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MULTISYMPLECTIC COMPOSITION INTEGRATORS OF HIGH ORDER *1)

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Abstract

A composition method for constructing high order multisymplectic integrators is presented in this paper. The basic idea is to apply composition method to both the time and the space directions. We also obtain a general formula for composition method.

Key words: Multisymplectic integrators, Composition method.

1. Introduction

Composition method originates from the construction of symplectic integrators for separable Hamiltonian systems. Yoshida applied symmetric composition method to obtaining high order decomposition of vector field and suggested the composition method for Hamiltonian systems [12]. Based on the theory of Lie series and formal vector field, Qin and Zhu systematically suggested the composition method for general ordinary differential equations [8]. Suzuki established the general theory of high order decomposition of exponential operators [10]. We will show in this paper that Suzuki's theory can be used to obtain a general formula for composition method.

Recently multisymplectic Hamiltonian systems and multisymplectic integrators are drawing a lot of attention [1, 2, 9, 3, 4]. Bridges first introduced the concept of multisymplectic Hamiltonian systems which possess a completely local multisymplectic conservation law [1]. Bridges and Reich suggested the concept of multisymplectic integrators which preserve a discrete version of multisymplectic conservation law [2]. Reich showed that Gauss-Legendre collocation in space and time leads to multisymplectic integrators [9]. However, in high order case, the multisymplectic integrators obtained by Reich are very difficult to implement. Therefore, we suggest a composition method for high order multisymplectic integrators. The resulting high order multisymplectic integrators are very easy to implement.

An outline of the paper is as follows. In §2, we present the composition method for ordinary differential equations. The basic formula for composition method is obtained. §3 is devoted to developing composition method for constructing high order multisymplectic integrators. We present numerical experiments in §4. Some conclusions are included in §5.

2. Composition Method for ODEs

We first present the composition method for ordinary differential equations (ODEs). We know that every one-step integrator for $y' = f(y)$ can be written

$$y_{n+1} = s(\tau)y_n,$$

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where $s(\tau)$ is the operator corresponding to the integrator, and τ is the step length.

Definition 1. *Suppose there are n integrators whose corresponding operators are* $s_1(\tau)$, $s_2(\tau)$, $\ldots$, $s_n(\tau)$ *respectively, and their corresponding order is* $p_1, p_2, \ldots, p_n$ *respectively. If there exist constants* $c_1, c_2, \ldots, c_n$ *such that the order of the integrator whose operator is the composition* $s_1(c_1\tau)s_2(c_2\tau)\cdots s_n(c_n\tau)$ *is* $m, m > max(p_i), 1 \leq i \leq n$, *then the new integrator is called composition integrator of the original n integrators. This method which is used to construct higher order integrators from the lower ones is called composition method.*

In the discussion as follows, we also need the concept of adjoint operator and self-adjoint operator.

Definition 2. *An operator* $s^*(\tau)$ *is called the adjoint operator of* $s(\tau)$, *if*

$$s^*(-\tau)s(\tau) = s(\tau)s^*(-\tau) = I, \tag{2.1}$$

where I *is the identity operator.*

Definition 3. *We call an operator* $s(\tau)$ *is self-adjoint, if* $s^*(\tau) = s(\tau)$.

The development of composition method relies on the theory of Lie series [5, 7, 11] and the following theorem [6].

Theorem 1. *Every operator* $s(\tau)$ *has a formal exponential representation*

$$s(\tau) = exp(\tau A + \tau^2 B + \tau^3 C + \tau^4 D + \cdots),$$

where $A, B, C, D, \ldots$ *are first order differential operators.*

According to the definition of composition method, constructing higher order integrator $s_1(c_1\tau)s_2(c_2\tau)\cdots s_n(c_n\tau)$ is to determine constants $c_1, c_2, \ldots, c_n$ such that the scheme $s_1(c_1\tau)$ s_2 $(c_2\tau)\cdots s_n(c_n\tau)$ has order m. Now we will deduce the basis formula for determining the constants $c_i (i = 1, \ldots, n)$. By theorem 1, we have

$$s_j(\tau) = \exp(\tau w_{j1} + \tau^2 w_{j2} + \tau^3 w_{j3} + \cdots + \tau^{p_j} w_{jp_j} + \tau^{p_j+1} w_{jp_j+1} + \cdots).$$

Since $s_j(\tau)$ has order p_j, $w_{j1} = L_f, w_{j2} = w_{j3} = \cdots = w_{jp_j} = 0, w_{jp_j+1} \neq 0$. Here L_f is the differential operator corresponding to $y' = f(y)$ [8]. As in [10], we introduce *symmetrization operator* S

$$S(x^p z^q) = \frac{p!q!}{(p+q)!} \sum_{P_m} P_m(x^p z^q),$$

where x, z are arbitrary noncommutable operators, P_m denotes the summation of all the operators obtained in all possible ways of permutation.

We also introduce *time-ordering operator* P

$$P(x_i x_j) = \begin{cases} x_i x_j, & \text{if } i < j; \\ x_j x_i & \text{if } j < i, \end{cases}$$

where x_i, x_j are noncommutable operators.

Letting $G_m(\tau) = s_1(c_1\tau)\cdots s_n(c_n\tau)$, we have

$$\begin{aligned}
G_m(\tau) &= \prod_{j=1}^{n} s_j(c_j\tau) \\
&= \prod_{j=1}^{n} \exp\left[(c_j\tau)w_{j1} + (c_j\tau)^2 w_{j2} + (c_j\tau)^3 w_{j3} + \cdots\right] \\
&= P\exp\left[\sum_{j=1}^{n}((c_j\tau)w_{j1} + (c_j\tau)^2 w_{j2} + (c_j\tau)^3 w_{j3} + \cdots)\right] \\
&= P\exp(\tau x_1 + \tau^2 x_2 + \tau^3 + \cdots) \\
&= PS(e^{\tau x_1} e^{\tau^2 x_2} e^{\tau^3 x_3} \cdots) \\
&= \sum_{n_1,n_2,n_3,\cdots} \frac{\tau^{n_1+2n_2+3n_3+\cdots}}{n_1! n_2! n_3! \cdots} PS(x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots),
\end{aligned} \tag{2.2}$$

where $x_1 = \sum_{j=1}^{n} w_{j1} = (\sum_{j=1}^{n}(c_j))L_f$, $x_l = \sum_{j=1}^{n} c_j^l w_{jl}$.

If $G_m(\tau)$ has order m, then

$$G_m(\tau) = e^{\tau L_f} + \mathcal{O}(\tau^{m+1}).$$

Therefore, we obtain the condition on which G_m has order m

$$\begin{aligned}
&PS(x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots) = 0, \\
&n_1 + 2n_2 + 3n_3 + \cdots \leq m, \\
&\text{excluding } \ n_2 = n_3 = \cdots = 0, \\
&\sum_{i=1}^{n} c_i = 1.
\end{aligned} \tag{2.3}$$

This is the basic formula for composition method. From this formula we can determine the constants $c_i \, (i = 1, \ldots, n)$.

In what follows, we also use the exponential representation of adjoint operators.

Let

$$s_j(\tau) = \exp(\tau w_{j1} + \tau^2 w_{j2} + \tau^3 w_{j3} + \cdots), \tag{2.4}$$

then

$$s_j^*(\tau) = \exp(\tau \hat{w}_{j1} + \tau^2 \hat{w}_{j2} + \tau^3 \hat{w}_{j3} + \cdots), \tag{2.5}$$

where $w_{j(2m-1)} = \hat{w}_{j(2m-1)}, \ w_{j(2m)} = -\hat{w}_{j(2m)}$.

The expression (2.5) can be derived from the definition of adjoint operator. Next, we propose some applications of the basic formula (2.3).

Application 1. Choosing

$$s_1(\tau) = s_3(\tau) = \cdots = s(\tau), s_2(\tau) = s_4(\tau) = \cdots = s^*(\tau),$$

where $s(\tau)$ is the Euler scheme, then we have

$$G_m(\tau) = s(c_1\tau)s^*(c_2\tau)s(c_3\tau)s^*(c_4\tau)\cdots s(c_{n-1}\tau)s^*(c_n\tau).$$

In this case,

$$x_{2l} = \sum_{j=1}^{n} c_j^{2l} w_{2l}, \quad x_{2l-1} = \sum_{j=1}^{n} c_j^{2l-1} w_{2l-1}, \quad w_{2lj} = w_{2l}, \quad w_{(2l-1)j} = w_{2l-1}.$$

We only consider the case with $m = 2,$ namely

$$G_2(\tau) = s(c_1\tau)s^*(c_2\tau)\cdots s(c_{n-1}\tau)s^*(c_n\tau). \tag{2.6}$$

From the basic formula, it follows that

$$\begin{aligned} &\sum_{j=1}^{n} c_i = 1, \\ &c_1^2 + c_3^2 + \cdots = c_2^2 + c_4^2 + \cdots . \end{aligned} \tag{2.7}$$

With $n = 2,$ (2.7) becomes

$$\begin{aligned} &c_1 + c_2 = 1, \\ &c_1^2 = c_2^2. \end{aligned} \tag{2.8}$$

From (2.8), we obtain $c_1 = c_2 = 1/2$ and (2.6) becomes

$$G_2(\tau) = s(\frac{1}{2}\tau)s^*(\frac{1}{2}\tau). \tag{2.9}$$

The integrator with operator (2.9) is just the mid-point scheme.
Application 2. Choosing $s_1(\tau) = s_2(\tau) = s_3(\tau) = G_2(\tau),\ c_1 = c_3$, we have

$$G_4(\tau) = G_2(c_1\tau)G_2(c_2\tau)G_3(c_1\tau),$$

where $G_2(\tau)$ is a self-adjoint scheme of order 2.

In this case, the basic formulas takes the form

$$\begin{aligned} &2c_1 + c_2 = 1, \\ &2c_1^3 + c_2^3 = 0. \end{aligned}$$

Thus, we get $c_1 = \frac{1}{2-2^{1/3}}, c_2 = -\frac{2^{1/3}}{2-2^{1/3}}$. For example, if $G_2(\tau)$ is the mid-point scheme, we obtain

$$\begin{aligned} y_{1/3} &= y_n + \frac{1}{2-2^{1/3}}\tau f(\frac{y_n + y_{1/3}}{2}), \\ y_{2/3} &= y_{1/3} + \frac{-2^{1/3}}{2-2^{1/3}}\tau f(\frac{y_{1/3} + y_{2/3}}{2}), \\ y_{n+1} &= y_{2/3} + \frac{1}{2-2^{1/3}}\tau f(\frac{y_{2/3} + y_{n+1}}{2}). \end{aligned} \tag{2.10}$$

This is a one-step three-stage integrator of order 4. Generally, we have [8] **Corollary 1.** *Let $s(\tau)$ be a self-adjoint integrator of order $2n$, then the scheme $s(c_1\tau)s(c_2\tau)s(c_3\tau)$, with c_1, c_2 satisfying*

$$2c_1^{2n+1} + c_2^{2n+1} = 0, \quad 2c_1 + c_2 = 1$$

is of order $2n + 2$.

3. High Order Multisymplectic Integrators by Composition

We first present the concept of multisymplectic integrators introduced by Bridges in [2]. A large class of PDEs (for simplicity, we only consider one space dimension) can be cast into a system of the form

$$Mz_t + Kz_x = \bigtriangledown_z S(z),\ z \in \mathbf{R}^n,\ (x,t) \in \mathbf{R}^2, \tag{3.1}$$

where M and K are skew-symmetric matrices on $\mathbf{R}^n, n \geq 3$ and $S : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. We call (3.1) a multisymplectic Hamiltonian system.

For example, consider the non-linear wave equation

$$u_{tt} - u_{xx} + V'(u) = 0. \tag{3.2}$$

With new variables, v, w, and p, (3.2) is equivalent to

$$\begin{aligned}
&v = u_t + p_x,\\
&w = -u_x - p_t,\\
&w_t + v_x = 0,\\
&v_t + w_x + v'(u) = 0,
\end{aligned}$$

or with $z = (u, v, w, p)^T \in \mathbf{R}^4$

$$Mz_t + Kz_x = \bigtriangledown_z S(z), \tag{3.3}$$

where

$$M = \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{bmatrix},$$

and

$$S(z) = \frac{1}{2}(v^2 - w^2) + V(u).$$

Associated with (3.1) are the pair of differential two forms defined by

$$\omega(U,V) = \langle MU, V\rangle, \quad \kappa(U,V) = \langle KU, V\rangle, \quad \forall U, V \in \mathbf{R}^n,$$

where$\langle \cdot, \cdot \rangle$ stands for the standard inner product.

A fundamental geometric property of systems of the form (3.1) is that they conserve the following multisymplectic conservation law

$$\frac{\partial}{\partial t}\omega(U,V) + \frac{\partial}{\partial x}\kappa(U,V) = 0,$$

where U and V are any pair of solutions of the variational equation associated with (3.1)

$$Mdz_t + Kdz_x = D_{zz}S(z)dz.$$

A multisymplectic integrator for (3.1) is a numerical scheme for approximating (3.1) which also conserves a discrete version of multisymplectic conservation law.

We can formulate the numerical discretization of (3.1) as

$$M\partial_t^{i,j}z_{i,j} + K\partial_x^{i,j}z_{i,j} = \nabla_z S(z_{i,j}), \tag{3.4}$$

where $z_{i,j} = z(x_i, t_j)$, and $\partial_t^{i,j}$ and $\partial_x^{i,j}$ are discretizations of the derivatives ∂_t and ∂_x respectively.

Using the same discretization as in (3.4), a discrete version of multisymplectic conservation law can be written as

$$\partial_t^{i,j}\omega_{i,j} + \partial_x^{i,j}\kappa_{i,j} = 0,$$

where

$$\omega_{i,j} = \langle MU_{i,j}, V_{i,j}\rangle, \quad \kappa_{i,j} = \langle KU_{i,j}, V_{i,j}\rangle.$$

$U_{i,j}$ and $V_{i,j}$ satisfy the discrete variational equations

$$M\partial_t^{i,j}dz_{i,j} + K\partial_x^{i,j}dz_{i,j} = D_{zz}^{i,j}dz_{i,j}.$$

We now use composition method to construct higher order multisymplectic integrators from the lower ones. From the definition of composition method, we can know that composition method keeps the group property of the original integrator.

Given a multisymplectic integrator for (3.1) with accuracy of $\mathcal{O}(\tau^p + \hat{\tau}^q)$

$$M(s(\tau)z_{i,j}) + K(\hat{s}(\hat{\tau})z_{i,j}) = \nabla_z(\tilde{z}_{i,j}), \tag{3.5}$$

where $s(\tau)$ and $\hat{s}(\hat{\tau})$ are discrete operators in t-direction and x-direction respectively, and τ and $\hat{\tau}$ are time step and space step respectively. $\tilde{z}_{i,j} = f_{s,\hat{s}}(z_{i,j})$ is a function of $z_{i,j}$ corresponding the operators $s(\tau)$ and $\hat{s}(\hat{\tau})$.

Suppose $G_m(\tau)$ is the composition operator of $s(\tau)$ with accuracy of $\mathcal{O}(\tau^m)$, and $\hat{G}_n(\hat{\tau})$ is the composition operator of $\hat{s}(\hat{\tau})$ with accuracy of $\mathcal{O}(\hat{\tau}^n)$, then the multi-symplectic integrator

$$M(G_m(\tau)z_{i,j}) + K(\hat{G}_n(\hat{\tau})z_{i,j}) = \nabla_z S(\tilde{z}_{i,j}) \tag{3.6}$$

has accuracy of $\mathcal{O}(\tau^m + \hat{\tau}^n)$.

For example, if $s(\tau)$ and $\hat{s}(\hat{\tau})$ are both the discrete operators corresponding the mid-point scheme, then (3.6) takes the form

$$\begin{aligned} &M\left(\frac{z_{i+1/2,j+1} - z_{i+1/2,j}}{\tau}\right) + K\left(\frac{z_{i+1,j+1/2} - z_{i,j+1/2}}{\hat{\tau}}\right) \\ &= \nabla_z S(z_{i+1/2,j+1/2}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} z_{i+1/2,j} &= \frac{1}{2}(z_{i,j} + z_{i+1,j}), \\ z_{i,j+1/2} &= \frac{1}{2}(z_{i,j} + z_{i,j+1}), \\ z_{i+1/2,j+1/2} &= \frac{1}{4}(z_{i,j} + z_{i+1,j} + z_{i,j+1} + z_{i+1,j+1}). \end{aligned}$$

Bridges and Reich have proved that this integrator is multisymplectic and the integrator is equivalent to the Preissman box scheme [2].

Let $G_4(\tau) = s(c_1\tau)s(c_2\tau)s(c_1\tau)$, $c_1 = \frac{1}{2-2^{1/3}}, c_2 = \frac{-2^{1/3}}{2-2^{1/3}}$. Here $s(\tau)$ is the discrete operator corresponding to the mid-point scheme. Then the following three-stage multisymplectic integrator has accuracy of $\mathcal{O}(\tau^4 + \hat{\tau}^2)$.

$$\begin{aligned} &M\left(\frac{z_{i+1/2,1/3} - z_{i+1/2,j}}{c_1\tau}\right) + K\left(\frac{z_{i+1,t_1} - z_{i,t_1}}{\hat{\tau}}\right) = \nabla_z S(z_{i+1/2,t_1}), \\ &M\left(\frac{z_{i+1/2,2/3} - z_{i+1/2,1/3}}{c_2\tau}\right) + K\left(\frac{z_{i+1,t_2} - z_{i,t_2}}{\hat{\tau}}\right) = \nabla_z S(z_{i+1/2,t_2}), \\ &M\left(\frac{z_{i+1/2,j+1} - z_{i+1/2,2/3}}{c_1\tau}\right) + K\left(\frac{z_{i+1,t_3} - z_{i,t_3}}{\hat{\tau}}\right) = \nabla_z S(z_{i+1/2,t_3}), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} z_{i,t_1} &= \frac{1}{2}(z_{i,1/3} + z_{i,j}), \\ z_{i,t_2} &= \frac{1}{2}(z_{i,2/3} + z_{i,1/3}), \\ z_{i,t_3} &= \frac{1}{2}(z_{i,j+1} + z_{i,2/3}). \end{aligned} \tag{3.9}$$

Similarly, let $\hat{G}_4(\hat{\tau}) = \hat{s}(c_1\hat{\tau})\hat{s}(c_2\hat{\tau})\hat{s}(c_1\hat{\tau})$, $c_1 = \frac{1}{2-2^{1/3}}, c_2 = \frac{-2^{1/3}}{2-2^{1/3}}$. Here $\hat{s}(\hat{\tau})$ is the discrete operator corresponding to the mid-point scheme. Then the following three-stage multisymplectic integrator has accuracy of $\mathcal{O}(\tau^2 + \hat{\tau}^4)$.

$$\begin{aligned} &M\left(\frac{z_{m_1,j+1} - z_{m_1,j}}{\tau}\right) + K\left(\frac{z_{1/3,j+1/2} - z_{i,j+1/2}}{c_1\hat{\tau}}\right) = \nabla_z S(z_{m_1,j+1/2}), \\ &M\left(\frac{z_{m_2,j+1} - z_{m_1,j}}{\tau}\right) + K\left(\frac{z_{2/3,j+1/2} - z_{1/3j+1/2}}{c_2\hat{\tau}}\right) = \nabla_z S(z_{m_2,j+1/2}), \\ &M\left(\frac{z_{m_3,j+1} - z_{m_3,j}}{\tau}\right) + K\left(\frac{z_{i+1,j+1/2} - z_{2/3,j+1/2}}{c_1\hat{\tau}}\right) = \nabla_z S(z_{m_3,j+1/2}), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} z_{m_1,j} &= \frac{1}{2}(z_{1/3,j} + z_{i,j}), \\ z_{m_2,j} &= \frac{1}{2}(z_{2/3,j} + z_{1/3,j}), \\ z_{m_3,j} &= \frac{1}{2}(z_{i+1,j} + z_{2/3,j}). \end{aligned} \tag{3.11}$$

If we use the composition procedure in both x and t directions, we obtain the following nine-stage multi-symplectic integrator with accuracy of $\mathcal{O}(\tau^4 + \hat{\tau}^4)$.

$$K(G_4(\tau)z_{i,j}) + M(\hat{G}_4(\hat{\tau})z_{i,j}) = \nabla_z S(\tilde{z}_{i,j}). \tag{3.12}$$

4. Numerical Experiments

In this section, we perform numerical experiments with multisymplectic integrators. We consider nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{4.1}$$

Using $u = p + iq$, we can rewrite (4.1) as a pair of real-valued equations

$$\begin{aligned} p_t + q_{xx} + 2(p^2+q^2)q &= 0, \\ q_t - p_{xx} - 2(p^2+q^2)p &= 0. \end{aligned} \tag{4.2}$$

Introducing a pair of conjugate momenta $v = p_x, w = q_x$, (4.2) is equivalent to the multi-symplectic system

$$\begin{aligned} q_t - v_x &= 2(p^2+q^2)p, \\ -p_t - w_x &= 2(p^2+q^2)q, \\ p_x &= v, \\ q_x &= w. \end{aligned} \tag{4.3}$$

Now we discretize (4.3) using mid-point scheme in time direction and symplectic Euler scheme in space direction and obtain

$$\begin{aligned} \frac{q_m^{n+1}-q_m^n}{\Delta t} - \frac{v_{m+1}^{n+\frac12}-v_m^{n+\frac12}}{\Delta x} &= 2((p_m^{n+\frac12})^2+(q_m^{n+\frac12})^2)p_m^{n+\frac12}, \\ -\frac{p_m^{n+1}-p_m^n}{\Delta t} - \frac{w_{m+1}^{n+\frac12}-w_m^{n+\frac12}}{\Delta x} &= 2((p_m^{n+\frac12})^2+(q_m^{n+\frac12})^2)q_m^{n+\frac12}, \\ \frac{p_m^{n+\frac12}-p_{m-1}^{n+\frac12}}{\Delta x} &= v_m^{n+\frac12}, \\ \frac{q_m^{n+\frac12}-q_{m-1}^{n+\frac12}}{\Delta x} &= w_m^{n+\frac12}, \end{aligned} \tag{4.4}$$

where $p_m^n \approx p(m\Delta x, n\Delta t), q_m^{n+\frac12} = \frac12(q_m^{n+1}+q_m^n)$, etc. Δx and Δt are the space step length and time step length respectively.

(4.4) is a multisymplectic integrator with accuracy of $\mathcal{O}(\Delta t^2 + \Delta x^2)$ since by eliminating v and w and using $u = p + iq$, (4.4) is equivalent to a scheme for (4.1)

$$i\frac{u_m^{n+1}-u_m^n}{\Delta t} + \frac{u_{m+1}^{n+\frac12}-2u_m^{n+\frac12}+u_{m-1}^{n+\frac12}}{\Delta x^2} + 2|u_m^{n+\frac12}|^2 u_m^{n+\frac12} = 0. \tag{4.5}$$

We first perform numerical experiments with the integrator (4.4). The following initial condition is used.

$$u(x,0) = \operatorname{sech}(x-100)\exp(2ix) + 1.5\operatorname{sech}(1.5x+150)\exp(-2ix). \tag{4.6}$$

The computation is done for $0 \le t \le 60, -160 < x < 160$, with a time step $\Delta t = 0.02$ and $\Delta x = 0.1$. From Fig.1, we can see that the collision of the two soliton is well simulated by (4.4).

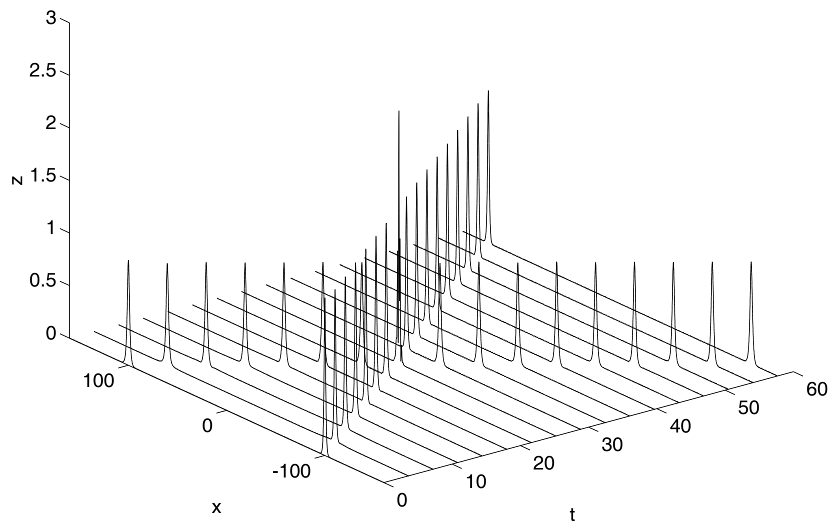

Fig.1 The collision of two solitons.

Now we use composition method in t direction for (4.4) and obtain

$$\begin{aligned} \frac{q_m^{n+1}-q_m^n}{c_i\Delta t}-\frac{v_{m+1}^{n+\frac12}-v_m^{n+\frac12}}{\Delta x}&=2((p_m^{n+\frac12})^2+(q_m^{n+\frac12})^2)p_m^{n+\frac12},\\ -\frac{p_m^{n+1}-p_m^n}{c_i\Delta t}-\frac{w_{m+1}^{n+\frac12}-w_m^{n+\frac12}}{\Delta x}&=2((p_m^{n+\frac12})^2+(q_m^{n+\frac12})^2)q_m^{n+\frac12},\\ \frac{p_m^{n+\frac12}-p_{m-1}^{n+\frac12}}{\Delta x}&=v_m^{n+\frac12},\\ \frac{q_m^{n+\frac12}-q_{m-1}^{n+\frac12}}{\Delta x}&=w_m^{n+\frac12}, \end{aligned} \tag{4.7}$$

where $i=1,2,3$ and $c_1=c_3=\frac{1}{2-2^{1/3}}, c_2=-\frac{2^{1/3}}{2-2^{1/3}}$.

The composition integrator (4.7) is of order $\mathcal{O}(\Delta t^4+\Delta x^2)$. To demonstrate this, we perform numerical experiments using one-soliton initial condition

$$u(x,0)=\operatorname{sech}(x)\exp(-2ix). \tag{4.8}$$

We use $\Delta x=0.01$ and $\Delta t=0.1$ so that the error mainly comes from Δt. Table 1 shows the maximal error between numerical solution and exact solution at different time levels. The figures roughly indicate the accuracy of (4.4) and (4.7).

Table 1

	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
Integrator (4.4)	0.0371	0.0760	0.0683	0.0594	0.0728
Integrator (4.7)	0.0006	0.0016	0.0023	0.0028	0.0042

5. Conclusions

In this paper, we obtain a general formula of composition method for ODEs. Based on the composition method for ODEs, a composition method for constructing high order multisymplectic integrators is presented. The high order multisymplectic integrators obtained by composition are very easy to implement, compared with the multisymplectic integrators obtained by Reich [9]. Numerical experiments are also reported.

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