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GLOBAL SUPERCONVERGENCE OF THE MIXED FINITE ELEMENT METHODS FOR 2-D MAXWELL EQUATIONS *1)

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Abstract

Superconvergence of the mixed finite element methods for 2-d Maxwell equations is studied in this paper. Two order of superconvergent factor can be obtained for the k-th Nedelec elements on the rectangular meshes.

Key words: Maxwell equations, Mixed finite element, Superconvergence, Postprocessing.

1. Introduction

Superconvergence of the mixed finite element methods for 3-d Maxwell equations was first considered by Monk [8]. In 1999, Lin and Yan [4] used the integral identity technique to study this problem once more and improved Monk's result. One order of superconvergent factor was obtained by them for k-th Nedelec elements on the cubic meshes. The similar result was proved for 2-d Maxwell equations by Lin and Yan [5] and Brandts [1]. In this paper, we improve the Brandts' result. If the domain is rectangular, two order of superconvergent factor which is one order higher than Brandts' result can be obtained for the k-th ($k \geq 1$) Nedelec elements on the rectangular meshes.

The paper is organized as follows: In section 2, the mixed finite element formulation for solving 2-d Maxwell equations is introduced. In section 3, we will consider the k-th ($k \geq 1$) Nedelec elements on the rectangular meshes and prove some basic estimates. In section 4, the mixed elliptic projection operator is defined and the error between the interpolation operator and the projection operator is estimated by utilizing the method introduced in [1]. In section 5, we obtain the superclose result. In the last section, the global superconvergence is obtained by the postprocessing.

2. Formulation

Consider the following two-dimension Maxwell equations

$$\mathbf{E}_t - \mathbf{rot} H = -\mathbf{J} \qquad \text{in } \Omega \times (0,T), \tag{1}$$

$$H_t + \mathrm{curl}\mathbf{E} = 0 \qquad \text{in } \Omega \times (0,T), \tag{2}$$

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{3}$$

$$\mathbf{E}(0) = \mathbf{E}_0, \quad H(0) = H_0, \tag{4}$$

where $\mathbf{E} = (E_1, E_2)$, $\mathbf{rot} H = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})$, $\mathrm{curl}\mathbf{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}$, $\mathbf{n}\times\mathbf{E} = E_2 n_1 - E_1 n_2$, $\mathbf{n} = (n_1, n_2)$ is the unit outward norm of $\partial\Omega$, $\Omega \subset \mathbf{R}^2$ is a bounded domain. In the following, we will use

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the notations

$$\|\cdot\|_0,\ \|\cdot\|_{0,e}\ \text{ for }\ L^2(\Omega),\ L^2(e)-\text{norm},$$

and

$$\|\cdot\|_k,\ \|\cdot\|_{k,e}\ \text{ for }\ H^k(\Omega),\ H^k(e)-\text{norm}.$$

Let

$$\mathbf{H}_0(\text{curl};\Omega)=\{\mathbf{v}=(v_1,v_2)\in(L^2(\Omega))^2;\text{curl}\mathbf{v}\in L^2(\Omega),\mathbf{n}\times\mathbf{v}\mid_{\partial\Omega}=0\},$$

with norm

$$\|\mathbf{v}\|_{\mathbf{H}(\text{curl};\Omega)}=\{\|\mathbf{v}\|^2_{0,\Omega}+\|\text{curl}\mathbf{v}\|^2_{0,\Omega}\}^{1/2}.$$

The variational formulation based on (1)-(4) reads as: find $(\mathbf{E},H)\in\mathbf{H}_0(\text{curl};\Omega)\times L^2(\Omega)$ such that

$$(\mathbf{E}_t,\mathbf{\Phi})-(H,\text{curl}\mathbf{\Phi})=-(\mathbf{J},\mathbf{\Phi})\quad\forall\mathbf{\Phi}\in\mathbf{H}_0(\text{curl};\Omega),\tag{5}$$

$$(H_t,\psi)+(\text{curl}\mathbf{E},\psi)=0\qquad\forall\psi\in L^2(\Omega),\tag{6}$$

$$\mathbf{E}(0)=\mathbf{E}_0,\ \ H(0)=H_0.\tag{7}$$

Let T_h be a regular partition of Ω and $\mathbf{V}_h\times W_h\subset\mathbf{H}_0(\text{curl};\Omega)\times L^2(\Omega)$ be the finite element space. Then the finite element approximation based on (5)-(7) reads as follows ([7]): find $(\mathbf{E}_h,H_h)\in\mathbf{V}_h\times W_h$ such that

$$((\mathbf{E}_h)_t,\mathbf{\Phi})-(H_h,\text{curl}\mathbf{\Phi})=-(\mathbf{J},\mathbf{\Phi})\quad\forall\mathbf{\Phi}\in\mathbf{V}_h,\tag{8}$$

$$((H_h)_t,\psi)+(\text{curl}\mathbf{E}_h,\psi)=0\qquad\forall\psi\in W_h,\tag{9}$$

$$\mathbf{E_h}(0)=R_h\mathbf{E}_0,\ \ H(0)=R_hH_0,\tag{10}$$

where R_h is the mixed elliptic projection which is given by (22)-(25). Since (8)-(10) is an ordinary differential equations with respect to time t, there exists a unique solution. In this paper, we will consider the k-th ($k\ge 1$) Nedelec finite element spaces [9].

3. Nedelec Finite Element Spaces

Let Ω be a polygon with boundaries parallel to the axes. $T_h=\{e\}$ is a rectangulation of Ω, where

$$e=[x_e-h_e,x_e+h_e]\times[y_e-k_e,y_e+k_e]$$

and $h=\max_e\{h_e,k_e\}$. T_h is called regular if

$$C_0h^2\le\text{meas}(e)\le C_1h^2\quad\forall e\in T_h.$$

The Nedelec finite element spaces come from Raviart-Thomas finite element spaces [10]. We first list some properties on these two finite element spaces.

Raviart-Thomas finite element spaces $(\mathbf{V}_1)_h\times W_h$ is defined by

$$(\mathbf{V}_1)_h=\{\mathbf{v}=(v_1,v_2)\in\mathbf{H}_0(\text{div};\Omega);\mathbf{v}\mid_e\in Q_{k+1,k}(e)\times Q_{k,k+1}(e),e\in T_h\},\tag{11}$$

$$W_h=\{w\in L^2(\Omega);w\mid_e\in Q_k(e),e\in T_h\},\tag{12}$$

where

$$\mathbf{H}_0(\text{div};\Omega)=\left\{\mathbf{v}\in L^2(\Omega)^2;\text{div}\mathbf{v}=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}\in L^2(\Omega),\mathbf{v}\cdot\mathbf{n}\mid_{\partial\Omega}=0\right\}$$

with norm

$$\|\mathbf{v}\|_{\mathbf{H}(\text{div};\Omega)}=\{\|\mathbf{v}\|_{0,\Omega}+\|\text{div}\|^2_{0,\Omega}\}^{1/2},$$

$Q_{i,j}(e) = span\{x^t y^s; 0 \le t \le i, 0 \le s \le j\}$ and $Q_k = Q_{k,k}$. For $\mathbf{v} \in (\mathbf{V}_1)_h$, by the definition, we know that the normal component $\mathbf{v} \cdot \mathbf{n}$ is continuous on the shared-edge of the two adjacent elements.

Nedelec finite element spaces $\mathbf{V}_h \times W_h$ is defined by

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} = (v_1, v_2); (v_2, -v_1) \in (\mathbf{V}_1)_h\} \\ &= \{\mathbf{v} \in \mathbf{H}_0(\mathrm{curl};\Omega); \mathbf{v}\mid_e \in Q_{k,k+1}(e) \times Q_{k+1,k}(e), e \in T_h\}, \\ W_h &= \{w \in L^2(\Omega); w\mid_e \in Q_k(e), e \in T_h\}. \end{aligned} \tag{13}$$

For $\mathbf{v} \in \mathbf{V}_h$, by the definition, we know that the tangential component $\mathbf{v} \times \mathbf{n}$ is continuous on the shared-edge of the two adjacent elements.

For the Raviart-Thomas elements, two of the most important properties are [10]

$$\|\mathbf{v}\|_{L^2(\Omega)} = \|\mathbf{v}\|_{\mathbf{H}(\mathrm{div};\Omega)} \quad \forall \mathbf{v} \in \mathbf{K}_1, \tag{14}$$

where $\mathbf{K}_1 = \{\mathbf{v} \in (\mathbf{V}_1)_h; (\mathrm{div}\mathbf{v}, w) = 0, \forall w \in W_h\}$.

And

$$\inf_{0 \ne w \in W_h} \sup_{0 \ne \mathbf{v} \in (\mathbf{V}_1)_h} \frac{(\mathrm{div}\mathbf{v}, w)}{\|\mathbf{v}\|_{\mathbf{H}(\mathrm{div};\Omega)}} \ge C_0 > 0, \tag{15}$$

where C_0 is a constant independent of h.

Accordingly, for the Nedelec elements, we also have

$$\|\mathbf{v}\|_{L^2(\Omega)} = \|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl};\Omega)} \quad \forall \mathbf{v} \in \mathbf{K}, \tag{16}$$

where $\mathbf{K} = \{\mathbf{v} \in \mathbf{V}_h; (\mathrm{curl}\mathbf{v}, w) = 0, \forall w \in W_h\}$.

And

$$\inf_{0 \ne w \in W_h} \sup_{0 \ne \mathbf{v} \in \mathbf{V}_h} \frac{(\mathrm{curl}\mathbf{v}, w)}{\|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl};\Omega)}} \ge C_0 > 0, \tag{17}$$

The equations (16) and (17) ensure the existence and uniqueness of the mixed elliptic projection operator R_h which we will define in §4.

Now, we define the interpolation operator

$$P_h : \mathbf{H}_0(\mathrm{curl};\Omega) \times L^2(\Omega) \mapsto \mathbf{V}_h \times W_h.$$

For $(\mathbf{v}, w) \in \mathbf{H}_0(\mathrm{curl};\Omega) \times L^2(\Omega)$, the interpolation function $(P_h\mathbf{v}, P_h w)$ is defined by

$$\begin{aligned} &\textstyle\int_{l_i} (\mathbf{v} - P_h\mathbf{v}) \times \mathbf{n} q ds = 0 \quad \forall q \in P_k(l_i), i = 1,2,3,4, \\ &\textstyle\int_e (\mathbf{v} - P_h\mathbf{v}) \cdot \mathbf{\Phi} dxdy = 0 \quad \forall \mathbf{\Phi} \in Q_{k,k-1}(e) \times Q_{k-1,k}(e), \\ &\textstyle\int_e (w - P_h w) q dxdy = 0 \quad \forall q \in W_h, \end{aligned}$$

where l_i $(i = 1,2,3,4)$ are the four edges of e, $P_k(l_i)$ is the set of the polynomials of degree$\le k$ on l_i. For such a special interpolation, we have the following lemmas.

Lemma 1. *Assume that* $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl};\Omega)$. *Then we have*

$$(\mathrm{curl}(\mathbf{u} - P_h\mathbf{u}), q) = 0 \quad \forall q \in W_h.$$

Proof. Note that $q\mid_e \in Q_k$, $\mathbf{rot}q\mid_e \in Q_{k,k-1}(e) \times Q_{k-1,k}(e)$. By Green formulation and the definition of interpolation operator, we get

$$\int_e \mathrm{curl}(\mathbf{u} - P_h\mathbf{u}) q dxdy = \int_e (\mathbf{u} - P_h\mathbf{u}) \cdot \mathbf{rot} q dxdy - \int_{\partial e} (\mathbf{u} - P_h\mathbf{u}) \times \mathbf{n} q ds = 0.$$

Lemma 2. *Assume that* $w \in L^2(\Omega)$. *Then we have*

$$(w - P_h w, \mathrm{curl}\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

Proof. Note that curl$\mathbf{v} \in W_h$. The lemma follows from the definition of interpolation operator.

Lemma 3. *Assume that* $\mathbf{u} \in (H^{k+3}(\Omega))^2 \cap \mathbf{H}_0(\text{curl};\Omega)$ $(k \geq 1)$. *Then we have*

$$(\mathbf{u} - P_h\mathbf{u}, \mathbf{v}) = O(h^{k+3})\|\mathbf{u}\|_{k+3}\|\mathbf{v}\|_{\mathbf{H}(\text{curl};\Omega)} \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

Especially, if curl$\mathbf{v} = 0$, *then we have*

$$(\mathbf{u} - P_h\mathbf{u}, \mathbf{v}) = O(h^{k+3})\|\mathbf{u}\|_{k+3}\|\mathbf{v}\|_0.$$

Proof. Let $\mathbf{u} = (u_1, u_2)$, $P_h\mathbf{u} = (P_hu_1, P_hu_2)$, $\mathbf{v} = (v_1, v_2)$. We first prove

$$\int_e (u_1 - P_hu_1)v_1 dxdy = -\frac{k_e^{2k+2}}{(2k+3)!!(2k+1)!!}\int_e \frac{\partial^{k+2}u_1}{\partial y^{k+2}}\frac{\partial^k v_1}{\partial y^k}dxdy + O(h^{k+3})\|\mathbf{u}\|_{k+3,e}\|\mathbf{v}\|_{0,e} \quad \forall \mathbf{v} \in \mathbf{V}_h. \tag{18}$$

Let $\hat{e} = [-1,1] \times [-1,1]$ be the reference element and $F : e \mapsto \hat{e}$ be a map defined by

$$F : (x,y) \mapsto (\hat{x}, \hat{y}), \hat{x} = \frac{x - x_e}{h_e}, \hat{y} = \frac{y - y_e}{k_e}.$$

Let

$$\hat{u}_1(\hat{x},\hat{y}) = u_1(x,y), \hat{v}_1(\hat{x},\hat{y}) = v_1(x,y), \widehat{P_hu_1}(\hat{x},\hat{y}) = P_hu_1(x,y).$$

Consider the bilinear functional over $\hat{e}$

$$B(\hat{u}_1, \hat{v}_1) = \int_{\hat{e}} (\hat{u}_1 - \widehat{P_hu_1})\hat{v}_1 d\hat{x}d\hat{y} + \frac{1}{(2k+3)!!(2k+1)!!}\int_{\hat{e}} \frac{\partial^{k+2}\hat{u}_1}{\partial \hat{y}^{k+2}}\frac{\partial^k \hat{v}_1}{\partial \hat{y}^k}d\hat{x}d\hat{y}.$$

By the inverse inequality [3], we have

$$|B(\hat{u}_1, \hat{v}_1)| \leq C\|\hat{u}_1\|_{k+3,\hat{e}}|\hat{v}_1|_{0,\hat{e}} \quad \forall \hat{v}_1 \in Q_{k,k+1}(\hat{e}). \tag{19}$$

If $\hat{u}_1$ takes

$$\hat{x}^{k+1}, \hat{x}^{k+2}, \hat{x}^{k+1}\hat{y}, \hat{y}^{k+2}$$

respectively, then the corresponding $\hat{u}_1 - \widehat{P_hu_1}$ will be

$$\frac{(k+1)!}{(2k+2)!}\frac{d^{k+1}(\hat{x}^2-1)^{k+1}}{d\hat{x}^{k+1}}, \qquad \frac{(k+2)!}{(2k+4)!}\frac{d^{k+2}(\hat{x}^2-1)^{k+2}}{d\hat{x}^{k+2}},$$
$$\frac{(k+1)!}{(2k+2)!}\frac{d^{k+1}(\hat{x}^2-1)^{k+1}}{d\hat{x}^{k+1}} \cdot \hat{y}, \qquad \frac{(k+2)!}{(2k+2)!}\frac{d^{k}(\hat{y}^2-1)^{k+1}}{d\hat{y}^{k}}.$$

This can be demonstrated as follows. As $\hat{u}_1 = \hat{x}^{k+1}$, let $p(\hat{x}) = \frac{(k+1)!}{(2k+2)!}\frac{d^{k+1}(\hat{x}^2-1)^{k+1}}{d\hat{x}^{k+1}}$. Since $p(\hat{x})$ is the $(k+1)$-th order Legendre polynomial. $p(\hat{x})$ is orthogonalized to the polynomials of degree$\leq k$ in $L^2(-1,+1)$. Furthermore, the first term of $p(\hat{x})$ is $\hat{x}^{k+1}$, so $p(\hat{x}) - \hat{x}^{k+1}$ belongs to $Q_{k,k+1}(\hat{e})$. Thus, by the definition of interpolation operator, we conclude that $p(\hat{x}) = \hat{u}_1 - \widehat{P_hu_1}$. For other three cases, we can use the same reason to justify. Note that

$$\int_{-1}^{1}\left(\frac{d^n(x^2-1)^n}{dx^n}\right)^2 dx = \frac{2^{2n+1}}{(2n+1)}(n!)^2.$$

As $\hat{u}_1$ taking

$$\hat{x}^{k+1}, \hat{x}^{k+2}, \hat{x}^{k+1}\hat{y}, \hat{y}^{k+2},$$

respectively, a direct calculation shows

$$B(\hat{u}_1, \hat{v}_1) = 0 \quad \forall \hat{v}_1 \in Q_{k,k+1}(\hat{e}).$$

Now Bramble-Hilbert Lemma [3] and (19) gives

$$|B(\hat{u}_1,\hat{v}_1)| \le C|\hat{u}_1|_{k+3,\hat{e}}|\hat{v}_1|_{0,\hat{e}} \quad \forall \hat{v}_1 \in Q_{k,k+1}(\hat{e}). \tag{20}$$

The inverse map of F

$$F^{-1}: \hat{e} \mapsto e, x = x_e + h_e\hat{x}, y = y_e + k_e\hat{y}$$

yields (18).

Note that $\frac{\partial^k v_1}{\partial y^k}$ can be rewritten as

$$\frac{\partial^k v_1}{\partial y^k} = -\frac{\partial^{k-1}(\mathrm{curl}\mathbf{v})}{\partial y^{k-1}} + \frac{\partial^k v_2}{\partial x \partial y^{k-1}}.$$

The inverse inequality and integration by parts lead to

$$\begin{aligned}
&\sum_e k_e^{2k+2}\int_e \frac{\partial^{k+2}u_1}{\partial y^{k+2}}\frac{\partial^k v_1}{\partial y^k}dxdy \\
=\ &\sum_e k_e^{2k+2}\int_e \frac{\partial^{k+2}u_1}{\partial y^{k+2}}\left(-\frac{\partial^{k-1}(\mathrm{curl}\mathbf{v})}{\partial y^{k-1}} + \frac{\partial^k v_2}{\partial x\partial y^{k-1}}\right)dxdy \\
=\ &\sum_e O(h^{k+3})|u_1|_{k+2,e}|\mathrm{curl}\mathbf{v}|_{0,e} \\
&+\sum_e k_e^{2k+2}\left\{-\int_e \frac{\partial^{k+3}u_1}{\partial x\partial y^{k+2}}\frac{\partial^{k-1}v_2}{\partial y^{k-1}}dxdy + (\int_{l_{e,1}} - \int_{l_{e,2}})\frac{\partial^{k+2}u_1}{\partial y^{k+2}}\frac{\partial^{k-1}v_2}{\partial y^{k-1}}dy\right\},
\end{aligned} \tag{21}$$

where $l_{e,1}$ and $l_{e,2}$ are the right and left edges of e. Recall that $\mathbf{v}\times\mathbf{n}$ is continuous on the shared-edge of the two adjacent elements. Hence, v_2 is continuous on the $l_{e,1}$ and $l_{e,2}$, so is the $\frac{\partial^{k-1}v_2}{\partial y^{k-1}}$. Furthermore, since $\mathbf{v}\times\mathbf{n}\ |_{\partial\Omega}= 0$, so $v_2 = 0$ on $l_{e,1}$ or $l_{e,2}$ if $l_{e,1}\cap\partial\Omega = l_{e,1}$ or $l_{e,2}\cap\partial\Omega = l_{e,2}$. Thus the line integrals in (21) disappear and the inverse estimates lead to

$$\sum_e k_e^{2k+2}\int_e \frac{\partial^{k+2}u_1}{\partial y^{k+2}}\frac{\partial^k v_1}{\partial y^k}dxdy = O(h^{k+3})\|\mathbf{u}\|_{k+3}\|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl};\Omega)} \quad \forall \mathbf{v}\in\mathbf{V}_h$$

By (18), we have

$$(u_1 - P_h u_1, v_1) = O(h^{k+3})\|\mathbf{u}\|_{k+3}\|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl};\Omega)} \quad \forall\mathbf{v}\in\mathbf{V}_h.$$

In like manner, we can get the error estimate for $(u_2 - P_h u_2, v_2)$. Combining these two results, we complete the proof of Lemma 3.

4. The Mixed Elliptic Projection

In this section, we will define the mixed elliptic projection operator [1]. Meanwhile, we will estimate the error between interpolation and the mixed elliptic projection operator. Now we construct the mixed elliptic projection operator

$$R_h: \mathbf{H}_0(\mathrm{curl};\Omega)\times L^2(\Omega)\mapsto \mathbf{V}_h\times W_h.$$

Let $(\mathbf{X}, Y)\in\mathbf{H}_0(\mathrm{curl};\Omega)\times L^2(\Omega)$ be the solution of the following equations

$$(\mathbf{X},\mathbf{\Phi}) - (Y,\mathrm{curl}\mathbf{\Phi}) = (\mathbf{f},\mathbf{\Phi}) \quad \forall\mathbf{\Phi}\in\mathbf{H}_0(\mathrm{curl};\Omega), \tag{22}$$

$$(\mathrm{curl}\mathbf{X},\psi) = (g,\psi) \qquad \forall\psi\in L^2(\Omega). \tag{23}$$

where $\mathbf{f}\in(L^2(\Omega))^2$, $g\in L^2(\Omega)$. The mixed elliptic projection function $(R_h\mathbf{X}, R_hY)$ is defined by

$$(R_h\mathbf{X},\mathbf{\Phi}) - (R_hY,\mathrm{curl}\mathbf{\Phi}) = (\mathbf{f},\mathbf{\Phi}) \quad \forall\mathbf{\Phi}\in\mathbf{V}_h, \tag{24}$$

$$(\mathrm{curl}R_h\mathbf{X},\psi) = (g,\psi) \qquad \forall\psi\in W_h. \tag{25}$$

By the theory of mixed finite element methods [2] and (16)-(17), we know that (24)-(25) exists a unique solution. Thus the mixed elliptic projection operator R_h is well defined.

Assume that $\mathbf{X}$, Y, $\mathbf{f}$ and g are all the function of time t. Differentiating the four equations (22)-(25) with respect to time t, we will find that $(R_h\frac{\partial \mathbf{X}(t)}{\partial t}, R_h\frac{\partial Y(t)}{\partial t})$ and $(\frac{\partial}{\partial t}R_h\mathbf{X}(t), \frac{\partial}{\partial t}R_hY(t))$ satisfy the same equations. By the uniqueness of the solution, we have

$$\frac{\partial}{\partial t}R_h\mathbf{X}(t) = R_h\frac{\partial \mathbf{X}(t)}{\partial t}, \frac{\partial}{\partial t}R_hY(t) = R_h\frac{\partial Y(t)}{\partial t}. \tag{26}$$

Now, we estimate the error between the interpolation and the mixed elliptic projection operator.

Lemma 4. *Assume that* $(\mathbf{X}, Y) \in \mathbf{H}_0(\text{curl};\Omega) \cap (H^{k+3}(\Omega))^2 \times L^2(\Omega) \cap H^{k+2}(\Omega)$ $(k \geq 1)$ *and satisfies (22) and (23), P_h and R_h are the interpolation and the mixed elliptic projection operator. Then we have*

$$\|R_h\mathbf{X} - P_h\mathbf{X}\|_0 \leq Ch^{k+3}\|\mathbf{X}\|_{k+3}, \tag{27}$$

$$\|R_hY - P_hY\|_0 \leq Ch^{k+3}\|\mathbf{X}\|_{k+3}. \tag{28}$$

Proof. By (23) and (25), we have

$$(\text{curl}(\mathbf{X} - R_h\mathbf{X}), \psi) = 0 \quad \forall \psi \in W_h.$$

Lemma 1 gives

$$(\text{curl}(\mathbf{X} - P_h\mathbf{X}), \psi) = 0 \quad \forall \psi \in W_h.$$

Above two equations yield

$$(\text{curl}(P_h\mathbf{X} - R_h\mathbf{X}), \psi) = 0 \quad \forall \psi \in W_h.$$

Especially, taking $\psi = \text{curl}(P_h\mathbf{X} - R_h\mathbf{X})$, we get

$$\text{curl}(P_h\mathbf{X} - R_h\mathbf{X}) = 0. \tag{29}$$

By (22), (24) and Lemma 2, we deduce that

$$\begin{aligned}(R_h\mathbf{X} - P_h\mathbf{X}, \mathbf{\Phi}) &= (\mathbf{X} - P_h\mathbf{X}, \mathbf{\Phi}) - (Y - R_hY, \text{curl}\mathbf{\Phi}) \\ &= (\mathbf{X} - P_h\mathbf{X}, \mathbf{\Phi}) - (P_hY - R_hY, \text{curl}\mathbf{\Phi}) \quad \forall \mathbf{\Phi} \in \mathbf{V}_h.\end{aligned} \tag{30}$$

Taking $\mathbf{\Phi} = R_h\mathbf{X} - P_h\mathbf{X}$, and applying (29) and Lemma 3, we obtain

$$\|R_h\mathbf{X} - P_h\mathbf{X}\|_0^2 \leq Ch^{k+3}\|\mathbf{X}\|_{k+3}\|R_h\mathbf{X} - P_h\mathbf{X}\|_0,$$

so, (27) holds.

By B-B condition (17), we have

$$\|R_hY - P_hY\|_0 \leq C \sup_{0 \neq \mathbf{\Phi} \in \mathbf{V}_h} \frac{(\text{curl}\mathbf{\Phi}, R_hY - P_hY)}{\|\mathbf{\Phi}\|_{\mathbf{H}(\text{curl};\Omega)}}. \tag{31}$$

By (30), Lemma 3 and (27), we get

$$\begin{aligned}(\text{curl}\Phi, R_hY - P_hY) &= (\mathbf{X} - P_h\mathbf{X}, \mathbf{\Phi}) - (R_h\mathbf{X} - P_h\mathbf{X}, \mathbf{\Phi}) \\ &\leq Ch^{k+3}\|\mathbf{X}\|_{k+3}\|\mathbf{\Phi}\|_{\mathbf{H}(\text{curl};\Omega)} \quad \forall \mathbf{\Phi} \in \mathbf{V}_h.\end{aligned}$$

Combining (31), we obtain (28) and complete the proof of the Lemma.

Differentiating both sides of (6) with respect to time t, we have

$$(\text{curl}\mathbf{E}_t, \psi) = -(H_{tt}, \psi) \quad \forall \psi \in L^2(\Omega). \tag{32}$$

By (5), (32) and the definition of the mixed elliptic projection operator, we know that $(R_h \frac{\partial \mathbf{E}}{\partial t}, R_h H)$ satisfies

$$(R_h \frac{\partial \mathbf{E}}{\partial t}, \mathbf{\Phi}) - (R_h H, \text{curl}\mathbf{\Phi}) = -(\mathbf{J}, \mathbf{\Phi}) \quad \forall \mathbf{\Phi} \in \mathbf{V}_h, \tag{33}$$

$$(\text{curl} R_h \frac{\partial \mathbf{E}}{\partial t}, \psi) = -(H_{tt}, \psi) \quad \forall \psi \in W_h. \tag{34}$$

Similarly, integrating both sides of (5) from 0 to t, we obtain

$$(\mathbf{E}, \mathbf{\Phi}) - (\int_0^t H(t)dt, \text{curl}\mathbf{\Phi}) = (\mathbf{E}_0 - \int_0^t \mathbf{J}(t)dt, \mathbf{\Phi}) \quad \forall \mathbf{\Phi} \in \mathbf{H}_0(\text{curl}; \Omega). \tag{35}$$

By (35), (6) and the definition of the mixed elliptic projection operator, we know that $(R_h \mathbf{E}, R_h \int_0^t H(t)dt) \in \mathbf{V}_h \times W_h$ satisfies

$$(R_h \mathbf{E}, \mathbf{\Phi}) - (R_h \textstyle\int_0^t H(t)dt, \text{curl}\mathbf{\Phi}) = (\mathbf{E}_0 - \textstyle\int_0^t \mathbf{J}(t)dt, \mathbf{\Phi}) \quad \forall \mathbf{\Phi} \in \mathbf{V}_h, \tag{36}$$

$$(\text{curl} R_h \mathbf{E}, \psi) = -(H_t, \psi) \quad \forall \psi \in W_h. \tag{37}$$

By (5)-(6), (32)-(37) and Lemma 4, we can easily derive the following corollary.

Corollary 1. *Assume that* $(\mathbf{E}, H) \in \mathbf{H}_0(\text{curl}; \Omega) \times L^2(\Omega)$ *is the solution of (5)-(7), and* $\mathbf{E}$, $\mathbf{E}_t$, $\mathbf{E}_{tt}$, $\mathbf{E}_{ttt} \in (H^{k+3}(\Omega))^2$ $(k \geq 1)$. *Then we have*

$$\|P_h H - R_h H\|_0 \leq C h^{k+3} \|\mathbf{E}_t\|_{k+3}, \tag{38}$$

$$\|P_h \mathbf{E} - R_h \mathbf{E}\|_0 \leq C h^{k+3} \|\mathbf{E}\|_{k+3}, \tag{39}$$

$$\left\| P_h \frac{\partial H}{\partial t} - R_h \frac{\partial H}{\partial t} \right\|_0 \leq C h^{k+3} \|\mathbf{E}_{tt}\|_{k+3}, \tag{40}$$

$$\left\| P_h \frac{\partial^2 H}{\partial t^2} - R_h \frac{\partial^2 H}{\partial t^2} \right\|_0 \leq C h^{k+3} \|\mathbf{E}_{ttt}\|_{k+3}. \tag{41}$$

5. The Superclose Estimate

In this section, we will estimate the difference between $(\mathbf{E}_h, H_h)$ and $(P_h \mathbf{E}, P_h H)$. Suppose that $(\mathbf{E}_h, H_h) \in \mathbf{V}_h \times W_h$ is the solution of (8)-(10), and R_h is the mixed elliptic projection operator. Then we have

Lemma 5.

$$\left(\frac{\partial \mathbf{E}_h}{\partial t} - R_h \frac{\partial \mathbf{E}}{\partial t} \right)(0) = 0, \quad \left(\frac{\partial}{\partial t} H_h - \frac{\partial}{\partial t} R_h H \right)(0) = (P_h - R_h) \frac{\partial H}{\partial t}(0).$$

Proof. Subtracting (33) from (8), we get

$$(\frac{\partial \mathbf{E}_h}{\partial t} - R_h \frac{\partial \mathbf{E}}{\partial t}, \mathbf{\Phi}) - (H_h - R_h H, \text{curl}\mathbf{\Phi}) = 0 \quad \forall \mathbf{\Phi} \in \mathbf{V}_h.$$

Note that $H_h(0) = R_h H(0)$. Taking $t = 0$ and $\mathbf{\Phi} = (\frac{\partial \mathbf{E}_h}{\partial t} - R_h \frac{\partial \mathbf{E}}{\partial t})(0)$, we obtain the first equation of Lemma 5.

By (9), (37) and the definition of interpolation operator, we have

$$(\text{curl}(\mathbf{E}_h - R_h \mathbf{E}), \psi) = (H_t - (H_h)_t, \psi) = (P_h H_t - (H_h)_t, \psi) \quad \forall \psi \in W_h.$$

Note that $\mathbf{E}_h(0) = R_h \mathbf{E}(0)$. Taking $t = 0$ and $\psi = (P_h H_t - (H_h)_t)(0)$, we obtain

$$\frac{\partial H_h}{\partial t}(0) = P_h \frac{\partial H}{\partial t}(0).$$

Combining with (26), we get the second equation of the lemma.

Now we can state the superclose result.

Theorem 1. *Assume that* $(\mathbf{E}, H) \in \mathbf{H}_0(\text{curl};\Omega) \times L^2(\Omega)$, $(\mathbf{E}_h, H_h) \in \mathbf{V}_h \times W_h$ *are the solution of (5)-(7) and (8)-(10) respectively. Furthermore,* $\mathbf{E}$, $\mathbf{E}_t$, $\mathbf{E}_{tt}(0) \in (H^{k+3}(\Omega))^2$, $\mathbf{E}_{ttt} \in L^2(0,T;(H^{k+1}(\Omega))^2)$ $(k \geq 1)$ *and* R_h, P_h *are the mixed elliptic projection and interpolation operator respectively. Then we have*

$$\|\mathbf{E}_h(t) - R_h\mathbf{E}(t)\|_0 \leq Ch^{k+3}Q^{1/2}, \tag{42}$$

$$\|H_h(t) - R_hH_h(t)\|_0 \leq Ch^{k+3}Q^{1/2}, \tag{43}$$

$$\|\mathbf{E}_h(t) - P_h\mathbf{E}(t)\|_0 \leq Ch^{k+3}\left\{Q + \|\mathbf{E}(t)\|_{k+3}^2\right\}^{1/2}, \tag{44}$$

$$\|H_h(t) - P_hH(t)\|_0 \leq Ch^{k+3}\left\{Q + \left\|\frac{\partial \mathbf{E}}{\partial t}\right\|_{k+3}^2\right\}^{1/2}, \tag{45}$$

where

$$Q = \int_0^t \left\|\frac{\partial^3\mathbf{E}}{\partial t^3}\right\|_{k+3}^2 dt + \left\|\frac{\partial^2\mathbf{E}}{\partial t^2}(0)\right\|_{k+3}^2.$$

Proof. Differentiating both sides of equations (8) and (33) with respect to time t, we obtain

$$\left(\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}_h}{\partial t} - R_h\frac{\partial \mathbf{E}}{\partial t}\right), \mathbf{\Phi}\right) - \left(\frac{\partial H_h}{\partial t} - \frac{\partial R_hH}{\partial t}, \text{curl}\mathbf{\Phi}\right) = 0 \quad \forall \mathbf{\Phi} \in \mathbf{V}_h. \tag{46}$$

Differentiating both sides of (9) with respect to time t and subtracting the result equation from (34), by the definition of interpolation operator, we get

$$\begin{aligned}
&\left(\text{curl}\left(\frac{\partial \mathbf{E}_h}{\partial t} - R_h\frac{\partial \mathbf{E}}{\partial t}\right), \psi\right) + \left(\frac{\partial^2 H_h}{\partial t^2} - \frac{\partial^2 R_hH}{\partial t^2}, \psi\right) \\
= &\left(\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 R_hH}{\partial t^2}, \psi\right) = \left(P_h\frac{\partial^2 H}{\partial t^2} - R_h\frac{\partial^2 H}{\partial t^2}, \psi\right) \quad \forall \psi \in W_h.
\end{aligned} \tag{47}$$

Taking $\mathbf{\Phi} = (\mathbf{E}_h)_t - R_h\mathbf{E}_t$ and $\psi = (H_h)_t - (R_hH)_t$ in (46) and (47) and applying Corollary 1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{\Phi}\|_0^2 + \|\psi\|_0^2\right) = \left(P_h\frac{\partial^2 H}{\partial t^2} - R_h\frac{\partial^2 H}{\partial t^2}, \psi\right) \\
\leq\ &Ch^{k+3}\left\|\frac{\partial^3\mathbf{E}}{\partial t^3}\right\|_{k+3}\|\psi\|_0 \leq Ch^{2k+6}\left\|\frac{\partial^3\mathbf{E}}{\partial t^3}\right\|_{k+3}^2 + C\|\psi\|_0^2.
\end{aligned} \tag{48}$$

Applying the Gronwall inequality , Lemma 5 and Corollary 1, we have

$$\begin{aligned}
&\left\|\frac{\partial \mathbf{E}_h}{\partial t} - R_h\frac{\partial \mathbf{E}}{\partial t}\right\|_0 + \left\|\frac{\partial H_h}{\partial t} - \frac{\partial R_hH}{\partial t}\right\|_0 \\
\leq\ &Ch^{k+3}\left\{\int_0^t \left\|\frac{\partial^3\mathbf{E}}{\partial t^3}\right\|_{k+3}^2 dt + \left\|\frac{\partial^2\mathbf{E}}{\partial t^2}(0)\right\|_{k+3}^2\right\}^{1/2}.
\end{aligned} \tag{49}$$

Observe that

$$\frac{d}{dt}\|f\|_0^2 = \frac{d}{dt}(f,f) = 2(f_t, f) \leq 2\|f_t\|_0\|f\|_0,$$

so $\frac{d}{dt}\|f\|_0 \leq \|\frac{\partial}{\partial t}f\|_0$. Hence

$$\frac{d}{dt}(\|\mathbf{E}_h - R_h\mathbf{E}\|_0 + \|H_h - R_hH\|_0) \leq \left\|\frac{\partial}{\partial t}(\mathbf{E}_h - R_h\mathbf{E})\right\|_0 + \left\|\frac{\partial}{\partial t}(H_h - R_hH)\right\|_0.$$

Integrating the above equation from 0 to t, by (26) (49) and (10), we can obtain (42) and (43). The last two equations of the theorem follow from (38) and (39).

6. Global Superconvergence

Theorem 1 means that $(\mathbf{E}_h, H_h)$ is superclose to $(P_h\mathbf{E}, P_hH)$. The purpose, however, is to superclose $(\mathbf{E}, H)$. This can be done by constructing the postprocessing operator $(\mathbf{I}_{2h}, J_{2h})$.

Let us assume that T_h has been obtained from T_{2h} by dividing each element into four congruent rectangles. Let $\tau = \bigcup_{i=1}^4 e_i \in T_{2h}$ with $e_i \in T_h$, and $(\mathbf{I}_{2h}, J_{2h})$ a) is defined as: For $(\mathbf{v}, w) \in \mathbf{H}_0(\text{curl};\tau) \times L^2(\tau)$,

$$\begin{cases} \mathbf{I}_{2h}\mathbf{v}\Big|_\tau \in Q_{2m+1,2m+2} \times Q_{2m+2,2m+1}; \\ \int_{l_i}(\mathbf{I}_{2h}\mathbf{v} - \mathbf{v}) \times \mathbf{n} q dl = 0, \forall q \in P_m(l_i), i = 1, \cdots, 12; \\ \int_{e_i}(\mathbf{I}_{2h}\mathbf{v} - \mathbf{v}) \cdot \mathbf{\Phi} dxdy = 0, \forall \mathbf{\Phi} \in Q_{m,m-1} \times Q_{m-1,m}(e_i), i = 1, 2, 3, 4; \end{cases}$$

and

$$\begin{cases} J_{2h}w\Big|_\tau \in Q_{2m+1}; \\ \int_{e_i}(J_{2h}w - w) q dxdy = 0, \forall q \in Q_m(e_i), i = 1, 2, 3, 4; \end{cases}$$

where l_i $(i = 1, \cdots, 12)$ are all edges of four e_i $(i = 1, 2, 3, 4)$ and

$$m = \begin{cases} (k+2)/2, & \text{if } k \text{ is even}, \\ (k+1)/2, & \text{else}. \end{cases}$$

It can be proved that (see [6]) $(\mathbf{I}_{2h}, J_{2h})$ satisfying

$$\begin{cases} \|\mathbf{I}_{2h}\mathbf{v} - \mathbf{v}\|_0 \le Ch^{k+3}\|\mathbf{v}\|_{k+3}; \\ \|\mathbf{I}_{2h}\mathbf{v}\|_0 \le C\|\mathbf{v}\|_0, \quad \forall \mathbf{v} \in \mathbf{V}_h; \\ \mathbf{I}_{2h}\mathbf{v} = \mathbf{I}_{2h}P_h\mathbf{v}. \end{cases} \tag{50}$$

And

$$\begin{cases} \|J_{2h}w - w\|_0 \le Ch^{k+3}\|w\|_{k+3}; \\ \|J_{2h}w\|_0 \le C\|w\|_0, \quad \forall w \in W_h; \\ J_{2h}w = J_{2h}P_hw. \end{cases} \tag{51}$$

Now, the global superconvergence can be obtained by Theorem 1 and the postprocessing operator.

Theorem 2. *Under the assumption of Theorem 1, we have*

$$\|\mathbf{I}_{2h}\mathbf{E}_h(t) - \mathbf{E}(t)\|_0 \le Ch^{k+3}\left\{Q + \|\mathbf{E}(t)\|_{k+3}^2\right\}^{\frac{1}{2}}, \tag{52}$$

$$\|J_{2h}H_h(t) - H(t)\|_0 \le Ch^{k+3}\left\{Q + \left\|\frac{\partial \mathbf{E}}{\partial t}\right\|_{k+3}^2\right\}^{1/2}. \tag{53}$$

where Q is defined as in Theorem 1.

Proof. We only prove the first equation, the second one can be proved in a similar way. Note that

$$\mathbf{I}_{2h}\mathbf{E}_h - \mathbf{E} = \mathbf{I}_{2h}(\mathbf{E}_h - P_h\mathbf{E}) + (\mathbf{I}_{2h}P_h\mathbf{E} - \mathbf{E}). \tag{54}$$

By (50) and (35), we get

$$\|\mathbf{I}_{2h}(\mathbf{E}_h - P_h\mathbf{E})\|_0 \le C\|\mathbf{E}_h - P_h\mathbf{E}\|_0 \le Ch^{k+3}\left\{Q + \|\mathbf{E}(t)\|_{k+3}^2\right\}^{\frac{1}{2}}, \tag{55}$$

and

$$\|\mathbf{I}_{2h}P_h\mathbf{E} - \mathbf{E}\|_0 = \|\mathbf{I}_{2h}\mathbf{E} - \mathbf{E}\|_0 \le Ch^{k+3}\|\mathbf{E}\|_{k+3}. \tag{56}$$

Hence, by (54)–(56), we obtain Theorem 2.

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