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A COMPUTATIONAL APPROACH FOR OPTIMAL CONTROL SYSTEMS GOVERNED BY PARABOLIC VARIATIONAL INEQUALITIES *

N.H. Sweilam L.F. Abd-Elal
(*Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt*)

Abstract

Iterative techniques for solving optimal control systems governed by parabolic variational inequalities are presented. The techniques we use are based on linear finite elements method to approximate the state equations and nonlinear conjugate gradient methods to solve the discrete optimal control problem. Convergence results and numerical experiments are presented.

Key words: Parabolic variational inequalities, finite elements method, nonlinear conjugate gradient methods.

1. Introduction

In the theory of variational inequalities and their approximation by finite elements methods the dam problem models hold a particular place (see for example [3], [14], and references therein). Such models are of a great practical interest in the development and management of water resources. Knowledge of the amount of seepage is essential for water conservation practice. The main goal of the present paper is to study numerical approximations for the following optimal control problem (CP):

$$\begin{cases} \text{Find} \quad q_{min} \in U_{ad} \subset L^2(0,T) \quad \text{such that} \\ J_j(q_{min}) = \min_{\forall q \in U_{ad}} J_j(q), \quad j = 1,2,...,m, \end{cases} \tag{1.1}$$

subjected to

$$\dot{H}_{j-1}(t) + \tilde{\alpha} H_{j-1}(t) = q(t), \quad j = 1,2,...,m. \tag{1.2}$$

The cost functionals are defined by

$$J_j(q) := \frac{1}{2}\int_{Q_j} [w_j(q) - w_j^d]^2\, dx\, dt + \frac{N}{2}\int_0^T [q]^2\, dt, \quad j = 1,2,...,m, \tag{1.3}$$

where N is a nonnegative real. Let us denote by $q(t)$ the control variable, by U_{ad} a closed convex subset in $L^2(0,T)$ and by $w_j^d(x,t)$ a given functions in $L^2(Q_j)$. Also this problem subjected to the following parabolic system of variational inequalities (Problem (P)):

$$\begin{cases} w_j(t) \geq 0 \; a.e., w_j \in L^2(0,T;H^2(D_j)), \; \dfrac{\partial w_j}{\partial t} \in L^2(0,T;H^1(D_j)), \; (\text{then}\, w_j \in C^0(\bar{Q}_j)\,), \\ \nu\,(\dfrac{\partial w_j}{\partial t}, v_j - w_j(t))_{L^2(D_j)} + a(w_j(t), v_j - w_j(t)) \geq -(1, v_j - w_j(t))_{L^2(D_j)}, \\ \forall v_j \in L^2(D_j),\, a.e., \quad w_j(t_{j-1}) = \tilde{w}_{j-1}, \quad w_j(t) \in K_j(t), \quad v_j \geq 0 \quad j = 1,2,...m, \end{cases}$$

where

$$K_j(t) = \{v_j \in H^1(D_j) : v_j = G_j \text{ on } \Gamma_{dj},\, j = 1,2,...,m\},$$

G_j are any functions in $H^2(Q_j)$ such that the value of G_j on the boundary Γ_{n_j}, $\hat{g}_j$, is assumed to have a zero derivative and $G_j = g_j$ on $\Gamma_{d_j} \times]t_{j-1}, T[$ such that $g_j \geq 0,\; j = 1,2,...,m,$

$$\begin{cases} g_j(a_j,y,t_j) := \displaystyle\int_{t_{j-1}}^{t_j} [H_{j-1}(\tau) - (y + t_j - \tau)]^+ d\tau + \frac{1}{2}[(H_{j-1}(t_{j-1}) - y - t_j)^+]^2\,, \\ g_j(b_j,y,t_j) := \displaystyle\int_{t_{j-1}}^{t_j} [H_j(\tau) - (y + t_j - \tau)]^+ d\tau + \frac{1}{2}[(H_j(t_j) - y - t_j)^+]^2\,, \\ g_j(x,e_j,t_j) := 0, \quad m^+ := \dfrac{(|m| + m)}{2}, \end{cases}$$

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and $a(u,v) := (\bigtriangledown u, \bigtriangledown v)_{L^2(D_j)}$ is the bilinear, coercive form in $H^1(D_j)$.

Physically, the above model describes (see [27]) the optimal control evolution system of earth dams where we consider an unsteady flow, say water, moving through **m** homogeneous porous rectangular earth dams. The dams have the following domains $D_j := \{(x,y) \,|\, 0 < a_j < x < b_j,\ 0 < y < e_j\ ,\ j = 1,2,...,m\}$, respectively, with vertical walls $x = a_j,\ x = b_j,\ \ j = 1,2,...,m$. We suppose that the water levels $H_0(t), H_m(t)$ are given real numbers, $H_0(t) > H_m(t) > 0$ and $H_j(t)$ are the intermediate water levels between j^{th} and $(j+1)^{\text{th}}$ dams, $j = 1,2,...,m-1$ (see Figure 1), $w_j(q)$ are the weak solutions of the evolution dams problem when the initial data follows from the stationary dam problem (see [6],[10],[28]) and $\tilde{w}_{j-1}$ are the solutions of the stationary dams problem (see [3]). The constant $\nu \geq 0$ is called the retentivity cofficient, the case $\nu = 0$ is related to an incompressible fluid. We will denote by $t_j \in [0,T], 0 < T < +\infty$ the time intervals during which we want to study the filtration dams, $j = 1,2,...,m$, $t_0 = 0$. Let $Q_j = D_j \times (t_{j-1}, t_j),\ j = 1,2,...,m,$

$$\Gamma_{nj} = \{(x,y) : a_j < x < b_j, y = 0\}, \quad \Gamma_{d_j} := \Gamma_j - \Gamma_{nj}.$$

where Γ_j are the smooth boundary of D_j. Optimal control problems in connection with variational inequalities contain many difficulties, e.g., [4], [5], [12], [16], [17], [18] and [19] or more recently [1], [7] and the references therein. The control problem $(1.1)-(1.3)$ is in general a non-convex and non-differentiable optimization problem, see [18]. It has been proved in ([27]) that by controlling the amount of fluid that may go out of each dam the free boundary in each dam can be controlled. Also in ([27]) regularizing the problem necessary optimality conditions were exhibited and obtained convergence results when the regularization parameter tends to zero. One justification to use these methods for solving variational inequalities numerically is the fact that inequalities are replaced by equations (see for example [23] and [26]). For simplicity we will write w_j instead of $w_j(q)$. Also we write

$$g_j(q) := g_j(a_j, y, t_j)\,, \quad \forall y \in D_j,\, t_j \in [0,T]\,, \quad j = 1,2,...,m \tag{1.4}$$

which is continuous functions. This under the assumptions $|H_j(t)| \leq \tilde{C}_1, j = 1,3,5,...,m$, we have $|g_j| \leq \tilde{C}_2$ on the sides $x = b_j$ where $\tilde{C}_1$ and $\tilde{C}_2$ are posstive constant. In the sequel we do not care about the existence of an optimal solution of (1.1)-(1.3): one can refer to [27], where the above formulations and the following theorems in this section can be found.

Problem (P^ε). *Consider the ε-approximating problem for (P) as follows:*

$$\begin{cases} \forall\varepsilon > 0,\ \forall v \in V_j \quad \text{find the functions}\, w_j^\varepsilon(q_j) \ \ \text{such that,} \\ \nu\left(\dfrac{\partial w_j^\varepsilon}{\partial t}, v\right) + a\left(w_j^\varepsilon, v\right) + \dfrac{1}{\varepsilon}\left(\beta(w_j^\varepsilon, v\right) = \left(1, v\right) \quad a.e.,\ \ t \in [0,T]\,, \\ w_j^\varepsilon(t_{j-1}) = \tilde{w}_{j-1}^\varepsilon, \quad w_j^\varepsilon(q_j)|_{\Gamma_{d_j}} = g_j^\varepsilon(q_j)\,, \end{cases}$$

where $V_j = \{v \in H^1(D_j) : v = 0 \text{ on } \Gamma_{d_j}\}$, $g_j^\varepsilon(q)$ is a regularized function for $g_j(q)$, see [17] for example we can choose $g_j(q)$, as an exterior penalized function in $H^2(D_j \times (0,T))$. We can prove (iteratively) that this problem has a unique solution in $L^2(0,T,V_j) \cap H^1(0,T,V_j')$ (see Barbu [4] pp. 160) where V_j' is the dual of V_j. To emphasize the penalization method, therefore for any $v \in H^1(0,T;V)$ the metric projection is given by (see R. Scholz [23])

$$P(v) := v - v^+, \tag{1.5}$$

therefore we introduce a penalty function $\beta(v)$,

$$\beta(v) := v - P(v) = v^+, \tag{1.6}$$

$$\beta'(v(t)) = \begin{cases} v'(t) = \frac{d}{dt}v(t) & \text{for } \beta(v) \neq 0, \\ 0 & otherwise, \end{cases} \quad \left(v'(t), \beta(v)\right) = \frac{1}{2}\frac{\partial}{\partial t}\,\|\beta(v)\|^2_{L^2(D)}. \tag{1.7}$$

Lemma 1. *Let w_j^ε be the solutions of (P^ε), $\varepsilon > 0$, $j = 1,2,...,m$. Then the estimates*

$$\|\beta(w_j^\varepsilon)\|_{L^2(Q_j)} \leq C\varepsilon, \qquad \|\beta(w_j^\varepsilon)\|_{L^2(0,T,H^1(D_j))} \leq C\varepsilon^{\frac{1}{2}}, \tag{1.8}$$

$$\|w_j^\varepsilon\|_{L^\infty(0,T,L^2(D_j))} + \|w_j^\varepsilon\|_{L^2(0,T,V)} + \|\frac{\partial w_j^\varepsilon}{\partial t}\|_{L^2(0,T,V')} \leq C\, \|g^\varepsilon{}_j\|_{L^2(0,T,H^{\frac{1}{2}}(\Gamma_{d_j}))},$$

are hold and C is independent of ε.

Theorem A. *Let w_j, w_j^ε be the solutions of $(P), (P_\varepsilon)$ respectively. Then the estimate*

$$\|w_j - w_j^\varepsilon\|_{L^2(0,T,L^2(D_j))} + \|w_j - w_j^\varepsilon\|_{L^2(0,T,V)} \leq C\varepsilon^{\frac{1}{2}}, \tag{1.9}$$

hold and C is independent of ε. Moreover, for any small $\delta > 0$ the directional derivative z_j^ε of w_j^ε are the solutions of

$$\nu\left(\frac{\partial}{\partial t} z_j^\varepsilon, \mathring{w}_j\right) + a\left(z_j^\varepsilon, \mathring{w}_j\right) + \frac{1}{\varepsilon}\left(\beta'(w_j^\varepsilon) z_j^\varepsilon, \mathring{w}_j\right) = 0, \quad \forall \mathring{w}_j \in H^1(D_j),\ a.a.\ t \in [0,T], \tag{1.10}$$

$$\frac{g_j^\varepsilon(q_\delta) - g_j^\varepsilon(q)}{\delta} \to z_j^\varepsilon \qquad on \quad \Gamma_{d_j}. \tag{1.11}$$

Proof. See [27].

Problem (CP^ε). *Find $q_\varepsilon \in U_{ad}$ such that $J_\varepsilon(q_\varepsilon) = \min_{\forall q \in U_{ad}} J_\varepsilon(q)$* , subjected to the problem (P^ε) and (1.2), where

$$J_j^\varepsilon(q) := \frac{1}{2}\|w_j^\varepsilon(q) - w_j^d\|^2_{L^2(Q_j)} + \frac{N}{2}\|q\|^2_{L^2(0,T)}. \tag{1.12}$$

Theorem B. *If q_ε is a solution of (CP_ε) and $w_j^\varepsilon = w_j^\varepsilon(q_\varepsilon)$, then there exists an adjoint state $p_j^\varepsilon(t) \in L^\infty(0,T;H^2(D_j) \cap H_0^1(D_j))$ for every $t \in [0,T]$, such that the triple $(q_\varepsilon, w_j^\varepsilon, p_j^\varepsilon)$ satisfies $\forall \varepsilon > 0,\ \forall v \in V_j$ the following optimality system*

$$\begin{cases} w_j^\varepsilon \in L^2(0,T,V_j) \cap H^1(0,T,V_j') \subset C^0([0,T], L^2(D_j)), \\ \nu(\frac{\partial}{\partial t} w_j^\varepsilon, v) + a(w_j^\varepsilon, v) + \frac{1}{\varepsilon}(\beta(w_j^\varepsilon), v) = -(1,v), \\ w_j^\varepsilon = g_j^\varepsilon(q_\varepsilon) \quad on \quad \Gamma_{d_j} \end{cases} \tag{1.13}$$

$$\begin{cases} -\nu \frac{\partial}{\partial t} p_j^\varepsilon + \Delta p_j^\varepsilon - \frac{1}{\varepsilon}(\beta'(w^\varepsilon{}_j) p_j^\varepsilon) = (w_j^\varepsilon - w_j^d) \qquad a.e.\ in\ the\ regions\ where \quad w_j^\varepsilon > 0, \\ p_j^\varepsilon(x,t) = 0 \quad a.e.\ in\ the\ regions\ where \quad w_j^\varepsilon = 0, \qquad p_j^\varepsilon(x,T) = 0, \quad in \quad Q_j \\ \int_0^T \int_{\Gamma_{n_j}} \bigtriangledown\ g_j^\varepsilon(q_\varepsilon)\ \frac{\partial p_j^\varepsilon}{\partial n}(\sigma,t) d\sigma dt + N q_\varepsilon \geq 0 \qquad \forall q_\varepsilon \in U_{ad}, \end{cases} \tag{1.14}$$

where $\mathbf{n}$ is the unit outward normal at the boundary Γ_{d_j}.

Proof. See [27].

There are many works on parabolic equations discretization process (see for example [22]). The full discrete approximations for our parabolic control problem and for the necessary optimality condition are given in section 2. In section 3 numerical approximations of this problem were obtained. In section 4 numerical experiments are presented, and finially end in section 5 with conclusions.

2. Discretization of the State Equations

In the following, a finite difference scheme is used for the discretization of the time variables. For the computational work, let M be a positive integer. Consider the uniform partition of $[0,T]$ defined by

$$t_0 = 0 < t_1 < t_2 < ... < t_M = T,$$

$$t_k = k\Delta t, \quad (k = 0,1,...,M), \quad \text{where} \quad \Delta t = \frac{T}{M}.$$

For any time-dependent function Z we define

$$\dot{Z} \simeq \frac{1}{2}\frac{(Z^{k+1} - Z^{k-1})}{\Delta t}.$$

A piecewise linear finite elements method in space (see [10]) is used for the discretization of the time vatiables. To define the approximating spaces, let τ_{jh} be a family of triangulations of D_j

with $h \in (0,1]$, corresponding to the mesh size. The triangulation τ_{jh} is assumed to be regular (see [9]). Let V_{jh} be closed convex finite dimensional subspaces of $H^1(D_j)$ defined by

$$V_{jh} := \{v_{jh} \in L^2(0,T,(H^1(D_j) \cap C^0(\overline{D_j})) \mid v_{jh} \text{ is linear over any } K \in \tau_{jh}\}. \tag{2.1}$$

We can introduce the discrete penalty problem to the problem (P^ε) as follows:

Problem (P^ε_{jh}). *Find* $w^\varepsilon_{jh} \in V_{jh}$*:*

$$\begin{cases} \nu\,(\dot{w}^\varepsilon_{jh}, v_{jh})_{L^2(D_j)} + a(w^\varepsilon_{jh}(q_j), v_{jh}) + \dfrac{1}{\varepsilon}(\beta(w^\varepsilon_{jh}(q_j)), v_{jh}) = -(1, v_{jh}) \quad \forall v_{jh} \in V_{jh}, \\ w^\varepsilon_{jh}(q_j) = g^\varepsilon_{jh}(q_j) \quad on \quad \Gamma_{dj} \quad w^\varepsilon_{jh}(t_{j-1}) = \tilde{w}^\varepsilon_{(j-1)h}, \end{cases} \tag{2.2}$$

where $g^\varepsilon_{jh}(q_j)$ *is the discrete boundary conditions.*

Theorem 1. *Let* $w^\varepsilon_j \in L^2(0,T,H^1(D_j))$ *and* $w^\varepsilon_{jh} \in V_{jh}$ *be the solutions of* (P^ε) *and* (P^ε_{jh}) *respectively. Then the estimate*

$$\|w^\varepsilon_j - w^\varepsilon_{jh}\|_{L^2(0,T,H^1(D_j))} + \varepsilon^{-\frac{1}{2}}\|\beta(w^\varepsilon_j) - \beta(w^\varepsilon_{jh})\|_{L^2(Q_j)} \le C(h + \varepsilon^{-\frac{1}{2}}h^2), \tag{2.3}$$

is hold, where C *is independent of* ε, h.

Before proving Theorem 1 we prove first the following Corollary which show us the relation between the penalty parameter ε and the discretization parameter h.

Corollary 1. *Let* w_j, $w^\varepsilon_j \in L^2(0,T,H^1(D_j))$ *be solutions of problem* (P), (P^ε) *respectively and let* $w^\varepsilon_{jh} \in V_{jh}$ *be the solution of problem* (P^ε_{jh}). *Then the estimate*

$$\|w_j - w^\varepsilon_{jh}\|_{L^2(0,T,H^1(D_j))} \le 3\,Ch, \tag{2.4}$$

holds for $\varepsilon = h^2$.

Proof. According to Theorem A, for every $t \in (0,T)$, $\varepsilon = h^2$ we claim that

$$\|w_j - w^\varepsilon_{jh}\|_{L^2(0,T,H^1(D_j))} \le \|w_j - w^\varepsilon_j\|_{L^2(0,T,H^1(D_j))} + \|w^\varepsilon_j - w^\varepsilon_{jh}\|_{L^2(0,T,H^1(D_j))} \le 3\,Ch.$$

Proof of Theorem 1. Let $w^\varepsilon_{jh} \in V_{jh}$ be a solution of (P^h_ε), we have (see [23]) from the properties which are characteristic for linear finite elements spaces with respect to regular triangulations of D_j with mesh size h

$$\begin{aligned} \|w^\varepsilon - v_h\|_{H^k(D_j)} &\le Ch^{2-k}\|w^\varepsilon\|_{H^2(D_j)}, \quad k = 0,1 \\ &\le Ch^{2-k} \quad \forall v_h \in V_{jh}. \end{aligned} \tag{2.5}$$

Let P_{jh} denotes the L^2- projection onto V_{jh} and $\phi^\varepsilon{}_{jh} \in V_{jh}$. By the monotonicity of β and the coercivity as well as the continuity of $a(.,.)$ we get:

$$\begin{aligned} &\frac{\nu}{2}\frac{\partial}{\partial t}\|w^\varepsilon_j - w^\varepsilon_{jh}\|^2{}_{L^2(D_j)} + \|w^\varepsilon_j - w^\varepsilon_{jh}\|^2{}_{H^1(D_j)} + \frac{1}{\varepsilon}\|\beta(w^\varepsilon_j) - \beta(w^\varepsilon_{jh})\|^2{}_{L^2(D_j)} \\ &\le \nu(\dot{w}^\varepsilon_j - \dot{w}^\varepsilon_{jh}, w^\varepsilon_j - w^\varepsilon_{jh}) + a(w^\varepsilon_j - w^\varepsilon_{jh}, w^\varepsilon_j - w^\varepsilon_{jh}) + \frac{1}{\varepsilon}(\beta(w^\varepsilon_j) - \beta(w^\varepsilon_{jh}), w^\varepsilon_j - w^\varepsilon_{jh}) \\ &\le \nu(\dot{w}^\varepsilon_j - \dot{w}^\varepsilon_{jh}, w^\varepsilon_j - \phi^\varepsilon_{jh}) + a(w^\varepsilon_j - w^\varepsilon{}_{jh}, w^\varepsilon_j - \phi^\varepsilon_{jh}) + \frac{1}{\varepsilon}(\beta(w^\varepsilon_j) - \beta(w^\varepsilon_{jh}), w^\varepsilon_j - \phi^\varepsilon_{jh}), \end{aligned}$$

If especially $\phi^\varepsilon{}_{jh} := P_{jh}\,w^\varepsilon_{jh}$ is chosen, we have

$$(\dot{w}^\varepsilon_j - \dot{w}^\varepsilon_{jh}, w^\varepsilon_j - \phi^\varepsilon_{jh}) = (\dot{w}^\varepsilon_j - \dot{\phi}^\varepsilon_{jh}, w^\varepsilon_j - phi^\varepsilon_{jh}) = \frac{1}{2}\frac{\partial}{\partial t}\|w^\varepsilon_j - w^\varepsilon_{jh}\|^2{}_{L^2(D_j)}$$

then we have

$$\begin{aligned} &\frac{\nu}{2}\frac{\partial}{\partial t}\|w^\varepsilon_j - w^\varepsilon_{jh}\|^2{}_{L^2(D_j)} + \|w^\varepsilon_j - w^\varepsilon_{jh}\|^2{}_{H^1(D_j)} + \frac{1}{\varepsilon}\|\beta(w^\varepsilon_j) - \beta(w^\varepsilon_{jh})\|^2{}_{L^2(D_j)} \\ &\le \nu\frac{\partial}{\partial t}\|w^\varepsilon_j - \phi^\varepsilon_{jh}\|^2{}_{L^2(D_j)} + \|w^\varepsilon_j - \phi^\varepsilon_{jh}\|^2{}_{H^1(D_j)} + \frac{1}{\varepsilon}\|w^\varepsilon_j - \phi^\varepsilon{}_{jh}\|^2{}_{L^2(D_j)}. \end{aligned}$$

Using (2.2) we get (2.3) by integrating the result from the last equation with respect to the time t.

To construct an approximation to the regularized control problem based on the discretization of the variational inequalities. Assume that problem (CP^ε) is approximated by problem (CP^ε_{jh}), where $\varepsilon \in (0,1)$ is an arbitrary parameter and $h > 0$. We introduce the following discrete space U^{jh}_{ad} of U_{ad} in V_{jh} endowed with the norm in $L^2(0,T)$. The discrete control problem (CP^ε_{jh}) can be written as follows

$$\begin{cases} \text{Find } \hat{q^\varepsilon} \in U^{jh}_{ad} \text{ such that} \\ J^\varepsilon_{jh}(\hat{q^\varepsilon}) = \min\limits_{\forall q \in U^{jh}_{ad}} J^\varepsilon_{jh}(q), \end{cases} \tag{2.6}$$

subjected to (P_h^ε) and (1.2), where the discrete cost function, in the interval $I_j^n = (t_j^{n-1}, t_j^n)$, $n = 1, 2, \ldots M, j = 1, 2, ..., m$ is given by

$$J_{jh}^\varepsilon(q) = \frac{1}{2}\sum_{n=1}^{M}(\Delta t)_j \|w_{njh}^\varepsilon - w_{jh}^d\|_{L^2(D_j)}^2 + \frac{N}{2}\sum_{n=1}^{M}(\Delta t)_j (q^n)^2$$

and $w_{jh}^d \in V_{jh}$ is an approximation of w_j^d, such that $\lim_{h\to 0} \|w_{jh}^d - w_j^d\|_{L^2(Q_j)} = 0$.

Theorem 2. *Let $\varepsilon > 0$, h be fixed, then problem (CP_{jh}^ε) admits an optimal solution $\hat{q}^\varepsilon \in U_{ad}^{jh}$.*

Proof. The aim is to show that the functional J_ε^{jh} is weakly l.s.c. on U_{ad}^h. Let $\{q_n\} \in U_{ad}^h$ be any sequence such that $q_n \to q$ weakly in U_{ad}^h as $n \to \infty$. Then for every time step the sequence $w_{jh}^\varepsilon(q_n)$ is bounded in $H^1(D_j)$ by a constant independent on n satisfying (2.2) (see Barbu ([4]), p. 169). Moreover the map from $q_n \to w_{jh}^\varepsilon(q_n)$ is continuous (weak) from U_{ad}^h to $L^2(0,T; H^1(D_j))$. Consequently, the functional J_ε^h is weakly l.s.c. and therefore the existence of optimal control for $((CP)_\varepsilon^h)$ is proved.

Theorem 3. *The mapping from the control and the discrete state observation is Lipschitz continuous.*

Proof. Corresponding to the perturbed data at the levels H_{01} and H_{02}. We can assume that the corresponding amount of water from each dam is q_{11} and q_{12}, respectively. We can choose the test function $v = w_{jh}^\varepsilon(q_{12})$ or $v = w_{jh}^\varepsilon(q_{11})$, from equation (2.2) we have in the discrete form:

$$\nu\Big(\dot{w}_{jh}^\varepsilon(q_{12}), \xi_{jh}^\varepsilon\Big) - \Big(\Delta w_{jh}^\varepsilon(q_{12}), \xi_{jh}^\varepsilon\Big) + \int_{\Gamma_j}\Big(\xi_{jh}^\varepsilon\Big) \nabla w_{jh}^\varepsilon(q_{12}) \cdot n\, d\sigma + \frac{1}{\varepsilon}\Big(\beta(w_{jh}^\varepsilon(q_{12})), \xi_{jh}^\varepsilon\Big) = \Big(-1, \xi_{jh}^\varepsilon\Big),$$

$$\nu\Big(\dot{w}_{jh}^\varepsilon(q_{11}), \xi_{jh}^\varepsilon\Big) - \Big(\Delta w_{jh}^\varepsilon(q_{11}), \xi_{jh}^\varepsilon\Big) + \int_{\Gamma_j}\Big(\xi_{jh}^\varepsilon\Big) \nabla w_{jh}^\varepsilon(q_{11}) \cdot n\, d\sigma + \frac{1}{\varepsilon}\Big(\beta(w_{jh}^\varepsilon(q_{11})), \xi_{jh}^\varepsilon\Big) = \Big(-1, \xi_{jh}^\varepsilon\Big),$$

where

$$\xi_{jh}^\varepsilon = w_{jh}^\varepsilon(q_{12}) - w_{jh}^\varepsilon(q_{11}),$$
$$w_{jh}^\varepsilon(q_{1i}) = g_{jh}^\varepsilon(q_{1i}) \quad \text{on } \Gamma_{nj},\ i = 1, 2,$$

By subtracting the two equations, we get

$$\nu(\dot{\xi^\varepsilon}_{jh}, \xi_{jh}^\varepsilon) - (\Delta\xi_{jh}^\varepsilon, \xi_{jh}^\varepsilon) - \frac{1}{\varepsilon}(\beta(w_{jh}^\varepsilon(q_{11}) - \beta(w_{jh}^\varepsilon(q_{12}), \xi_{jh}^\varepsilon) = -\int_{\Gamma_j} \xi_{jh}^\varepsilon(\xi_{jh}^\varepsilon.n)d\sigma,$$

By the monotonicity of β and Young's inequality we derive

$$\frac{\nu}{2}\frac{\partial}{\partial t}\|\xi_{jh}^\varepsilon\|_{L^2(D_j)}^2 + \|\xi_{jh}^\varepsilon\|_{H^2(D_j)}^2 \le \frac{\alpha}{2}\int_{\Gamma_j}|\xi_{jh}^\varepsilon)|^2 d\sigma + \frac{1}{2\alpha}\int_{\Gamma_j}|\nabla \xi_{jh}^\varepsilon|^2 d\sigma,$$

where α is positive constant. Integrating with respect to t and noting that on the boundary $w_{jh}^\varepsilon(q) = g_{jh}^\varepsilon(q)$, we get

$$\|w_{jh}^\varepsilon(q_{12}) - w_{jh}^\varepsilon(q_{11})\|_{L^\infty(0,T,H^1(D_j))} \le \|q_{12} - q_{11}\|_{L^\infty(0,T)}. \tag{2.7}$$

Remark 1. Let l be a function such that $\hat{q}^\delta = \hat{q}^\varepsilon + \delta l \in U_{ad}^{jh}$ for any $\delta > 0$. Then the map $\hat{q}^\varepsilon \to J_{jh}^\varepsilon(\hat{q}^\varepsilon)$ is differentiable:

$$\frac{J_h^\varepsilon(\hat{q}^\delta) - J_h^\varepsilon(\hat{q}^\varepsilon)}{\delta} \to \int_{Q_j}\frac{w_{jh}^\varepsilon(\hat{q}^\delta) - w_{jh}^\varepsilon(\hat{q}^\varepsilon)}{\delta}(w_{jh}^\varepsilon(\hat{q}^\delta) - w_{jh}^\varepsilon(\hat{q}^\varepsilon))dxdt + N(\hat{q}^\varepsilon, l)_{L^2(0,T)}$$
$$\to \int_{Q_j} z_{jh}^\varepsilon(w_{jh}^\varepsilon(\hat{q}^\delta) - w_{jh}^\varepsilon(\hat{q}^\varepsilon))dxdt + N(\hat{q}^\varepsilon, l)_{L^2(0,T)},$$

where z_{jh}^ε is defined in Theorem A.

Corollary 2. *As a consequence of Theorem A the estimates*

$$|J(q) - J_h^\varepsilon(q)| \le Ch, \quad |J_h^\varepsilon(q^\varepsilon) - J_h^\varepsilon(q)| \le Ch, \quad |J_h^\varepsilon(q^\varepsilon) - J(q)| \le Ch \tag{2.8}$$

hold for all $j = 1, 2, ..., m$, where C is independent of ε.

Proof. From Theorem A we have by coupling the penalty parameter ε and the discretization parameter h by $\varepsilon = h^2$,

$$|J(q) - J_h^\varepsilon(q)| \le |J(q) - J^\varepsilon(q)| + |J^\varepsilon(q) - J_h^\varepsilon(q)|,$$

but using Theorem A we derive

$$|J^\varepsilon(q) - J_h^\varepsilon(q)| \le C_1 \|w_{jh}^\varepsilon(q) - w_{jh}^\varepsilon(q)\|_{L^2(0,T,H^1(D_j))} \le Ch\,,$$

where C_1 is a constant independent of ε, then we get the first inequality. By the same idea we can derive the others.

Let us define the discrete adjoint state as the solution of the following problem (see [16] chapter 3):

Problem (AP_{jh}^ε).

$$\begin{cases} \nu\,\dot{p}_{jh}^\varepsilon + \Delta p_{jh}^\varepsilon - \dfrac{1}{\varepsilon}(\beta'(w_{jh}^\varepsilon)p_{jh}^\varepsilon) = (w_{jh}^\varepsilon - w_{jh}^d) \quad a.e.\ in\ the\ regions \quad w_{jh}^\varepsilon(x,t) > 0, \\ p_{jh}^\varepsilon(x,t) = 0 \qquad a.e\ in\ the\ regions\ where \quad w_{jh}^\varepsilon(x,t) = 0, \end{cases}$$

with the discrete maximum principle

$$\int_{t_{i-1}}^{t_i} \int_{\Gamma_j} \nabla\, g_{jh}^\varepsilon(q^\varepsilon)\, \frac{\partial p_{jh}^\varepsilon}{\partial n}(\sigma,t) d\sigma dt + N\, q^\varepsilon \ge 0 \qquad \forall q^\varepsilon \in U_{ad}^{jh}\,, \tag{2.9}$$

where $\mathbf{n}$ *is the outward unit normal at the boundary* Γ_j *and* t_i *is the discrete time,* $i = 1, 2, \cdots, M$.

3. Method of Solution

In what follows, we present an algorithm to solve the discrete approximation of the control problem (CP_{jh}^ε). This algorithm consists of four parts. In part two we used a combined approach of a linear finite elements method for the space variable, finite difference discretization to the time variable and the projected successive over-relaxation method for solving the linear complementarity problem. In part three the technique of nonlinear conjugate gradient methods is applied to minimize the effect of zigzagging. The conjugate gradient method is the conjugate direction method that is obtained by selecting the successive direction vectors as a conjugate version of the successive gradients obtained as the method progresses, see [13],[14], [15]. We introduce a nonlinear conjugate-gradient algorithm with exact line searches to solve the control problem for two reasons. One is that their storage needs are low. The other is that the work required to compute a search direction is also low. This algorithm based on Armijo's rule in the unconstrained optimization theory (see [11], [20]). For more details on the global convergence properties of nonlinear conjugate gradient methods see [13]. convergence properties of nonlinear conjugate gradient methods see [13].

Algorithm.

1. **Initialization:** Let ε, $h > 0$ be fixed, let i denote the iteration index, $\theta \in (0.5, 1)$, $\tilde{\alpha} \in (0,1)$, $q^\varepsilon := q$, $w_j := w_{jhn}^\varepsilon$ and $g_j := g_{jhn}^\varepsilon$. Take w_{jh}^d be any continuous functions. Set $i = 0, j = 1$, choose the initial functions $H_0^{(0)}, H_m^{(0)}$, such that $H_0^{(0)} > H_m^{(0)}$ for all $t > 0$.

2. **Optimality system computations:**

Step 0. Compute the distance between the dams $l_j = a_{j+1} - b_j$, $j = 1, 2, ..., m-1$. By solving (1.2) find $q_j^{(i)}$, then find the intermediate water level $H_j^{(i)} = \frac{q_j^{(i)}}{l_j}$ between j^{th} and $(j+1)^{\text{th}}$ dams.

Step 1. Compute w_j^i by solving the parabolic problem (P_{jh}^ε); **Step 2**. Compute $p_j^{(i)}$ by solving the the parabolic adjoint equation $((AP)_{jh}^\varepsilon)$;

Step 3. For $n = 1, 2, ..., M$ compute $d_{ij}{}^{(n)} = -(N\, q_{ij}{}^{(n)} + (\nabla g_j, p_{ij}{}^{(n)})_{\Gamma_{n_j}})$;

3. **Nonlinear conjugate gradient technique:**

Step 4. If $\sum_{n=1}^M (\Delta t)_j \|d_{ij}^{(n)}\|_{L^2(D_j)} = 0$, stop; otherwise continue;

Step 5. Set $d^{(-1)} = 0$; $\alpha^{(0)} = 0$ and for $n > 0$ set $\alpha^{(n)} = \dfrac{\|d_{ij}^{(n)}\|_{L^2(D_j)}}{\|d_{ij}^{(n-1)}\|_{L^2(D_j)}}$;

Step 6. Set $d_{ij}^{(n)} := d_{ij}^{(n)} + \alpha^{(n)}\, d_{ij}^{(n)}$;
Step 7. set $A_1 = 1$;
4. **Update and stopping criteria:**
Step 8. Compute $\delta := J_h^\varepsilon(q_j^{(i)} + A_1 d_{ij}^{(n)}) - J_h^\varepsilon(q_j^{(i)}) + \sum_{n=1}^{M} (\Delta t)_j \, \|d_{ij}^{(n)}\|_{L^2(D_j)}$;
Step 9. If $\delta \le 0$ go to step 10, otherwise $A_1 = \theta \times A_1$, go to step 8;
Step 10. We define $q^{(i+1)}$ from $q^{(i)}$ using the relation: $q^{(i+1)} = q^{(i)} + A_1(d_{ij}^{(n)})$
Step 11. We define $H_j^{(i+1)}$ from $H_j^{(i)}$ using the relation: $H_j^{(i+1)} = H_j^{(i)} + A_1(d_{ij}^{(n)})$;
Step 12. Set $i := i+1$ and return to the step 1 untill the stopping rules (see section 4) are satisfied.
Step 13. Set $j := j+1$ and return to the step 0.

Convergence of the Algorithm

To prove the convergence of this algorithm in the framwork of the general theory in $C\grave{e}a$([8], chapter 3). For any $q \in U_{ad}^{jh} \subset L^2(0,T)$, the gradient $G(q)$ of the functional $J_h^\varepsilon(q_j) := J(q_j)$ is defined as

$$(J'(q))(l) := (G(q), l) \quad \forall l \in U_{ad}^{jh}\,, \tag{3.1}$$

where $J'(q,l)$ is the Gâteaux derivative of J for every $j = 1, 2, ..., m$. From Theorem B we have $G(q) = \sum_{n=1}^{M} (Nq^n + (\bigtriangledown g_j^\varepsilon, p_j^n)_\Gamma)\chi^n(t)$ where $\chi^n(t)$ is the characteristic function of the interval $I^n = (t^{n-1}, t^n)$, $n = 1, 2, \ldots M$. We will show that the gradient $G(q)$ satisfies the Lipschitz condition on every bounded subset $B \subset L^2(0,T)$, i.e. there exist a constant $C_B > 0$ such that

$$\|G_j(q_1) - G_j(q_2)\|_{L^2(0,T)} \le C_B \|q_1 - q_2\|_{L^2(0,T)}\,. \tag{3.2}$$

Remark 2. As a result of Proposition 1, we have

$$p_j^\varepsilon \to p_j \quad \text{weakly star in } L^\infty(0,T,H_0^1(D_j))\,,$$
$$p_j^\varepsilon \beta'(w_j(q_\varepsilon)) \quad \text{is bounded in} \quad (L^\infty(Q_j))^*.$$

Proof. (The proof idea is similar to the proof of Lemma 6.2 in [4], p. 235.) We take the scalar product of the first equation in (1.14) with p_j^ε we get

$$\nu\,(\dot p_j^\varepsilon, p_j^\varepsilon) + (\Delta p_j^\varepsilon, p_j^\varepsilon) - \frac{1}{\varepsilon}((\beta'(w_j^\varepsilon)p_j^\varepsilon, p_j^\varepsilon)) = ((w_j^\varepsilon - w_j^d, p_j^\varepsilon)) \quad \text{in} \quad D_j\,, \tag{3.3}$$

Taking into account that $(w_j^\varepsilon - w_j^d)$ is bounded, $(\beta'(w_j^\varepsilon) > 0$,

$$a(z,z) + \alpha\|z\|_{L^2(D_j)}^2 \ge \omega\|z\|_{H^1(D_j)}^2 \quad \forall z \in H^1(D_j),$$

where α and ω are constant. Integrate on $[t_{j-1}, t_j]$ we derive:

$$\nu\,\|p_j^\varepsilon\|_{L^2(D_j)} + \int_{t_{j-1}}^{t_j} \|p_j^\varepsilon\|_{H^1(D_j)} \le C, \qquad \nu\,\|\frac{\partial}{\partial t} p_j^\varepsilon\|_{L^2(0,T,H^{-1}(D_j))}^2 \le C.$$

We claim that p_j^ε is weakly compact in $L^2(0,T,H^1(D_j))$ and weak star in $L^\infty(0,T,L^2(D_j))$

Remark 3. $\{\frac{\partial p^\varepsilon{}_j}{\partial n}\}$ is bounded in $L^\infty(0,T,H_0^1(D_j)) \quad \forall v \in H^2(D)$.

Proof. From (1.14) and by Green's formula we have

$$\nu\,(\dot p_j^\varepsilon, v) - a(p_j^\varepsilon, v) + \int_\Gamma v\,\frac{\partial p_j^\varepsilon}{\partial n} + \frac{1}{\varepsilon}(\beta'(w_j)\,p^\varepsilon{}_j, v) = (w_j^\varepsilon - w_j^d, v) \quad \text{in} \quad D_j$$

or

$$\int_\Gamma v\,\frac{\partial p^\varepsilon{}_j}{\partial n} = (w_j^\varepsilon - w_j^d, v) + \frac{1}{\varepsilon}(\beta'(w_j))\,p_j^\varepsilon, v) + a(p_\varepsilon, v) \le C\|v\|_{H^1(D_j)} \quad \forall v \in H^2(D_j)\,.$$

Then by the trace theorem we get the desired result.

Remark 4. $\|G_j(q_1) - G_j(q_2)\|_{L^2(0,T)} \le C_B\|q_1 - q_2\|_{L^2(0,T)} \quad \forall q_1, q_2 \in U_{ad}\,.$

Proof. By using the above remarks we can derive

$$\|G_j(q_1) - G_j(q_2)\|_{L^2(0,T)} = \|Nq_1 + (\bigtriangledown g^\varepsilon(q_1), p^n)_{\Gamma_{d_j}} - Nq_2 + (\bigtriangledown g^\varepsilon(q_2), p^n)_{\Gamma_{d_j}}\|_{L^2(0,T)}.$$
$$\le N\,\|q_1 - q_2\|_{L^2(0,T)} + C\,\|\bigtriangledown g^\varepsilon(q_1) - \bigtriangledown g^\varepsilon(q_2)\|_{L^2(0,T)}.$$

The last inequality is due to Remark 3. By noting that g^ε is a continuous function we get the desired result.

By $C\grave{e}a$([8], chapter 3) and the above remarks we can prove the following convergence theorem

Theorem 4. *The sequence constructed in the algorithm,* $\{q\}$*, is either a finite sequence, or it converges to some element* $q^* \in L^2(0,T)$ *such that for* q^**, the necessary condition is satisfied. Moreover, the iteration steps number* $7-9$ *can be done in finitely many steps.*

4. Numerical Experiments

The algorithm presented above was tested with different examples. The numerical example given here shows in some detail selected results when the initial height of the flow level and the function w^d_{jh} are periodic functions in time. To find the approximate solutions of the control problem (2.6) we note that the control problem has in general no unique solution. The stopping rules for the program are (I) For physical reasons, the height of the seepage surface is equal to 0.255 for each dam. (II) $\sum_{n=1}^{M} (\Delta t)_j \, \|d^n_j\|_{L^2(D_j)} \leq T \times 10^{-4}$, $j = 1, 2, ..., m$.

Example. Consider an evolution system of eart dams consists of three homogeneous porous rectangular dams. Let

$$w^d_{jh} := \hat{w}_{0jh}\,(1 + 0.01\,cos(\frac{\pi \times t}{2 \times T})), \quad 0 < t \leq T, \quad T = 0.05 \quad \Delta t = .005,$$

where $\hat{w}_{0jh}$ is the discrete solution of the stationary dam problem corresponding to the initial height of water H^0_0. In tables $1a, 1b$ and $1c$ we illustrate the computional results in the fourth, seventh and tenth time steps by using the above algorithm with the initial iteration function $H^0_0(t) = 1.3 + 0.1\cos(\frac{\pi\, t}{2 \times T})$. The stopping rule as $(I), (II)$. The reduction are computed by

$$\text{Reduction} = \frac{\text{Initial value} - \text{Final value}}{\text{Initial value}} \times 100\%.$$

We have the following results (see figure 1), where we assume that

$$\mathbf{N} = 1.e-2, \quad \theta \in (0.5, 1), \quad \theta = 0.51, \tilde{\alpha} = 0.2, l_1 = a_2 - b_1 = 1 = a_3 - b_2 = l_2,$$

- after 4 instants of time, the initial function is $H^{(0)}_0 = 1.374$, then $q^{(0)}_1 = 1.271$. By applying the above algorithm we have, after 7 iteration steps the stopping rule for the first dam is satisfied and the level of water when J is minimum is at height 1.172 and the corresponding amount of water is 1.070. Then the first intermediate water level is $H^{(0)}_1$ is 1.271 (see step 0),

Table 1a

	Initial value	Final value	Reduction
$\sum_{n=1}^{M} \Delta t \|d^n\|_{L^2(\Gamma_j)}$	8.247E-6	6.243E-6	24.298 %
$\|w^h_\varepsilon(q) - w^h_d\|^2_{L^2(Q_j)}$	2.007	0.969	51.699%
$J^h_\varepsilon(q)$	1.844	0.799	56.643%

- after 7 instants of time, the initial height of water for the second dam is $\mathbf{H^{(0)}_1} = 1.271$, then $q^{(0)}_2 = 1.010$. By applying the above algorithm we have, after 17 iteration steps the stopping rule is satisfied for the second dam and the level of water when J is minimum is at height equal 0.917 , and the corresponding amount of water is 0.650,

Table 1b

	Initial value	Final value	Reduction
$\sum_{n=1}^{M} \Delta t \|d^n\|_{L^2(\Gamma_j)}$	1.676E-5	6.6119E-6	63.497 %
$\|w^h_\varepsilon(q) - w^h_d\|^2_{L^2(Q_j)}$	7.000	0.892	87.260%
$J^h_\varepsilon(q)$	4.994	0.868	82.624%

- after 10 instants of time, the initial height for the third dam is $\mathbf{H^{(0)}_2} = 1.010$, then $q^{(0)}_3 = 0.577$. By applying the above algorithm we have, after 13 iteration steps the stopping rule is satisfied and the level of water when J is minimum is at height $= 0.934$, and the corresponding amount of water is 0.465

Table 1c

	Initial value	Final value	Reduction
$\sum_{n=1}^{M}\Delta t\|d^n\|_{L^2(\Gamma_j)}$	2.002E-5	6.127E-6	69.696 %
$\|w_\varepsilon^h(q)-w_d^h\|^2_{L^2(Q_j)}$	15.834	1.801	88.627%
$J_\varepsilon^h(q)$	8.833	1.762	80.058%

In figure 1 we report on the discrete free boundary representation for each dam with the data $a_1b_1 = 1$, $a_1e_1 = 1.5$, $a_2b_2 = 0.75$, $a_2e_2 = 1.125$, $a_3b_3 = 0.50$, $a_3e_3 = 0.75$. The picture is the representation of the free boundary in different time steps where the retentivity cofficient $\nu = 1$, (see [10], [25]) and we started from the stationary situation for each dam.

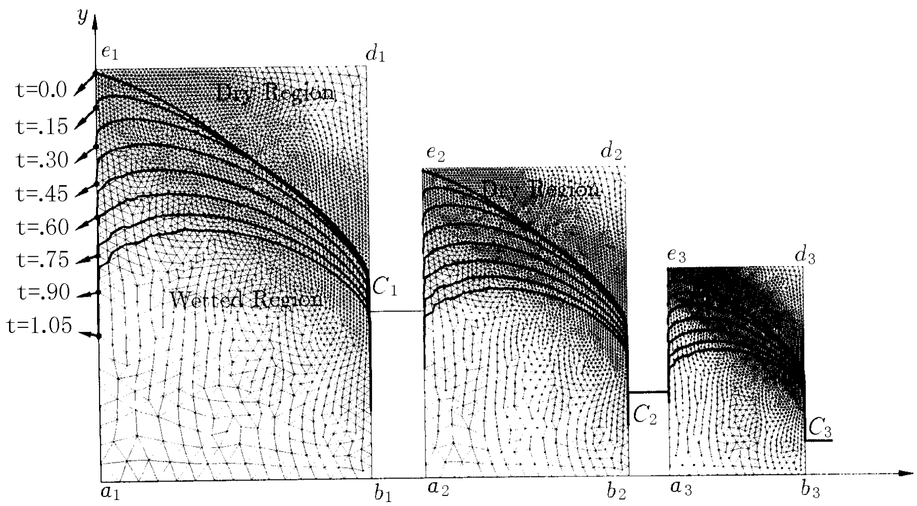

Figure 1: The Evolution System of Earth Dams

5. Conclusions

In this article we presented a full discrete approximations for a pratical control problem governed by a system of parabolic variational inequalities. The nonlinear conjugate gradient method investigated here is found to be highly stable and always converges even with large sized problems, compare with the usual gradient type methods (see [26]) which are the most convenient in such problems. The material in this paper can be generalized to three dimensions models.

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