicit function theorem to G, we get the following matrix equation

$$\begin{bmatrix} \dfrac{d^4a_m}{d\theta^4} \\ \\ \dfrac{d^4b_m}{d\theta^4} \end{bmatrix}_{\theta=0} = -6J_1^{-1}J_3 \begin{bmatrix} \dfrac{d^2a_m}{d\theta^2} \\ \\ \dfrac{d^2b_m}{d\theta^2} \end{bmatrix}_{\theta=0} - J_1^{-1} \begin{bmatrix} \dfrac{\partial^4 G_\rho}{\partial\theta^4} \\ \\ \dfrac{\partial^4 G_{\mathrm{i}}}{\partial\theta^4} \end{bmatrix}_{\theta=0}, \tag{4.16}$$

where

$$J_3 = \begin{bmatrix} \dfrac{\partial^3 G_\rho}{\partial\theta^2\partial a_m} & \dfrac{\partial^3 G_\rho}{\partial\theta^2\partial b_m} \\ \\ \dfrac{\partial^3 G_{\mathrm{i}}}{\partial\theta^2\partial a_m} & \dfrac{\partial^3 G_{\mathrm{i}}}{\partial\theta^2\partial b_m} \end{bmatrix}_{\theta=0}.$$

After a lengthy but straightforward calculation we obtain the fourth order derivative of a and b w.r.t. θ

$$\begin{bmatrix} \dfrac{d^4a_m}{d\theta^4} \\ \\ \dfrac{d^4b_m}{d\theta^4} \end{bmatrix}_{\theta=0} = \frac{-1}{15B_0^4} \begin{bmatrix} N_{24}m^4 + N_{23}m^3 + N_{22}m^2 + N_{21}m + N_{20} \\ \\ D_{24}m^4 + D_{22}m^2 + D_{21}m + D_{20} \end{bmatrix},$$

where

$$\begin{aligned}
N_{24} &= (8-53c)B_0^4 + 60cB_1B_0^3,\\
N_{23} &= 180c(B_0-B_1)B_0^2B_1,\\
N_{22} &= 10(9c(B_0-B_1)C_1 + (2+c)B_0K_1)B_0,\\
N_{21} &= 15(6l_2C_1 + K_2B_0)B_0,\\
N_{20} &= -3(1-c)(10K_1C_1 + 10l_2LB_0 - K_3B_0^2),\\
D_{24} &= (7-22c)B_0^4 + 30cB_1B_0^3,\\
D_{22} &= 10((2c+1)K_1B_0 - 9cC_1(B_0-B_1))B_0,\\
D_{21} &= -15c(6l_2C_1 + K_2B_0)B_0,\\
D_{20} &= -N_{20},\\
K_2 &= 4l_3B_1 - l_4B_0,\\
K_3 &= 5l_4B_1 - l_5B_0,\\
C_1 &= B_0B_2 - 2B_1^2,\\
L &= 3B_1B_2 - B_0B_3.
\end{aligned}$$

Using the parametric equations (4.11), we calculate the second derivative of $\Gamma_{m,c}$ at the double point P^c.

$$\Gamma''_{m,c}(1-c)|_{\theta=0} = \frac{N_1D_2 - N_2D_1}{3N_1^3}. \tag{4.17}$$

Table 1: Analysis of $N\tau(0)$-compatibility of some well known implicit LM methods

Methods	Order	$N\tau(0)-$compatible
Trapezoidal rule	2	Yes
Implicit Adams	3	No
Implicit Adams	4	No
BDF	1	No
BDF	2	No
Milne-Simpson	4	No
Milne-Simpson	5	No

5. Conclusions

It is well-known that an explicit method cannot be A-stable, so we restrict our analysis to implicit linear multistep methods. We have already seen in [13] that trapezoidal rule and

BDF of first and second order methods are $\tau(0)-$stable, when they are applied to delay differential equations of the form $y'(t) = ay(t) + by(t-1), \ \ t \geq 0, \ y(t) = g(t), \ t \in [-1, 0]$. We know that if a numerical method is not $\tau(0)-$stable then it is also not $N\tau(0)-$stable. So one only need to explore those linear multistep methods which are $\tau(0)-$stable. By applying the necessary condition (4.15), on $\tau(0)-$stable methods we find that trapezoidal rule is the only method which is $N\tau(0)-$compatible and BDF methods of first and second order methods are not $N\tau(0)-$compatible. Finally, the comparison of different methods with regard to compatibility are presented in Table 1. Figures 2-7 depict the stability regions for different implicit multistep methods with $c = 0.5$ and $c = -0.5$.

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References

[1] Al-Mutib, A. N., Stability properties of numerical methods for solving delay differential equations, *J. Comput. Appl. Math.,* **10** (1984), 71-79.

[2] Bellen, A., Jackiewicz, Z. and Zennaro, M., Stability analysis of one-step methods for neutral delay differential equations, *Numer. Math.*, **52** (1988), 605-619.

[3] Barwell, V. K., Special stability problems for functional differential equations, *BIT*, **15** (1975), 130-135.

[4] Baker, C. T. H. and Paul, C. A. H., Computing Stability regions-Runge Kutta methods for Delay differential Equations, *IMA J. Numer. Anal.,* **14** (1994), 347-362.

[5] Brayton, R. K. and Willoughby, R. A., On the numerical integration of a symmetric system of a difference-differential equation, *J. Math. Anal. Appl.,* **18** (1967), 182-189.

[6] Calvo, M. and Grande, T., On the Asymptotic stability of Θ–methods for Delay Differential Equations, *Numer. Math.*, **54** (1988), 257-269.

[7] Hu, G. D. and Mitsui, T., Stability analysis of numerical methods for systems of Neutral Delay Differential Equations, *BIT*, **35** (1995), 504-515.

[8] Guglielmi, N., Delay dependent stability regions of numerical methods for delay differential equations. Quaderno Matematico n.356. Dipartmento di Scienze Matematiche. Universitá di Trieste (1995).

[9] Guglielmi, N., On the asymptotic stability properties of Runge-Kutta methods for delay differential equations, *Numer. Math.*, **77** (1997), 467-485.

[10] Guglielmi, N., Delay dependent stability regions of Θ-methods for delay differential equations, *IMA J. Numer. Anal.*, **18** (1998), 399-418.

[11] Guglielmi, N. and Hairer, E., Order star and stability for delay differential equations, *Numer. Math.*, **83** (1999), 371-383.

[12] Hairer, E., Nørsett, S. P. and Wanner, G., Solving ordinary differential equations I, Non-stiff problems. Springer-Verlag, New York, 2nd edition (1993).

[13] Jaffer, S. K. and Liu, M. Z., Delay-dependent asymptotic stability of linear multistep methods for delay differential equations, *submitted.*

[14] Lambert, J. D., Numerical Methods for Ordinary Differential Equations, initial value problem, 1990 Ed.

[15] Sidibe, B. S. and Liu, M. Z., Asymptotic stability for Gauss methods for neutral delay differential equations, *J. Comp. Math.,* **20** (2002), 217-224.