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THE INVERSE PROBLEM FOR PART SYMMETRIC MATRICES ON A SUBSPACE *1)

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Abstract

In this paper, the following two problems are considered:

Problem I. Given $S \in R^{n\times p}, X, B \in R^{n\times m}$, find $A \in SR_{s,n}$ such that $AX = B$, where $SR_{s,n} = \{A \in R^{n\times n} | x^T(A - A^T) = 0, \text{ for all } x \in R(S)\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that $\|\hat{A} - A^*\| = \min_{A\in S_E} \|A - A^*\|$, where S_E is the solution set of Problem I.

The necessary and sufficient conditions for the solvability of and the general form of the solutions of problem I are given. For problem II, the expression for the solution, a numerical algorithm and a numerical example are provided.

Key words: Part symmetric matrix, Inverse problem, Optimal approximation.

1. Introduction

Let $R^{n\times m}, SR^{n\times n}, OR^{n\times n}$ denote the set of real $n \times m$ matrices, real $n \times n$ symmetric matrices and real $n \times n$ orthogonal matrices, respectively. The notation $R(A), N(A), A^+$ and $\|A\|$ stand for the column space, the null space, the Moore-Penrose generalized inverse and the Frobenius norm of a matrix A, respectively. I_k represents the identity matrix of order k. For $A = (a_{ij}) \in R^{n\times m}$ and $B = (b_{ij}) \in R^{n\times m}$, define $A * B = (a_{ij}b_{ij}) \in R^{n\times m}$ as Hardmard product of A and B.

Inverse problem for nonsymmetric matrices and symmetric matrices have studied in [1-5], and a series of perfect results have been obtained. However, inverse problem for matrices between above two kinds of matrices, i.e., inverse problem for part symmetric matrices on a subspace, have not been considered yet. In this paper, we will discuss this problem.

Let $SR_{s,n} = \{A \in R^{n\times n} | x^T(A - A^T) = 0, \text{ for all } x \in R(S)\}$. we considered the following problems:

Problem I. Given $S \in R^{n\times p}, X, B \in R^{n\times m}$, find $A \in SR_{s,n}$ such that $AX = B$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|\hat{A} - A^*\| = \min_{A\in S_E} \|A - A^*\|,$$

where S_E is the solution set of Problem I.

In Section 2, the necessary and sufficient conditions for the solvability of Problem I have been studied, and the general form of S_E has been given. In Section 3, the expression of the solution of Problem II has been provided, and a numerical algorithm and a numerical example are included.

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2. The Solution of Problem I

Let us first introduce some lemmas.

Lemma 1. *Suppose the Singular-Value Decomposition (SVD) of matrix S in Problem I is*

$$S = U_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} V_1^T = U_{11}\Lambda V_{11}^T, \tag{2.1}$$

where $U_1 = (U_{11},\ U_{12}) \in OR^{n\times n}$, $U_{11} \in R^{n\times r}$, $V_1 = (V_{11}, V_{12}) \in OR^{p\times p}$, $V_{11} \in R^{p\times r}$, $\Lambda = diag(\sigma_1, \sigma_2, \dots, \sigma_r) > 0$, and $r = rank(S)$. Let

$$U_1^T A U_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in R^{r\times r}. \tag{2.2}$$

Then $A \in SR_{s,n}$ if and only if $A_{11} \in SR^{r\times r}$ and $A_{12} = A_{21}^T \in R^{r\times(n-r)}$.

Proof. If $A \in SR_{s,n}$, then by $x^T(A - A^T) = 0$, for all $x \in R(S)$, we have

$$S^T(A - A^T) = 0. \tag{2.3}$$

Substitute (2.1) and (2.2) into (2.3), we have $V_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} - A_{11}^T & A_{12} - A_{12}^T \\ A_{21} - A_{21}^T & A_{22} - A_{22}^T \end{pmatrix} U_1^T = 0$, i.e., $\begin{pmatrix} \Lambda(A_{11} - A_{11}^T) & \Lambda(A_{12} - A_{12}^T) \\ 0 & 0 \end{pmatrix} = 0$. Hence $A_{11} \in SR^{r\times r}$ and $A_{12} = A_{21}^T \in R^{r\times(n-r)}$.

Conversely, for all $x \in R(S)$, there exists $y \in R^{p\times 1}$ such that $x = Sy = U_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} V_1^T y$. By $A_{11} = A_{11}^T, A_{12} = A_{21}^T$, we have

$$\begin{aligned} x^T(A - A^T) &= (V_1^T y)^T \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} U_1^T (A - A^T) \\ &= (V_1^T y)^T \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} - A_{11}^T & A_{12} - A_{21}^T \\ A_{21} - A_{12}^T & A_{22} - A_{22}^T \end{pmatrix} U_1^T \\ &= 0. \end{aligned}$$

Hence $A \in SR_{s,n}$.

Lemma 2[2]**.** *Given $Z \in R^{n\times k}, Y \in R^{m\times k}$, and the SVD of Z is*

$$Z = \tilde{U}_1 \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \tilde{V}_1 = \tilde{U}_{11}\Delta \tilde{V}_{11}^T, \tag{2.4}$$

where $\tilde{U}_1 = (\tilde{U}_{11},\ \tilde{U}_{12}) \in OR^{n\times n}$, $\tilde{U}_{11} \in R^{n\times r_0}$, $\tilde{V}_1 = (\tilde{V}_{11}, \tilde{V}_{12}) \in OR^{k\times k}$, $\tilde{V}_{11} \in R^{k\times r_0}$, $\Delta = diag(\delta_1, \delta_2, \dots, \delta_{r_0}) > 0, r_0 = rank(Z)$. Then there is a matrix $A \in R^{m\times n}$ such that $AZ = Y$ if and only if $Y\tilde{V}_{12} = 0$. In that case the general solution can be expressed as $A = YZ^+ + \tilde{G}\tilde{U}_{12}^T$, where $\tilde{G} \in R^{m\times(n-r_0)}$ is arbitrary matrix.

Lemma 3[2]**.** *Given $Z, Y \in R^{n\times k}$, and the SVD of Z is of the form (2.4). Then there is a matrix $A \in SR^{n\times n}$ such that $AZ = Y$ if and only if $Z^TY = Y^TZ$ and $Y\tilde{V}_{12} = 0$. In that case the general solution can be expressed as $A = YZ^+ + (YZ^+)^T(I_n - ZZ^+) + \tilde{U}_{12}\tilde{M}\tilde{U}_{12}^T$, where $\tilde{M} \in SR^{(n-r_0)\times(n-r_0)}$ is arbitrary matrix.*

Partition $U_1^T X$ and $U_1^T B$, where U_1 is the same as (2.1), into the following form

$$U_1^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, U_1^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, X_1, B_1 \in R^{r\times m}, X_2, B_2 \in R^{(n-r)\times m}. \tag{2.5}$$

Suppose the SVD of X_2 is

$$X_2 = U_2 \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} V_2^T = U_{21}\Gamma V_{21}^T \tag{2.6}$$

where $U_2 = (U_{21}, U_{22}) \in OR^{(n-r)\times(n-r)}, U_{21} \in R^{(n-r)\times k_1}, V_2 = (V_{21}, V_{22}) \in OR^{m\times m}, V_{21} \in R^{m\times k_1}, \Gamma = \mathrm{diag}(a_1, a_2, \dots, a_{k_1}) > 0, k_1 = \mathrm{rank}(X_2)$.

Suppose the SVD of (X_1V_{22}) is

$$X_1 V_{22} = U_3 \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} V_3^T = U_{31}\Omega V_{31}^T \tag{2.7}$$

where $U_3 = (U_{31}, U_{32}) \in OR^{r\times r}, U_{31} \in R^{r\times k_2}, V_3 = (V_{31}, V_{32}) \in OR^{(m-k_1)\times(m-k_1)}, V_{31} \in R^{(m-k_1)\times k_2}, \Omega = \text{diag}(b_1, b_2, \ldots, b_{k_2}) > 0, k_2 = \text{rank}(X_1V_{22})$.

Let

$$W_1 = B_1V_{22}(X_1V_{22})^+ + [B_1V_{22}(X_1V_{22})^+]^T[I_r - (X_1V_{22})(X_1V_{22})^+], \tag{2.8}$$

$$W_2 = (B_1 - W_1X_1)X_2^+ + [U_{22}U_{22}^TB_2V_{22}(X_1V_{22})^+]^T \tag{2.9}$$

and

$$W_3 = (B_2 - W_2^TX_1)X_2^+. \tag{2.10}$$

Then we have the following theorem.

Theorem 1. *Given $X, B \in R^{n\times m}, S \in R^{n\times p}$, and the SVD of S, X_2 and X_1V_{22} separately are of the form (2.1),(2.6) and (2.7), then Problem I is soluble if and only if*

(i). $B_1V_{22}V_{32} = 0$;

(ii). $V_{22}^TB_1^TX_1V_{22} = V_{22}^TX_1^TB_1V_{22}$;

(iii). $U_{21}^TB_2V_{22} - U_{21}^T(X_2^+)^T(B_1 - W_1X_1)^T(X_1V_{22}) = 0$;

(iv). $U_{22}^TB_2V_{22}V_{32} = 0$.

When the above conditions are satisfied , the general solution of Problem I can be represented as

$$A = U_1 \begin{pmatrix} W_1 + U_{32}MU_{32}^T & W_2 + U_{32}N^TU_{22}^T - U_{32}MU_{32}^TX_1X_2^+ \\ W_2^T + U_{22}NU_{32}^T & W_3 + HU_{22}^T - U_{22}NU_{32}^TX_1X_2^+ \\ -(U_{32}^TX_1X_2^+)^TMU_{32}^T & +(U_{32}^TX_1X_2^+)^TMU_{32}^TX_1X_2^+ \end{pmatrix} U_1^T, \tag{2.11}$$

where $M \in SR^{(r-k_2)\times(r-k_2)}, N \in R^{(n-r-k_1)\times(r-k_2)}, H \in R^{(n-r)\times(n-r-k_1)}$ are arbitrary matrix.

Proof. Necessity. Suppose there exists $A \in SR_{s,n}$ such that $AX = B$, then by Lemma 1, $U_1^TAU_1$ can be partitioned as

$$U_1^TAU_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}, A_{11} \in SR^{r\times r}, A_{22} \in R^{(n-r)\times(n-r)}. \tag{2.12}$$

Hence $AX = B$ is equivalent to

$$A_{12}X_2 = B_1 - A_{11}X_1 \tag{2.13}$$

and

$$A_{22}X_2 = B_2 - A_{12}^TX_1. \tag{2.14}$$

Applying Lemma 2 and (2.6) to (2.13), we get

$$A_{11}X_1V_{22} = B_1V_{22} \tag{2.15}$$

and

$$A_{12} = (B_1 - A_{11}X_1)X_2^+ + GU_{22}^T, \tag{2.16}$$

where $G \in R^{r\times(n-r-k_1)}$ is arbitrary matrix. Applying Lemma 3, (2.7) and (2.8) to (2.15), we get

$$V_{22}^TB_1^TX_1V_{22} = V_{22}^TX_1^TB_1V_{22}, \quad B_1V_{22}V_{32} = 0 \tag{2.17}$$

and

$$A_{11} = W_1 + U_{32}MU_{32}^T, \tag{2.18}$$

where $M \in SR^{(r-k_2)\times(r-k_2)}$ is arbitrary matrix. Substitute (2.18) into (2.16), and furthermore into (2.14), we have

$$A_{22}X_2 = B_2 - (X_2^+)^T(B_1 - W_1X_1)^TX_1 + (U_{32}^TX_1X_2^+)^TMU_{32}^TX_1 - U_{22}G^TX_1. \tag{2.19}$$

Notice that $U_{32}^TX_1V_{22} = 0$, and using Lemma 2 and (2.6) to (2.19), we get

$$U_{22}G^TX_1V_{22} = B_2V_{22} - (X_2^+)^T(B_1 - W_1X_1)^TX_1V_{22} \tag{2.20}$$

and

$$A_{22}=[B_2-(X_2^+)^T(B_1-W_1X_1)^TX_1+(U_{32}^TX_1X_2^+)^TMU_{32}^TX_1-U_{22}G^TX_1]X_2^++HU_{22}^T, \quad (2.21)$$

where $H\in R^{(n-r)\times(n-r-k_1)}$ is arbitrary matrix. But (2.20) is equivalent to

$$U_{21}^TB_2V_{22}-U_{21}^T(X_2^+)^T(B_1-W_1X_1)^TX_1V_{22}=0 \quad (2.22)$$

and

$$G^TX_1V_{22}=U_{22}^TB_2V_{22}. \quad (2.23)$$

By Lemma 2 and (2.23), we get

$$U_{22}^TB_2V_{22}V_{32}=0 \quad (2.24)$$

and

$$G^T=U_{22}^TB_2V_{22}(X_1V_{22})^++NU_{32}^T, \quad (2.25)$$

where $N\in R^{(n-r-k_1)\times(r-k_2)}$ is arbitrary matrix. Taking (2.18) and (2.25) into (2.16), we have

$$A_{12}=W_2+U_{32}N^TU_{22}^T-U_{32}MU_{32}^TX_1X_2^+, \quad (2.26)$$

where W_2 is the same as (2.9). Taking (2.25) into (2.21), we have

$$A_{22}=W_3+HU_{22}^T-U_{22}NU_{32}^TX_1X_2^++(U_{32}^TX_1X_2^+)^TMU_{32}^TX_1X_2^+, \quad (2.27)$$

where W_3 is the same as (2.10). Taking (2.18),(2.26) and (2.27) into (2.12), we have (2.11).

Sufficiency. Suppose the conditions (i)-(iv) are satisfied. Let

$$A_{11}=W_1=B_1V_{22}(X_1V_{22})^++[B_1V_{22}(X_1V_{22})^+]^T[I_r-(X_1V_{22})(X_1V_{22})^+],$$
$$A_{12}=(B_1-A_{11}X_1)X_2^++[U_{22}U_{22}^TB_2V_{22}(X_1V_{22})^+]^T$$

and

$$A_{22}=(B_2-A_{12}^TX_1)X_2^+,$$

then we have

$$\begin{aligned}A_{11}^T&=[B_1V_{22}(X_1V_{22})^+]^T+(B_1V_{22})(X_1V_{22})^+-(X_1V_{22})^{+T}(X_1V_{22})^T(B_1V_{22})(X_1V_{22})^+\\&=(B_1V_{22})(X_1V_{22})^++[(B_1V_{22})(X_1V_{22})^+]^T-(X_1V_{22})^{+T}(B_1V_{22})^T(X_1V_{22})(X_1V_{22})^+\\&=(B_1V_{22})(X_1V_{22})^++[(B_1V_{22})(X_1V_{22})^+]^T[I_r-(X_1V_{22})(X_1V_{22})^+]=A_{11},\end{aligned}$$

$$\begin{aligned}A_{11}X_1V_{22}&=B_1V_{22}(X_1V_{22})^+(X_1V_{22})+[B_1V_{22}(X_1V_{22})^+]^T[I_r-(X_1V_{22})(X_1V_{22})^+](X_1V_{22})\\&=B_1V_{22}(X_1V_{22})^+(X_1V_{22})=(B_1V_{22})-B_1V_{22}V_{32}V_{32}^T=B_1V_{22},\end{aligned}$$

$$\begin{aligned}A_{12}X_2&=(B_1-A_{11}X_1)X_2^+X_2+[U_{22}U_{22}^TB_2V_{22}(X_1V_{22})^+]^TX_2\\&=(B_1-A_{11}X_1)-(B_1-A_{11}X_1)V_{22}V_{22}^T=B_1-A_{11}X_1,\end{aligned}$$

$$\begin{aligned}(B_2-A_{12}^TX_1)V_{22}V_{22}^T&=[B_2-X_2^{+T}(B_1-W_1X_1)^TX_1-U_{22}U_{22}^TB_2V_{22}(X_1V_{22})^+X_1]V_{22}V_{22}^T\\&=U_2\begin{pmatrix}U_{21}^TB_2V_{22}-U_{21}^TX_2^{+T}(B_1-W_1X_1)^TX_1V_{22}\\U_{22}^TB_2V_{22}-U_{22}^TB_2V_{22}(X_1V_{22})^+(X_1V_{22})\end{pmatrix}V_{22}^T\\&=U_2\begin{pmatrix}U_{21}^TB_2V_{22}-U_{21}^TX_2^{+T}(B_1-W_1X_1)^TX_1V_{22}\\U_{22}^TB_2V_{22}V_{32}V_{32}^T\end{pmatrix}V_{22}^T=0\end{aligned}$$

and

$$A_{22}X_2=(B_2-A_{12}^TX_1)-(B_2-A_{12}^TX_1)V_{22}V_{22}^T=B_2-A_{12}^TX_1.$$

Let

$$A=U_1\begin{pmatrix}A_{11}&A_{12}\\A_{12}^T&A_{22}\end{pmatrix}U_1^T,$$

then $A\in SR_{s,n}$ and

$$AX=U_1\begin{pmatrix}A_{11}&A_{12}\\A_{12}^T&A_{22}\end{pmatrix}\begin{pmatrix}X_1\\X_2\end{pmatrix}=U_1\begin{pmatrix}A_{11}X_1+A_{12}X_2\\A_{12}^TX_1+A_{22}X_2\end{pmatrix}=U_1\begin{pmatrix}B_1\\B_2\end{pmatrix}=B.$$

Hence Problem I is soluble.

3. The solution for Problem II

Lemma 4. *Given $C, D, H, K \in R^{n\times n}, T = diag(t_1, t_2, \ldots, t_n) > 0$, then the problem*

$$\varphi(G) \hat{=} \|G - C\|^2 + \|GT - D\|^2 + \|TG - H\|^2 + \|TGT - K\|^2 = \min \tag{3.1}$$

has an unique solution $\hat{G} \in SR^{n\times n}$,and

$$\hat{G} = F * (C + C^T + DT + TD^T + TH + H^T T + TKT + TK^T T) \tag{3.2}$$

where $F = (f_{ij}) \in R^{n\times n}, f_{ij} = \frac{1}{2(1+t_i^2+t_j^2+t_i^2 t_j^2)}$.

Proof. Since $G \in SR^{n\times n}$, we have from (3.1) that

$$\begin{aligned}\varphi(G) &= \sum_{1\le i,j\le n} [(g_{ij} - c_{ij})^2 + (t_j g_{ij} - d_{ij})^2 + (t_i g_{ij} - h_{ij})^2 + (t_i t_j g_{ij} - k_{ij})^2] \\ &= \sum_{1\le i<j\le n} [(g_{ij} - c_{ij})^2 + (g_{ij} - c_{ji})^2 + (t_j g_{ij} - d_{ij})^2 + (t_i g_{ij} - d_{ji})^2 \\ &\quad + (t_i g_{ij} - h_{ij})^2 + (t_j g_{ij} - h_{ji})^2 + (t_i t_j g_{ij} - k_{ij})^2 + (t_i t_j g_{ij} - k_{ji})^2] \\ &\quad + \sum_{1\le i\le n} [(g_{ii} - c_{ii})^2 + (t_i g_{ii} - d_{ii})^2 + (t_i g_{ii} - h_{ii})^2 + (t_i^2 g_{ii} - k_{ii})^2].\end{aligned}$$

By $\frac{\partial\varphi(G)}{\partial g_{ij}} = 0 \ \ (1 \le i, j \le n)$, we have

$$g_{ij} = \frac{1}{2(1 + t_i^2 + t_j^2 + t_i^2 t_j^2)}(c_{ij} + c_{ji} + t_i d_{ji} + t_j d_{ij} + t_i h_{ij} + t_j h_{ji} + t_i t_j k_{ij} + t_i t_j k_{ji}).$$

Hence

$$\hat{G} = F * (C + C^T + DT + TD^T + TH + H^T T + TKT + TK^T T).$$

Similar to the proof of the Lemma 4, we can prove the following lemma 5.

Lemma 5. *Given $A_i, B_j (i = 1, 2, \ldots, p, j = 1, 2, \ldots, q) \in R^{m\times n}, T = diag(t_1, t_2, \ldots, t_n) > 0$, then the problem*

$$\varphi(G) \hat{=} \sum_{1\le i\le p} \|G - A_i\|^2 + \sum_{1\le j\le q} \|GT - B_j\|^2 = \min \tag{3.3}$$

has an unique solution $\hat{G} \in R^{m\times n}$, and

$$\hat{G} = F * (\sum_{1\le i\le p} A_i + \sum_{1\le j\le q} B_j T) \tag{3.4}$$

where $F = (f_{ij}) \in R^{m\times n}, f_{ij} = \frac{1}{p+qt_j^2}$.

Similar to the proof of the lemma 7 in [7], we can prove the following lemma 6.

Lemma 6. *When the solution set S_E of Problem I is nonempty, then S_E is a convex cone, and the corresponding Problem II has an unique optimal approximate solution.*

Suppose the SVD of $(U_{32}^T X_1 V_{21} \Gamma^{-1})$ is

$$U_{32}^T X_1 V_{21} \Gamma^{-1} = U_4 \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V_4^T, \tag{3.5}$$

where $U_4 = (U_{41}, U_{42}) \in OR^{(r-k_2)\times(r-k_2)}$, $V_4 = (V_{41}, V_{42}) \in OR^{k_1\times k_1}$, $\Sigma = \text{diag}(\delta_1, \delta_2, \ldots, \delta_t) > 0$, $t = \text{rank}(U_{32}^T X_1 V_{21} \Gamma^{-1})$.

Let

$$U_1^T A^* U_1 = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, A_1 \in R^{r\times r}, A_4 \in R^{(n-r)\times(n-r)}, \tag{3.6}$$

$$\begin{aligned} Q_1 = &U_{41}^T U_{32}^T (A_1 + A_1^T - W_1 - W_1^T) U_{32} U_{41} + U_{41}^T U_{32}^T (W_2 - A_2) U_{21} V_{41} \Sigma \\ &+ \Sigma V_{41}^T U_{21}^T (W_2 - A_2)^T U_{32} U_{41} + \Sigma V_{41}^T U_{21}^T (W_2^T - A_3) U_{32} U_{41} \\ &+ U_{41}^T U_{32}^T (W_2^T - A_3)^T U_{21} V_{41} \Sigma + \Sigma V_{41}^T U_{21}^T (A_4 + A_4^T - W_3 - W_3^T) U_{21} V_{41} \Sigma, \end{aligned} \tag{3.7}$$

$$Q_2 = U_{41}^T U_{32}^T (A_1 + A_1^T - W_1 - W_1^T) U_{32} U_{42} + \Sigma V_{41}^T U_{21}^T (2W_2^T - A_3^T - A_2^T) U_{32} U_{42}, \tag{3.8}$$

$$Q_3 = \frac{1}{2} U_{42}^T U_{32}^T (A_1 + A_1^T - W_1 - W_1^T) U_{32} U_{42}, \tag{3.9}$$

$$Q_4 = U_{22}^T (A_3 - 2W_2^T + A_2^T) U_{32} U_{41} + U_{22}^T (A_4 - W_3) U_{21} V_{41} \Sigma, \tag{3.10}$$

and

$$Q_5 = \frac{1}{2} U_{22}^T (A_3 + A_2^T - 2W_2^T) U_{32} U_{42}. \tag{3.11}$$

Then we have the following theorem.

Theorem 2. *If the solution set S_E of Problem I is nonempty , then Problem II has a unique optimal approximate solution which can be represented as*

$$\hat{A} = U_1 \begin{pmatrix} W_1 + U_{32}\hat{M}U_{32}^T & W_2 + U_{32}\hat{N}^T U_{22}^T - U_{32}\hat{M}U_{32}^T X_1 X_2^+ \\ \begin{matrix} W_2^T + U_{22}\hat{N}U_{32}^T \\ -(U_{32}^T X_1 X_2^+)^T \hat{M} U_{32}^T \end{matrix} & \begin{matrix} W_3 + \hat{H}U_{22}^T - U_{22}\hat{N}U_{32}^T X_1 X_2^+ \\ +(U_{32}^T X_1 X_2^+)^T \hat{M} U_{32}^T X_1 X_2^+ \end{matrix} \end{pmatrix} U_1^T \tag{3.12}$$

where $\hat{M} = U_4 \begin{pmatrix} F_1 * Q_1 & F_2 * Q_2 \\ (F_2 * Q_2)^T & Q_3 \end{pmatrix} U_4^T, F_1 = (\psi_{ij}) \in R^{t\times t}, \psi_{ij} = \frac{1}{2(1+\delta_i^2+\delta_j^2+\delta_i^2\delta_j^2)}, F_2 = (\phi_{ij}) \in R^{t\times(r-k_2-t)}, \phi_{ij} = \frac{1}{2(1+\delta_i^2)}, \hat{N} = (F_3 * Q_4, Q_5)U_4^T, F_3 = (\rho_{ij}) \in R^{(n-r-k_1)\times t}, \rho_{ij} = \frac{1}{2+\delta_j^2}, \hat{H} = (A_4 - W_3)U_{22}.$

Proof. Since the solution set S_E of Problem I is nonempty , hence Problem II has a unique optimal approximate solution. Attention to $U_i, V_i (i = 1, 2, 3, 4)$ are orthogonal matrices, we have from (2.11) that

$$\begin{aligned} \|A - A^*\|^2 &= \| U_{32}MU_{32}^T - (A_1 - W_1) \|^2 + \| U_{32}N^T U_{22}^T - U_{32}MU_{32}^T X_1 X_2^+ - (A_2 - W_2) \|^2 \\ &+ \| U_{22}NU_{32}^T - (U_{32}^T X_1 X_2^+)^T M U_{32}^T - (A_3 - W_2^T) \|^2 \\ &+ \| HU_{22}^T - U_{22}NU_{32}^T X_1 X_2^+ + (U_{32}^T X_1 X_2^+)^T M U_{32}^T X_1 X_2^+ - (A_4 - W_3) \|^2 \\ &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} - U_3^T (A_1 - W_1) U_3 \right\|^2 \\ &+ \left\| \begin{pmatrix} 0 & 0 \\ -M(U_{32}^T X_1 V_{21} \Gamma^{-1}) & N^T \end{pmatrix} - U_3^T (A_2 - W_2) U_2 \right\|^2 \\ &+ \left\| \begin{pmatrix} 0 & -(U_{32}^T X_1 V_{21} \Gamma^{-1})^T M \\ 0 & N \end{pmatrix} - U_2^T (A_3 - W_2^T) U_3 \right\|^2 \\ &+ \left\| \begin{pmatrix} (U_{32}^T X_1 V_{21}\Gamma^{-1})^T M (U_{32}^T X_1 V_{21}\Gamma^{-1}) & U_{21}^T H \\ N(U_{32}^T X_1 V_{21}\Gamma^{-1}) & U_{22}^T H \end{pmatrix} - U_2^T (A_4 - W_3) U_2 \right\|^2. \end{aligned}$$

Hence $\|A - A^*\| = \min$ is equivalent to

$$\begin{aligned} &\|M - U_{32}^T (A_1 - W_1) U_{32}\|^2 + \|M(U_{32}^T X_1 V_{21}\Gamma^{-1}) - U_{32}^T (W_2 - A_2) U_{21}\|^2 \\ &+ \|(U_{32}^T X_1 V_{21}\Gamma^{-1})^T M - U_{21}^T (W_2^T - A_3) U_{32}\|^2 \\ &+ \|(U_{32}^T X_1 V_{21}\Gamma^{-1})^T M U_{32}^T X_1 V_{21}\Gamma^{-1} - U_{21}^T (A_4 - W_3) U_{21}\|^2 = \min, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\|N^T - U_{32}^T (A_2 - W_2) U_{22}\|^2 + \|N - U_{22}^T (A_3 - W_2^T) U_{32}\|^2 \\ &+ \|N(U_{32}^T X_1 V_{21}\Gamma^{-1}) - U_{22}^T (A_4 - W_3) U_{21}\|^2 = \min \end{aligned} \tag{3.14}$$

and

$$\|H - (A_4 - W_3) U_{22}\| = \min. \tag{3.15}$$

Write

$$U_4^T M U_4 = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}, M_{11} \in SR^{t\times t}, M_{22} \in SR^{(r-k_2-t)\times(r-k_2-t)}, \tag{3.16}$$

then (3.13) is equivalent to

$$\|M_{11}-U_{41}^TU_{32}^T(A_1-W_1)U_{32}U_{41}\|^2+\|M_{11}\Sigma-U_{41}^TU_{32}^T(W_2-A_2)U_{21}V_{41}\|^2$$
$$+\|\Sigma M_{11}-V_{41}^TU_{21}^T(W_2^T-A_3)U_{32}U_{41}\|^2+\|\Sigma M_{11}\Sigma-V_{41}^TU_{21}^T(A_4-W_3)U_{21}V_{41}\|^2\min, \quad (3.17)$$

$$\|M_{12}-U_{41}^TU_{32}^T(A_1-W_1)U_{32}U_{42}\|^2+\|M_{12}-U_{41}^TU_{32}^T(A_1-W_1)^TU_{32}U_{42}\|^2$$
$$+\|\Sigma M_{12}-V_{41}^TU_{21}^T(W_2^T-A_2^T)U_{32}U_{42}\|^2+\|\Sigma M_{12}-V_{41}^TU_{21}^T(W_2-A_3)^TU_{32}U_{42}\|^2=\min \quad (3.18)$$

and

$$\|M_{22}-U_{42}^TU_{32}(A_1-W_1)U_{32}U_{42}\|=\min. \quad (3.19)$$

Applying Lemma 4 and Lemma 5 to (3.17),(3.18) and (3.19), we have $M_{11}=F_1*Q_1, M_{12}=F_2*Q_2, M_{22}=Q_3$. Hence

$$M=U_4\begin{pmatrix} F_1*Q_1 & F_2*Q_2 \\ (F_2*Q_2)^T & Q_3 \end{pmatrix}U_4^T. \quad (3.20)$$

Write

$$NU_4=(N_{11},N_{12}), N_{11}\in R^{(n-r-k_1)\times t}, N_{12}\in R^{(n-r-k_1)\times(r-k_2-t)}, \quad (3.21)$$

then (3.14) is equivalent to

$$\|N_{11}-U_{22}^T(A_2-W_2)^TU_{32}U_{41}\|^2+\|N_{11}-U_{22}^T(A_3-W_2^T)U_{32}U_{41}\|^2$$
$$+\|N_{11}\Sigma-U_{22}^T(A_4-W_3)U_{21}V_{41}\|^2=\min \quad (3.22)$$

and

$$\|N_{12}-U_{22}^T(A_2-W_2)^TU_{32}U_{42}\|^2+\|N_{12}-U_{22}^T(A_3-W_2^T)U_{32}U_{42}\|^2=\min. \quad (3.23)$$

Applying Lemma 5 to (3.22) and (3.23), we have $N_{11}=F_3*Q_4, N_{12}=Q_5$. Hence

$$N=(F_3*Q_4,Q_5)U_4^T. \quad (3.24)$$

From (3.15),we have

$$H=(A_4-W_3)U_{22}. \quad (3.25)$$

Taking (3.20),(3.24) and (3.25) into (2.11), we have (3.12).

According to Theorem 1 and 2 , we now give an algorithm of finding the optimal approximate solution of Problem II as the following steps:
(1). Construct the SVD of S according to (2.1);
(2). According to (2.5) compute $X_i, B_i (i=1,2)$;
(3). Construct the SVD of X_2 and (X_1V_{22}) according to (2.6) and (2.7) ;
(4). If conditions (i)-(iv) are satisfied, go to step (5); else go to step (9);
(5). According to (2.8)-(2.10) calculate $W_i (i=1,2,3)$;
(6). Construct the SVD of $(U_{32}^TX_1V_{22}\Gamma^{-1})$ according to (3.5) ;
(7). According to (3.7)-(3.11) calculate $Q_i (i=1,2,3,4,5)$;
(8). According to (3.12) calculate $\hat{A}$;
(9). stop.

Example 1. Given

$$S=\begin{pmatrix} 1.2 & -0.9 & 0.7 & 0.2 & 2.4 & -2.1 \\ 0.7 & -0.9 & 1.8 & -0.9 & 1.4 & 1.6 \\ -1.1 & -1.7 & 0.2 & 1.5 & -2.2 & 0.6 \\ 0.8 & 0.4 & 0.4 & -0.8 & 1.6 & 0.4 \\ 0.9 & -0.7 & -1.6 & 2.3 & 1.8 & 1.6 \\ 1.3 & 0.6 & 1.1 & -1.7 & 2.6 & -0.8 \\ 0.7 & -0.9 & 1.8 & -0.9 & 1.4 & 1.6 \\ -1.2 & 0.9 & -0.7 & -0.2 & -2.4 & 2.1 \end{pmatrix},$$

$$X = \begin{pmatrix} -23.3 & 46.6 & -32.8 \\ 32.4 & -64.8 & 31.4 \\ -42.7 & 85.4 & -22.9 \\ 32.1 & -64.2 & 31.6 \\ -23.9 & 47.8 & 57.1 \\ -12.4 & 24.8 & -26.9 \\ 31.1 & -62.2 & 41.7 \\ 27.6 & -55.2 & -23.7 \end{pmatrix}, B = \begin{pmatrix} 10.1 & -20.2 & -11.5 \\ 49.9 & -99.8 & 128.0 \\ 153.6 & -307.2 & 142.2 \\ -46.0 & 92.0 & -51.2 \\ -20.3 & 40.6 & 25.3 \\ 14.5 & -29.0 & 76.6 \\ 30.2 & -60.4 & 57.1 \\ 17.5 & -35.0 & -47.2 \end{pmatrix}$$

and

$$A^* = \begin{pmatrix} 1.2 & 1.7 & 2.0 & 3.1 & 4.0 & -5.0 & -1.8 & 3.0 \\ -1.7 & 2.6 & 1.9 & -1.1 & -6.0 & 3.9 & 7.0 & -1.9 \\ 3.0 & -4.1 & -2.9 & 1.7 & 1.6 & 0.8 & -4.5 & 2.1 \\ 1.5 & 1.9 & 1.6 & -5.3 & -2.6 & -6.1 & 3.0 & -2.3 \\ -5.3 & -2.6 & -4.7 & 1.2 & -7.1 & -6.1 & 2.3 & 2.1 \\ 2.0 & 3.0 & 4.0 & 6.1 & 7.1 & 2.9 & 1.6 & -4.0 \\ 1.7 & 3.5 & -1.8 & -2.1 & 4.1 & 2.1 & -2.2 & 1.6 \\ 2.3 & -4.9 & 3.1 & -2.3 & 3.1 & -4.7 & -1.3 & -3.3 \end{pmatrix}.$$

It is easy to verify that the conditions of Theorem 1 are satisfied. Hence, the solution set S_E of Problem I is nonempty, and so the corresponding Problem II has an unique solution. According to the above calculating steps, we have $\hat{A}$ as follow:

$$\hat{A} = \begin{pmatrix} 0.9008 & 1.8419 & 0.6820 & 0.7353 & -0.7187 & 0.2878 & -0.4294 & -0.8366 \\ 1.5452 & 2.2434 & 1.4235 & 1.3189 & -0.7862 & 1.8938 & 3.2973 & -2.4056 \\ 0.2746 & 0.5241 & -2.2272 & 3.3022 & 1.3065 & 1.0075 & -1.1703 & 0.8025 \\ -0.1051 & 1.0686 & -0.0580 & -4.5006 & -1.2741 & 0.0221 & 2.1978 & -1.4392 \\ -0.6617 & -0.9651 & 1.9445 & -0.4280 & -0.8892 & -1.9493 & 1.9113 & -0.4524 \\ 0.2415 & 0.7372 & 3.2156 & 4.9577 & -2.3304 & -0.6214 & 0.2630 & -3.5241 \\ 0.1717 & 3.8400 & -1.7986 & -0.0045 & 2.1712 & 0.7874 & -4.2788 & 1.0019 \\ -0.8531 & -2.5220 & 1.9986 & 1.1638 & -0.6455 & -3.4778 & 0.2135 & 2.2425 \end{pmatrix}$$

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