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A CLASS OF ASYNCHRONOUS PARALLEL MULTISPLITTING RELAXATION METHODS FOR LARGE SPARSE LINEAR COMPLEMENTARITY PROBLEMS *[1)]

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Abstract

Asynchronous parallel multisplitting relaxation methods for solving large sparse linear complementarity problems are presented, and their convergence is proved when the system matrices are H-matrices having positive diagonal elements. Moreover, block and multi-parameter variants of the new methods, together with their convergence properties, are investigated in detail. Numerical results show that these new methods can achieve high parallel efficiency for solving the large sparse linear complementarity problems on multiprocessor systems.

Key words: Linear complementarity problem, Matrix multisplitting, Relaxation method, Asynchronous iteration, Convergence theory.

1. Introduction

Consider the linear complementarity problem LCP(M,q):

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz + q) = 0,$$

where $M = (m_{kj}) \in \mathbb{R}^{n\times n}$ and $q = (q_k) \in \mathbb{R}^n$ are given real matrix and vector, respectively. This problem usually arises in (linear and) convex quadratic programming, in the problem of finding a Nash equilibrium point of a bimatrix (e.g., Cottle and Dantzig [13] and Lemke [25]), and also in a number of free boundary problems of fluid mechanics (e.g., Cryer [17]). Therefore, it has various practical backgrounds. Many efficient iterative methods were established to get a numerical solution of the LCP(M,q) on sequential computer systems, and their convergence properties were discussed in depth (see [1], [10], [13], [14], [16], [17], [23], [24], [25], [26], [28] and [31]). For a systematic and comprehensive study one can refer to Cottle, Pang and Stone [15].

To solve the LCP(M,q) on a high-speed multiprocessor system, we proposed two classes of synchronous multisplitting relaxation methods by successively projecting the unknowns into $\mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n)^T \mid x_i \geq 0, i = 1, 2, \cdots, \alpha\}$ (see Bai [3]) and by equivalently transforming the LCP(M,q) into a system of fixed-point equations (see Bai and Evans [5] and Bai, Evans and Wang [6]). In a quite different way, Machida, Fukushima and Ibaraki [27] and Bai[4] recently presented and discussed another class of multisplitting iterative methods by implicit splittings of the system matrix. These methods have good parallel computational properties and are suitable for implementing on synchronous parallel computer systems. They can achieve high parallel efficiency provided the task is roughly evenly distributed among all processors.

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However, in practical applications, many problems are natural to be decomposed into subproblems of unequal sizes due to the special physical properties of the original problems. Hence, the above assumption about the balanced distribution of a task does not always hold in actual computations.

To overcome this shortcoming of the abovementioned synchronous parallel multisplitting iterative methods, and to reduce the idle time of each processor, so that the multiprocessor system can achieve high parallel efficiency, in this paper, we further present asynchronous and relaxed variants of the synchronous multisplitting iterative methods proposed in [27] and [4], in accordance with the principle of using sufficiently and communicating flexibly the currently available information. These asynchronous multisplitting relaxation methods can be implemented on MIMD multiprocessor system without any mutual wait among the processors, and hence, they can achieve high parallel computing efficiency in practical applications. When the system matrix $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix , we set up the convergence theories of these new methods under suitable conditions on both the multiple splittings and the relaxation parameters. Moreover, for the convenience of applications, some explicit variants of the new asynchronous multisplitting relaxation methods are presented by making use of the successive overrelaxation technique, and their convergence properties are investigated in detail. With some numerical experiments, we show that these new asynchronous multisplitting relaxation methods can solve large sparse linear complementarity problems on multiprocessor systems with high parallel efficiency.

At last, we remark that this work is also a further development of the asynchronous parallel matrix multisplitting relaxation methods and theories for linear systems of equations in Wang, Bai and Evans [33], Bai, Wang and Evans [9] and Evans, Wang and Bai [20]; In-depth studies on parallel synchronous and asynchronous relaxation methods based on operator projection and fixed-point transformation techniques for solving the large sparse linear complementarity problems can be found in [3], [5], [6], [11], [18] and [29]; Asynchronous variants of the synchronous multisplitting relaxation methods in [3], [5] and [6] were given in [3] and [8], respectively; And generalizations to nonlinear complementarity problem of the synchronous multisplitting relaxation methods in [3] were discussed in [2].

2. Establishments of the New Methods

Without loss of generality, we assume that the considered multiprocessor system consists of α processors, and the host processor may be chosen to be any one of them. For a matrix $M \in \mathbb{R}^{n\times n}$, let $M = B_i + C_i$ $(i = 1, 2, \cdots, \alpha)$ be α Q-splittings and $E_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \cdots, \alpha)$ be α nonnegative diagonal matrices satisfying $\sum_{i=1}^{\alpha} E_i = I$ (the $n \times n$ identity matrix). Then the collection of triples $(B_i, C_i, E_i)(i = 1, 2, \cdots, \alpha)$ is called a multisplitting of the matrix M, and the matrices $E_i(i = 1, 2, \cdots, \alpha)$ are called weighting matrices. To describe the new asynchronous multisplitting relaxation methods for the LCP(M,q), we introduce the following necessary notations: $N_0 = \{0, 1, 2, \cdots\}$; for $\forall p \in N_0$, $J(p)$ is a nonempty subset of the number set $\{1, 2, \cdots, \alpha\}$; and for $\forall i \in \{1, 2, \cdots, \alpha\}$ and $\forall p \in N_0$, $s_i(p)$ is an infinite sequence of nonnegative integers. Natural and standard conditions about $J(p)$ and $s_i(p)(i = 1, 2, \cdots, \alpha), p \in N_0$, in the convergence analysis of an asynchronous parallel iteration are:

(1) for $\forall i \in \{1, 2, \cdots, \alpha\}$, the set $\{p \in N_0 | i \in J(p)\}$ is infinite;

$M = (m_{kj}) \in \mathbb{R}^{n\times n}$ is called an H_+-matrix if $m_{kk} > 0 (k = 1, 2, \cdots, n)$ and there exist positive reals $w_k (k = 1, 2, \cdots, n)$ such that $\overline{M} = W^{-1}MW$, where $W = \mathrm{diag}(w_1, w_2, \cdots, w_n)$, is a diagonally dominant matrix. In this case, the matrix $M \in \mathbb{R}^{n\times n}$ is also called a generalized diagonal dominant matrix. (See the equivalent definition of H_+-matrix in Section 3.)

$M = B + C$ is called a Q-splitting if B is a Q-matrix. See the definition about a Q-matrix in Section 3.

(2) for $\forall i \in \{1, 2, \cdots, \alpha\}$ and $\forall p \in N_0$, it holds that $s_i(p) \leq p$; and
(3) for $\forall i \in \{1, 2, \cdots, \alpha\}$, it holds that $\lim\limits_{p\to\infty} s_i(p) = \infty$.

If we denote $s(p) = \min\limits_{1\leq i\leq\alpha} s_i(p)$, then it holds that $s(p) \leq p$ and $\lim\limits_{p\to\infty} s(p) = \infty$.

Let $(B_{i,p}, C_{i,p}, E_i)(i = 1, 2, \cdots, \alpha)$, $p \in N_0$, be a sequence of multisplittings of the matrix M. Then we consider the following asynchronous multisplitting relaxation method for solving the LCP(M,q).

Method 2.1. (Asynchronous Multisplitting Relaxation Method for the LCP(M,q)) *Given an initial vector $z^0 \in \mathbb{R}^n$. Suppose that we have obtained approximate solutions z^1, z^2, $\cdots$, z^p of the LCP(M,q). Then the next approximate solution z^{p+1} of the LCP(M,q) is computed according to the formulas:*

$$z^{p+1} = \sum_{i=1}^{\alpha} E_i z^{i,p} \tag{1}$$

and

$$z^{i,p} = \begin{cases} \beta\widehat{z}^{i,p} + (1-\beta)z^{s_i(p)}, & \text{if } i \in J(p), \\ z^p, & \text{otherwise,} \end{cases} \qquad i = 1, 2, \cdots, \alpha, \tag{2}$$

where $\widehat{z}^{i,p}$ is an arbitrary solution of the LCP$(B_{i,p}, q_{i,p})$:

$$z \geq 0, \quad B_{i,p}z + q_{i,p} \geq 0, \quad z^T(B_{i,p}z + q_{i,p}) = 0,$$

with $q_{i,p} = C_{i,p}z^{s_i(p)} + q$, and $\beta(\neq 0)$ is a relaxation factor.

In Method 2.1, each processor is allowed to update or retrieve the global approximate solution residing in the host processor at any time. Hence, new information can be used once it is available. Moreover, considerable savings in computational workload are possible, since a component of $z^{i,p}$ does not need to be computed if the corresponding diagonal entry of the weighting matrix E_i is zero. The role of the weighting matrices $E_i(i = 1, 2, \cdots, \alpha)$ may be regarded as determining the distribution of the computational work to individual processors. We remark that when $J(p) = \{1, 2, \cdots, \alpha\}$ and $s_i(p) = p$ hold for all $p \in N_0$ and all $i \in \{1, 2, \cdots, \alpha\}$, if $\beta = 1$, then Method 2.1 becomes the multisplitting method for the LCP(M,q) in [4] and [27], and if $\beta \neq 1$, then it turns out to be a relaxed variant of that multisplitting method.

However, at every iterate step p of Method 2.1, each processor needs to solve an implicit linear complementarity problem LCP$(B_{i,p}, q_{i,p})$. This makes Method 2.1 not so appealing in practical implementations. The subproblem LCP$(B_{i,p}, q_{i,p})$ could be solved by some parameter-free technique, like conjugate gradient, however, for the sake of convenient application in actual computation, in the following, we present an explicit variant of Method 2.1 by specific choices of the splitting matrices of $M \in \mathbb{R}^{n\times n}$.

To this end, we let γ and $\omega(\neq 0)$ be two real constants, and take

$$\begin{cases} B_{i,p} = \frac{1}{\omega}(D + \gamma L_{i,p}), \\ C_{i,p} = \frac{1}{\omega}((\omega - 1)D + (\omega - \gamma)L_{i,p} + \omega W_{i,p}), \end{cases} \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0,$$

where $D = \mathrm{diag}(M)$, and for $i = 1, 2, \cdots, \alpha$ and $p \in N_0$, $L_{i,p} = (l_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}$ are strictly lower triangular matrices, $W_{i,p} = (w_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}$ are matrices with zero-diagonal entries, such that

(1) $M = D + L_{i,p} + W_{i,p}$, $i = 1, 2, \cdots, \alpha$, $p \in N_0$; and
(2) $\det(D) \neq 0$.

Then the collections of triples $(D + L_{i,p}, W_{i,p}, E_i)(i = 1, 2, \cdots, \alpha)$, $p \in N_0$, are called triangular multisplittings of the matrix M. Now, with these special multisplittings and for $\beta = 1$, Method 2.1 naturally leads to the following asynchronous multisplitting accelerated overrelaxation (AOR) method (AMAOR-method) for solving the LCP(M,q).

Method 2.2. (AMAOR-method for the LCP(M,q))
Given an initial vector $z^0 = ([z^0]_1, [z^0]_2, \cdots, [z^0]_n)^T \in \mathbb{R}^n$. *Suppose that we have obtained approximate solutions* $z^1, z^2, \cdots, z^p$ *of the LCP(M,q). Then the next approximate solution*

$$z^{p+1} = ([z^{p+1}]_1, [z^{p+1}]_2, \cdots, [z^{p+1}]_n)^T$$

of the LCP(M,q) is computed according to the formulas:

$$z^{p+1} = \sum_{i=1}^{\alpha} E_i z^{i,p}$$

and

$$\begin{cases} z^{i,p} & = & ([z^{i,p}]_1, [z^{i,p}]_2, \cdots, [z^{i,p}]_n)^T, \\ [z^{i,p}]_k & = & \begin{cases} [\widehat{z}^{i,p}]_k, & \text{if } i \in J(p), \\ [z^p]_k, & \text{otherwise,} \end{cases} \qquad k = 1, 2, \cdots, n, \end{cases}$$

where

$$[\widehat{z}^{i,p}]_k = \begin{cases} 0, & \text{if} \quad \gamma \sum_{j=1}^{k-1} l_{kj}^{(i,p)} ([\widehat{z}^{i,p}]_j - [z^{s_i(p)}]_j) \\ & \quad + \omega [M z^{s_i(p)} + q]_k > m_{kk} [z^{s_i(p)}]_k, \\ [z^{s_i(p)}]_k + \frac{\gamma}{m_{kk}} \sum_{j=1}^{k-1} l_{kj}^{(i,p)} ([z^{s_i(p)}]_j - [\widehat{z}^{i,p}]_j) - \frac{\omega}{m_{kk}} [M z^{s_i(p)} + q]_k, \\ \quad \text{otherwise,} \end{cases}$$

and γ *and* $\omega (\neq 0)$ *are a relaxation and an acceleration factors, respectively.*

When $J(p) = \{1, 2, \cdots, \alpha\}$ and $s_i(p) = p$ hold for all $p \in N_0$ and all $i \in \{1, 2, \cdots, \alpha\}$, Method 2.2 becomes the multisplitting AOR method for the LCP(M,q) in [4]. If we choose the parameter pair (γ, ω) to be (ω, ω), $(1, 1)$ and $(0, 1)$, respectively, Method 2.2 gives the asynchronous multisplitting successive overrelaxation (SOR), Gauss-Seidel and Jacobi methods, correspondingly, for the LCP(M,q). Hence, an extensive sequence of asynchronous parallel multisplitting relaxation methods can be obtained, which is quite practical and efficient for solving the large sparse linear complementarity problems on MIMD multiprocessor systems. Moreover, suitable adjustments of the relaxation parameters (γ, ω) can substantially improve the convergence properties of the asynchronous multisplitting AOR method.

3. Preliminaries

First of all, we briefly review some necessary notations and concepts in [4] and [27]. A matrix $A = (a_{kj}) \in \mathbb{R}^{n \times n}$ is called a monotone matrix if it is nonsingular and satisfies $A^{-1} \geq 0$; an M-matrix if it is a monotone matrix and satisfies $a_{kj} \leq 0$ for $k \neq j$, $k, j = 1, 2, \cdots, n$; an H-matrix if its comparison matrix $\langle A \rangle$ is an M-matrix, where $\langle A \rangle = (\langle a_{kj} \rangle) \in \mathbb{R}^{n \times n}$ is defined by $\langle a_{kk} \rangle = |a_{kk}|$ for $k = 1, 2, \cdots, n$, and $\langle a_{kj} \rangle = -|a_{kj}|$ for $k \neq j$, $k, j = 1, 2, \cdots, n$; an H_+-matrix if it is an H-matrix having positive diagonal elements; and a Q-matrix if the LCP(A,b) has a solution for any $b \in \mathbb{R}^n$. A sufficient condition for $A \in \mathbb{R}^{n \times n}$ to be a Q-matrix is that either A is an H_+-matrix [5] or A is a strictly copositive matrix [15]. In the former case, the LCP(A,b) always has a unique solution for every $b \in \mathbb{R}^n$. For a given matrix $A \in \mathbb{R}^{n \times n}$, let $F, G \in \mathbb{R}^{n \times n}$ be such that $A = F + G$ and F is nonsingular. Then (F, G) is called a splitting of the matrix A. The splitting (F, G) is called a convergent splitting if the spectral radius of the matrix $(F^{-1}G)$ is less than one, i.e., $\rho(F^{-1}G) < 1$. It is called an M-splitting if F is an M-matrix and $G \leq 0$; an H-splitting if $\langle F \rangle - |G|$ is an M-matrix; an H-compatible splitting if $\langle A \rangle = \langle F \rangle - |G|$; and a Q-splitting if F is a Q-matrix. In particular, the splitting (F, G) is called an H_+-splitting and

H_+-compatible splitting if it is an H-splitting and H-compatible splitting, respectively, with F an H_+-matrix.

The H_+-matrix concept plays a key role in the convergence analyses of asynchronous parallel iterative methods (see [11] Section 6.3). From [32] we know that $A \in \mathbb{R}^{n\times n}$ being an H_+-matrix is actually equivalent to $A \in \mathbb{R}^{n\times n}$ being a generalized diagonal dominant matrix subject to positive diagonal scaling.

The following lemma, proved in [21], will be frequently used in the rest of this paper.

Lemma 1. *[21] Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix, $D = \mathrm{diag}(A)$, and $A = D - B$. Then:*
(a) A is nonsingular;
(b) $|A^{-1}| \le \langle A\rangle^{-1}$; and
(c) $|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.

For all $p \in N_0$, let $\mathcal{I}_p = \sum_{i\in J(p)} E_i$ and $\mathcal{P}_p = \mathcal{I}_p(\mathcal{I}_p)^+$, where $E_i (i = 1, 2, \cdots, \alpha)$ are the weighting matrices, and $(\mathcal{I}_p)^+$ denotes the Moore-Penrose generalized inverse of the matrix $\mathcal{I}_p$. Then the following useful relations are straightforward.

Lemma 2. *For $\forall p \in N_0$, the following conclusions hold:*
(1) $0 \le \mathcal{I}_p \le \mathcal{P}_p \le I$; $(\mathcal{P}_p)^2 = \mathcal{P}_p$;
(2) $\mathcal{P}_p\mathcal{I}_p = \mathcal{I}_p\mathcal{P}_p = \mathcal{I}_p$;
(3) $(I - \mathcal{P}_p)\mathcal{I}_p = \mathcal{I}_p(I - \mathcal{P}_p) = 0$;
(4) $(I - \mathcal{P}_p)(I - \mathcal{I}_p) = (I - \mathcal{I}_p)(I - \mathcal{P}_p) = I - \mathcal{P}_p$.

Here, we remark that for $\forall p \in N_0$, $\mathcal{I}_p$ and $\mathcal{P}_p$ are nonnegative diagonal matrices. In particular, the diagonal elements of $\mathcal{P}_p$ are either 0 or 1, and they are equal to 0 if and only if the corresponding entries of $\mathcal{I}_p$ are equal to 0.

Define a positive integer sequence $\{m_t\}_{t\in N_0}$ by induction according to the following rule: $m_0 = 0$, and for $t = 0, 1, 2, \cdots$, m_{t+1} is the least positive integer such that

$$\bigcup_{m_t \le s(p) \le p < m_{t+1}} J(p) = \{1, 2, \cdots, \alpha\}.$$

Then in light of the definitions of the subset sequence $\{J(p)\}_{p\in N_0}$ and the nonnegative integer sequence $\{s(p)\}_{p\in N_0}$, the positive integer sequence $\{m_t\}_{t\in N_0}$ is well-defined and possesses the following properties:

Lemma 3. *For $\forall t \in N_0$, it holds that*
(1) $\mathcal{Q}_t \equiv \sum_{p=m_t}^{m_{t+1}-1} \mathcal{P}_p$ is a positive diagonal matrix;
(2) $\mathcal{S}_t \equiv \prod_{p=m_t}^{m_{t+1}-1} (I - \mathcal{P}_p) = 0$.

Proof. Evidently, for $\forall t, p \in N_0$, $\mathcal{Q}_t$, $\mathcal{S}_t$, $\mathcal{I}_p$ and $\mathcal{P}_p$ are nonnegative diagonal matrices, and the diagonal elements of $\mathcal{P}_p$ are either 0 or 1. Denote

$$\begin{cases} \mathcal{I}_p &= \mathrm{diag}([\mathcal{I}_p]_{11}, [\mathcal{I}_p]_{22}, \cdots, [\mathcal{I}_p]_{nn}), \\ \mathcal{Q}_t &= \mathrm{diag}([\mathcal{Q}_t]_{11}, [\mathcal{Q}_t]_{22}, \cdots, [\mathcal{Q}_t]_{nn}), \\ \mathcal{S}_t &= \mathrm{diag}([\mathcal{S}_t]_{11}, [\mathcal{S}_t]_{22}, \cdots, [\mathcal{S}_t]_{nn}), \\ \mathcal{P}_p &= \mathrm{diag}([\mathcal{P}_p]_{11}, [\mathcal{P}_p]_{22}, \cdots, [\mathcal{P}_p]_{nn}). \end{cases}$$

Now, we first verify (1). Suppose that for some $k_0 \in \{1, 2, \cdots, n\}$ and some $t_0 \in N_0$ it holds that $[\mathcal{Q}_{t_0}]_{k_0 k_0} = \sum_{p=m_{t_0}}^{m_{t_0+1}-1} [\mathcal{P}_p]_{k_0 k_0} = 0$. Then we have $[\mathcal{P}_p]_{k_0 k_0} = 0$, $p = m_{t_0}, m_{t_0}+1, \cdots, m_{t_0+1}-1$. Hence, $[\mathcal{I}_p]_{k_0 k_0} = 0$, $p = m_{t_0}, m_{t_0}+1, \cdots, m_{t_0+1}-1$, or in other words, $\sum_{i \in J(p)} e_{k_0}^{(i)} = 0$, $p = m_{t_0}, m_{t_0}+1, \cdots, m_{t_0+1}-1$. These equalities and the definition of the integer sequence $\{m_t\}_{t \in N_0}$ straightforwardly imply that $e_{k_0}^{(i)} = 0$, $i = 1, 2, \cdots, \alpha$, i.e., $\sum_{i=1}^{\alpha} e_{k_0}^{(i)} = 0$. However, this obviously contradicts to $\sum_{i=1}^{\alpha} E_i = I$. Therefore, (1) is valid.

To verify (2), we notice that $\mathcal{P}_p(I - \mathcal{P}_p) = 0$ holds for $\forall p \in N_0$. Then, by direct calculations we immediately obtain

$$\begin{aligned} \mathcal{Q}_t \mathcal{S}_t &= \left(\sum_{p=m_t}^{m_{t+1}-1} \mathcal{P}_p \right) \left(\prod_{p=m_t}^{m_{t+1}-1} (I - \mathcal{P}_p) \right) \\ &= \sum_{p=m_t}^{m_{t+1}-1} \left(\mathcal{P}_p (I - \mathcal{P}_p) \prod_{m_t \le q < m_{t+1}, q \ne p} (I - \mathcal{P}_q) \right) = 0. \end{aligned}$$

From (1) we see that $\mathcal{Q}_t$ is a nonsingular diagonal matrix. Therefore, $\mathcal{S}_t = 0$.

4. Convergence Theories

In this section, we will discuss the convergence of Method 2.1 and Method 2.2 for the nonsymmetric linear complementarity problems, respectively. First of all, we establish the convergence theorem for Method 2.1.

Theorem 1. *Let $M \in \mathbb{R}^{n \times n}$ be an H_+-matrix, $D = \mathrm{diag}(M)$ and $B = M - D$. Assume that for each $p \in N_0$ and $i \in \{1, 2, \cdots, \alpha\}$, $M = B_{i,p} + C_{i,p}$ is an H_+-compatible splitting satisfying $\mathrm{diag}(B_{i,p}) = \mathrm{diag}(M)$. Then the sequence $\{z^p\}_{p \in N_0}$ generated by Method 2.1 converges to the unique solution of the LCP(M,q), provided the relaxation parameter β satisfies*

$$0 < \beta < \frac{2}{1 + \rho(D^{-1}|B|)}.$$

Proof. Because $M \in \mathbb{R}^{n \times n}$ is an H_+-matrix, the LCP(M,q) has a unique solution $z^* \in \mathbb{R}^n$. Hence, it holds that

$$z^* \ge 0, \qquad Mz^* + q \ge 0, \qquad (z^*)^T (Mz^* + q) = 0.$$

Since for each $i \in J(p)$ and each $p \in N_0$, $M = B_{i,p} + C_{i,p}$ is an H_+-compatible splitting, we know that $B_{i,p}$ is an H_+-matrix, and therefore, the $LCP(B_{i,p}, q_{i,p})$ has a unique solution $\widehat{z}^{i,p}$, where $q_{i,p} = C_{i,p} z^{s_i(p)} + q$. That is to say, it holds that

$$\widehat{z}^{i,p} \ge 0, \qquad B_{i,p}\widehat{z}^{i,p} + q_{i,p} \ge 0, \qquad (\widehat{z}^{i,p})^T (B_{i,p}\widehat{z}^{i,p} + q_{i,p}) = 0.$$

Now, we claim that the following estimate holds:

$$|\widehat{z}^{i,p} - z^*| \le \langle B_{i,p} \rangle^{-1} |C_{i,p}| |z^{s_i(p)} - z^*|, \qquad \forall i \in J(p), \quad \forall p \in N_0. \tag{3}$$

In fact, by denoting $D_{i,p} = \mathrm{diag}(B_{i,p})$ and $\overline{B}_{i,p} = B_{i,p} - D_{i,p}$, we can verify (3) in four cases.

First, when the k-th element of z^*, $[z^*]_k$, is positive and $[B_{i,p}\widehat{z}^{i,p} + q_{i,p}]_k = 0$, it clearly holds that

$$[B_{i,p}(\widehat{z}^{i,p} - z^*) + C_{i,p}(z^{s_i(p)} - z^*)]_k = 0,$$

or

$$[D_{i,p}(\widehat{z}^{i,p}-z^*)]_k=[-\overline{B}_{i,p}(\widehat{z}^{i,p}-z^*)-C_{i,p}(z^{s_i(p)}-z^*)]_k.$$

Therefore,

$$[D_{i,p}|\widehat{z}^{i,p}-z^*|]_k\le[|\overline{B}_{i,p}||\widehat{z}^{i,p}-z^*|+|C_{i,p}||z^{s_i(p)}-z^*|]_k,$$

and

$$[\langle B_{i,p}\rangle|\widehat{z}^{i,p}-z^*|]_k\le[|C_{i,p}||z^{s_i(p)}-z^*|]_k. \tag{4}$$

Second, when $[z^*]_k>0$ and $[B_{i,p}\widehat{z}^{i,p}+q_{i,p}]_k>0$, it holds that $[z^{i,p}]_k=0$ and

$$\begin{cases}[\overline{B}_{i,p}\widehat{z}^{i,p}+C_{i,p}z^{s_i(p)}+q]_k & > & 0,\\ [\overline{B}_{i,p}z^*+C_{i,p}z^*+q]_k & = & -[D_{i,p}z^*]_k.\end{cases}$$

Subtracting these two inequalities we obtain

$$[\overline{B}_{i,p}(\widehat{z}^{i,p}-z^*)+C_{i,p}(z^{s_i(p)}-z^*)]_k>[D_{i,p}z^*]_k.$$

Therefore,

$$\begin{aligned}[D_{i,p}|\widehat{z}^{i,p}-z^*|]_k &= [D_{i,p}z^*]_k<|[\overline{B}_{i,p}(\widehat{z}^{i,p}-z^*)+C_{i,p}(z^{s_i(p)}-z^*)]_k|\\ &\le [|\overline{B}_{i,p}||\widehat{z}^{i,p}-z^*|+|C_{i,p}||z^{s_i(p)}-z^*|]_k,\end{aligned}$$

and hence, (4) also holds.

Third, when $[z^*]_k=0$ and $[\widehat{z}^{i,p}]_k>0$, (4) can be demonstrated analogously to the second case.

Fourth, when $[z^*]_k=0$ and $[\widehat{z}^{i,p}]_k=0$, (4) holds automatically.

Therefore, we can conclude that (4) holds for all $k\in\{1,2,\cdots,n\}$, all $i\in J(p)$ and all $p\in N_0$.

Since $B_{i,p}(i=1,2,\cdots,\alpha,p\in N_0)$ are H_+-matrices, we immediately know that (3) is valid.

From (1) and (2) we have

$$\begin{aligned}|z^{p+1}-z^*| &= \left|\sum_{i=1}^{\alpha}E_i(z^{i,p}-z^*)\right|\le\sum_{i=1}^{\alpha}E_i|z^{i,p}-z^*|\\ &\le \sum_{i\in J(p)}E_i|[\beta\widehat{z}^{i,p}+(1-\beta)z^{s_i(p)}]-z^*|+\sum_{i\notin J(p)}E_i|z^p-z^*|\\ &\le \sum_{i\in J(p)}E_i[\beta|\widehat{z}^{i,p}-z^*|+|1-\beta||z^{s_i(p)}-z^*|]+\sum_{i\notin J(p)}E_i|z^p-z^*|.\end{aligned}$$

Substituting (3) into the above inequality yields

$$|z^{p+1}-z^*|\le\sum_{i\in J(p)}E_iH_{i,p}|z^{s_i(p)}-z^*|+\sum_{i\notin J(p)}E_i|z^p-z^*|,\qquad p\in N_0, \tag{5}$$

where

$$H_{i,p}=\beta\langle B_{i,p}\rangle^{-1}|C_{i,p}|+|1-\beta|I,\qquad i=1,2,\cdots,\alpha,\quad p\in N_0 \tag{6}$$

are nonnegative matrices. Because $M\in\mathbb{R}^{n\times n}$ is an H_+-matrix, from Lemma 1 we have $\rho(D^{-1}|B|)<1$. By the continuity of the spectral radius of a matrix (see [32]), it holds that $\rho_\varepsilon\equiv\rho(D^{-1}|B|+\varepsilon ee^T)<1$ for a sufficiently small $\varepsilon>0$, where $e=(1,1,\cdots,1)^T\in\mathbb{R}^n$. Hence, in accordance with the Perron-Frobenius theorem (see [32]), there exists a positive vector $u_\varepsilon=([u_\varepsilon]_1,[u_\varepsilon]_2,\cdots,[u_\varepsilon]_n)^T\in\mathbb{R}^n$ such that $J_\varepsilon u_\varepsilon=\rho_\varepsilon u_\varepsilon$, where $J_\varepsilon=(D^{-1}|B|+\varepsilon ee^T)$ is a positive matrix. Noticing that the region of the relaxation parameter β implies that $|1-\beta|+\beta\rho(D^{-1}|B|)<1$. Again, by the continuity of the spectral radius we can let ε be small enough such that $\theta_\varepsilon\equiv|1-\beta|+\beta\rho_\varepsilon<1$.

Since $M = B_{i,p} + C_{i,p} (i = 1, 2, \cdots, \alpha, p \in N_0)$ are H_+-compatible splittings, that is, it holds that

$$\langle M \rangle = \langle B_{i,p} \rangle - |C_{i,p}|, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0,$$

we have

$$\begin{aligned} H_{i,p} u_\varepsilon &= \left[\beta \langle B_{i,p} \rangle^{-1} |C_{i,p}| + |1-\beta| I\right] u_\varepsilon \\ &= \left[\beta \langle B_{i,p} \rangle^{-1} (\langle B_{i,p} \rangle - \langle M \rangle) + |1-\beta| I\right] u_\varepsilon \\ &= (\beta + |1-\beta|) u_\varepsilon - \beta \langle B_{i,p} \rangle^{-1} (D - |B|) u_\varepsilon \\ &\leq (\beta + |1-\beta|) u_\varepsilon - \beta \langle B_{i,p} \rangle^{-1} D (I - J_\varepsilon) u_\varepsilon \\ &= (\beta + |1-\beta|) u_\varepsilon - \beta \langle B_{i,p} \rangle^{-1} D (1 - \rho_\varepsilon) u_\varepsilon \\ &\leq (|1-\beta| + \beta \rho_\varepsilon) u_\varepsilon, \end{aligned}$$

where the last inequality is got from

$$\langle B_{i,p} \rangle^{-1} D \geq I, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0,$$

which is valid because $B_{i,p} (i = 1, 2, \cdots, \alpha, p \in N_0)$ are H_+-matrices and

$$\mathrm{diag}(B_{i,p}) = D, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0$$

implies

$$\langle B_{i,p} \rangle \leq D, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0.$$

Thus,

$$H_{i,p} u_\varepsilon \leq \theta_\varepsilon u_\varepsilon, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0, \tag{7}$$

where $\theta_\varepsilon \in [0, 1)$ and $u_\varepsilon \in \mathbb{R}^n$ is the positive vector defined above.

Now, let $\delta > 0$ be such that $|z^0 - z^*| \leq \delta u_\varepsilon$. By applying (5), (6) and (7) we can directly verify the validity of the inequalities $|z^p - z^*| \leq \delta u_\varepsilon$, $p = 0, 1, 2, \cdots$, through induction. Moreover, we can assert that it has

$$|z^p - z^*| \leq \Gamma^t \delta u_\varepsilon, \qquad \forall p \geq m_t, \quad \forall t \in N_0, \tag{8}$$

where

$$\Gamma = 1 - (1 - \theta_\varepsilon) e_{\min}, \quad e_{\min} = \min\{e_k^{(i)} > 0 \mid k = 1, 2, \cdots, n, \quad i = 1, 2, \cdots, \alpha\}, \tag{9}$$

and $e_k^{(i)}$ is the k-th diagonal element of the weighting matrix E_i.

In fact, (8) and (9) is trivial when $t = 0$. Suppose that (8) and (9) is true for some $t \geq 1$, we now prove that it is also true for $t + 1$. From $s_i(p) \leq p (i = 1, 2, \cdots, \alpha)$ and the induction hypothesis we easily know that $|z^{s_i(p)} - z^*| \leq \Gamma^t \delta u_\varepsilon (i = 1, 2, \cdots, \alpha)$. Therefore, by making use of (5) and (7) we obtain

$$\begin{aligned} |z^{p+1} - z^*| &\leq \sum_{i \in J(p)} E_i \theta_\varepsilon \Gamma^t \delta u_\varepsilon + \sum_{i \notin J(p)} E_i |z^p - z^*| \\ &= \mathcal{I}_p \theta_\varepsilon \Gamma^t \delta u_\varepsilon + (I - \mathcal{I}_p) |z^p - z^*|. \end{aligned} \tag{10}$$

For $\forall p \geq m_{t+1}$, from (10) and Lemma 2 we have

$$\begin{aligned} (I - \mathcal{P}_p) |z^{p+1} - z^*| &\leq (I - \mathcal{P}_p) \left[\mathcal{I}_p \theta_\varepsilon \Gamma^t \delta u_\varepsilon + (I - \mathcal{I}_p) |z^p - z^*|\right] \\ &= (I - \mathcal{P}_p) |z^p - z^*| \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathcal{P}_p |z^{p+1} - z^*| &\leq \mathcal{P}_p \left[\mathcal{I}_p \theta_\varepsilon \Gamma^t \delta u_\varepsilon + (I - \mathcal{I}_p) |z^p - z^*|\right] \\ &= \mathcal{I}_p \theta_\varepsilon \Gamma^t \delta u_\varepsilon + (\mathcal{P}_p - \mathcal{I}_p) |z^p - z^*| \\ &\leq (\mathcal{P}_p - (1 - \theta_\varepsilon) \mathcal{I}_p) \Gamma^t \delta u_\varepsilon, \end{aligned} \tag{12}$$

where the last inequality is obtained by the induction hypothesis, again.

(11) and (12) immediately lead to

$$\begin{aligned} |z^{p+1}-z^*| &= \mathcal{P}_p|z^{p+1}-z^*|+(I-\mathcal{P}_p)|z^{p+1}-z^*| \\ &\le (\mathcal{P}_p-(1-\theta_\varepsilon)\mathcal{I}_p)\Gamma^t\delta u_\varepsilon+(I-\mathcal{P}_p)|z^p-z^*|. \end{aligned}$$

Since for $\forall p \in N_0$, $\mathcal{P}_p = 0$ if and only if $\mathcal{I}_p = 0$, by using Lemma 2(1)-(2) we see that $(\mathcal{P}_p-(1-\theta_\varepsilon)\mathcal{I}_p) \le \Gamma\mathcal{P}_p$ $(\forall p \in N_0)$. Therefore,

$$|z^{p+1}-z^*| \le \mathcal{P}_p\Gamma^{t+1}\delta u_\varepsilon+(I-\mathcal{P}_p)|z^p-z^*|, \qquad \forall p \ge m_{t+1}. \tag{13}$$

Let $\overline{p}_k = \max\{q \mid [\mathcal{P}_q]_{kk} = 1, q = m_t, m_{t+1}, \cdots, p\}$. Because $[\mathcal{Q}_t]_{kk}$ is a positive integer by Lemma 3(1), we know that the positive integer $\overline{p}_k$ is well-defined. Recalling that $[\mathcal{P}_p]_{kk} \in \{0,1\}$ $(k = 1, 2, \cdots, n,\ p \in N_0)$, based on (13) we have

$$|[z^{p+1}]_k-[z^*]_k| \le [\mathcal{P}_{\overline{p}_k}]_{kk}\Gamma^{t+1}\delta[u_\varepsilon]_k+(1-[\mathcal{P}_{\overline{p}_k}]_{kk})|[z^{\overline{p}_k}]_k-[z^*]_k| = \Gamma^{t+1}\delta[u_\varepsilon]_k$$

for all $k \in \{1, 2, \cdots, n\}$ and all $p \ge m_{t+1}-1$. This shows that (8) and (9) is also valid for $t+1$. By induction, we have confirmed (8) and (9).

Because $\Gamma \in [0,1)$, by (8) and (9) we finally get $|z^p-z^*| \le \Gamma^t\delta u_\varepsilon \to 0 (p \to \infty)$, or in other words, $z^p \to z^* (p \to \infty)$. This completes the proof of this theorem.

We now give the convergence analysis of Method 2.2.

Theorem 2. *Let $M \in \mathbb{R}^{n\times n}$ be an H_+-matrix, $D = \mathrm{diag}(M)$ and $B = M - D$. Assume that for every $p \in N_0$,*

$$|L_{i,p}|+|W_{i,p}| = |B|, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0.$$

Then the sequence $\{z^p\}_{p\in N_0}$ generated by Method 2.2 converges to the unique solution of the LCP(M,q), provided the relaxation parameters γ and ω satisfy

$$0 \le \gamma \le \omega, \qquad 0 < \omega < \frac{2}{1+\rho(D^{-1}|B|)}.$$

Proof. Because, for each $i \in \{1, 2, \cdots, \alpha\}$ and each $p \in N_0$,

$$\begin{cases} \langle B_{i,p}\rangle &= \frac{1}{\omega}(D-\gamma|L_{i,p}|) := \widehat{\mathcal{B}}_{i,p}, \\ |C_{i,p}| &\le \frac{1}{\omega}[|1-\omega|D+(\omega-\gamma)|L_{i,p}|+\omega|W_{i,p}|] := \widehat{\mathcal{C}}_{i,p}, \end{cases}$$

we see that $\widehat{\mathcal{B}}_{i,p}$ is an M-matrix and $\widehat{\mathcal{C}}_{i,p}$ a nonnegative matrix. Moreover, it holds that

$$\begin{aligned} \widehat{\mathcal{B}}_{i,p}-\widehat{\mathcal{C}}_{i,p} &= \frac{1}{\omega}[(1-|1-\omega|)D-\omega(|L_{i,p}|+|W_{i,p}|)] \\ &= \frac{1}{\omega}[(1-|1-\omega|)D-\omega|B|] := \widehat{M}. \end{aligned}$$

Since $\rho(D^{-1}|B|) < 1$ when $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix, we can easily verify that $\widehat{M} \in \mathbb{R}^{n\times n}$ is also an H_+-matrix. Analogously to the proof of Theorem 1, we can demonstrate that $\{z^p\}_{p\in N_0}$ converges to z^*, the unique solution of the LCP(M,q).

5. Block and Multi-Parameter Variants

For the convenience of practical computation, in this section, we further generalize the asynchronous multisplitting AOR Method 2.2 and define its block and multi-parameter variants. For this purpose, we separate the number set $\{1, 2, \cdots, n\}$ into α nonempty subsets $J_i (i = 1, 2, \cdots, \alpha)$ such that $\cup_{i=1}^{\alpha} J_i = \{1, 2, \cdots, n\}$. Corresponding to this separation, for

$i \in \{1, 2, \cdots, \alpha\}$ we introduce matrices

$$
\begin{aligned}
L_{i,p} &= (\mathcal{L}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{L}_{kj}^{(i,p)} = \begin{cases} l_{kj}^{(i,p)}, & \text{for } k,j \in J_i \text{ and } k > j, \\ 0, & \text{otherwise}, \end{cases} \\
U_{i,p} &= (\mathcal{U}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{U}_{kj}^{(i,p)} = \begin{cases} u_{kj}^{(i,p)}, & \text{for } k,j \in J_i \text{ and } k < j, \\ 0, & \text{otherwise}, \end{cases} \\
W_{i,p} &= (\mathcal{W}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{W}_{kj}^{(i,p)} = \begin{cases} 0, & \text{for } k = j, \\ w_{kj}^{(i,p)}, & \text{otherwise}, \end{cases} \\
E_i &= \operatorname{diag}(e_k^{(i)}) \in \mathbb{R}^{n\times n}, \quad e_k^{(i)} = \begin{cases} e_k^{(i)} \geq 0, & \text{for } k \in J_i, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}
$$

Evidently, for $\forall p \in N_0$, $L_{i,p}(i = 1, 2, \cdots, \alpha)$ are strictly lower triangular matrices, $U_{i,p}(i = 1, 2, \cdots, \alpha)$ are strictly upper triangular matrices, $W_{i,p}(i = 1, 2, \cdots, \alpha)$ are zero-diagonal matrices, and $E_i(i = 1, 2, \cdots, \alpha)$ are nonnegative diagonal matrices. For the system matrix $M \in \mathbb{R}^{n\times n}$ of the LCP(M,q), let $D = \operatorname{diag}(M)$ be nonsingular, $B = M - D$, and the matrices $L_{i,p}$, $U_{i,p}$ and $W_{i,p}$, $i = 1, 2, \cdots, \alpha$, $p \in N_0$, satisfy

$$M = D + L_{i,p} + U_{i,p} + W_{i,p}, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0.$$

Then, we can establish the following block and multi-parameter variant of Method 2.2.

Method 5.1. *Given an initial vector $z^0 = ([z^0]_1, [z^0]_2, \cdots, [z^0]_n)^T \in \mathbb{R}^n$. Suppose that we have obtained approximate solutions $z^1, z^2, \cdots, z^p$ of the LCP(M,q). Then the next approximate solution $z^{p+1} = ([z^{p+1}]_1, [z^{p+1}]_2, \cdots, [z^{p+1}]_n)^T$ of the LCP(M,q) is computed according to the formulas:*

$$[z^{p+1}]_k = \sum_{i=1}^{\alpha} e_k^{(i)} [z^{i,p}]_k, \qquad k = 1, 2, \cdots, n,$$

and

$$[z^{i,p}]_k = \begin{cases} [\widehat{z}^{i,p}]_k, & \text{if } i \in J(p), \\ [z^p]_k, & \text{otherwise}, \end{cases} \qquad k = 1, 2, \cdots, n,$$

where for $k \in J_i$,

$$
\begin{cases}
[\widehat{z}^{i,p}]_k = \begin{cases} 0, & \text{if} \quad \gamma_2 \sum_{j=1}^{k-1} u_{kj}^{(i,p)} ([\widehat{z}^{i,p}]_j - [\overline{z}^{i,p}]_j) \\ & \quad + \omega_2 [M\overline{z}^{i,p} + q]_k > m_{kk} [\overline{z}^{i,p}]_k, \\ [\overline{z}^{i,p}]_k + \frac{\gamma_2}{m_{kk}} \sum_{j=1}^{k-1} u_{kj}^{(i,p)} ([\overline{z}^{i,p}]_j - [\widehat{z}^{i,p}]_j) - \frac{\omega_2}{m_{kk}} [M\overline{z}^{i,p} + q]_k, \\ & \quad \text{otherwise}, \end{cases} \\
[\overline{z}^{i,p}]_k = \begin{cases} 0, & \text{if} \quad \gamma_1 \sum_{j=1}^{k-1} l_{kj}^{(i,p)} ([\overline{z}^{i,p}]_j - [z^{s_i(p)}]_j) \\ & \quad + \omega_1 [M z^{s_i(p)} + q]_k > m_{kk} [z^{s_i(p)}]_k, \\ [z^{s_i(p)}]_k + \frac{\gamma_1}{m_{kk}} \sum_{j=1}^{k-1} l_{kj}^{(i,p)} ([z^{s_i(p)}]_j - [\overline{z}^{i,p}]_j) - \frac{\omega_1}{m_{kk}} [M z^{s_i(p)} + q]_k, \\ & \quad \text{otherwise}. \end{cases}
\end{cases}
$$

Here, $\gamma_j, j = 1, 2$, are relaxation factors, and $\omega_j (\neq 0)$, $j = 1, 2$, are acceleration factors.

In Method 5.1, two relaxation sweeps are introduced within each iteration, and each sweep possibly includes its own pair of relaxation parameters. Therefore, this new method covers all the known and generates a lot of novel practical and efficient relaxed asynchronous multisplitting

block methods following different choices of the relaxation parameters. For details, one can refer to case (c) in the next section. Moreover, suitable adjustments of the relaxation parameters can greatly improve the convergence properties of these new asynchronous multisplitting block multi-parameter relaxation methods. We have the following convergence theorems for these new methods.

Theorem 3. *Let $M \in \mathbb{R}^{n\times n}$ be an H_+-matrix, $D = \mathrm{diag}(M)$ and $B = M - D$. Assume that for every $p \in N_0$,*

$$|L_{i,p}| + |U_{i,p}| + |W_{i,p}| = |B|, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0. \tag{14}$$

Then the sequence $\{z^p\}_{p\in N_0}$ generated by Method 5.1 converges to the unique solution of the LCP(M,q), provided the relaxation parameters γ_j and ω_j, $j = 1, 2$, satisfy

$$0 \le \gamma_j \le \omega_j, \qquad 0 < \omega_j < \frac{2}{1 + \rho(D^{-1}|B|)}, \qquad j = 1, 2. \tag{15}$$

Proof. By introducing matrices

$$\begin{cases} B_{i,p}(\gamma_1, \omega_1) &= \frac{1}{\omega_1}(D + \gamma_1 L_{i,p}), \\ C_{i,p}(\gamma_1, \omega_1) &= \frac{1}{\omega_1}((\omega_1 - 1)D + (\omega_1 - \gamma_1)L_{i,p} + \omega_1(U_{i,p} + W_{i,p})), \end{cases} \tag{16}$$

and

$$\begin{cases} B'_{i,p}(\gamma_2, \omega_2) &= \frac{1}{\omega_2}(D + \gamma_2 U_{i,p}), \\ C'_{i,p}(\gamma_2, \omega_2) &= \frac{1}{\omega_2}((\omega_2 - 1)D + (\omega_2 - \gamma_2)U_{i,p} + \omega_2(L_{i,p} + W_{i,p})); \end{cases} \tag{17}$$

analogously to the derivation of (3) in the proof of Theorem 1, we can obtain

$$\begin{cases} |\overline{z}^{i,p} - z^*| &\le \langle B_{i,p}(\gamma_1, \omega_1)\rangle^{-1}|C_{i,p}(\gamma_1, \omega_1)||z^{s_i(p)} - z^*|, \\ |\widehat{z}^{i,p} - z^*| &\le \langle B'_{i,p}(\gamma_2, \omega_2)\rangle^{-1}|C'_{i,p}(\gamma_2, \omega_2)||\overline{z}^{i,p} - z^*|, \end{cases} \qquad i \in J(p), \quad p \in N_0,$$

where $z^* \in \mathbb{R}^n$ is the unique solution of the LCP(M,q). Therefore, the iteration sequence $\{z^p\}$ produced by Method 5.1 obeys

$$|z^{p+1} - z^*| \le \sum_{i\in J(p)} E_i H_{i,p}(\gamma_1, \omega_1; \gamma_2, \omega_2)|z^{s_i(p)} - z^*| + \sum_{i\notin J(p)} E_i|z^p - z^*|, \quad p \in N_0,$$

where for $i = 1, 2, \cdots, \alpha$ and $p \in N_0$,

$$H_{i,p}(\gamma_1, \omega_1; \gamma_2, \omega_2) = \langle B'_{i,p}(\gamma_2, \omega_2)\rangle^{-1}|C'_{i,p}(\gamma_2, \omega_2)|\langle B_{i,p}(\gamma_2, \omega_2)\rangle^{-1}|C_{i,p}(\gamma_2, \omega_2)|. \tag{18}$$

Again, from the proof of Theorem 1, we know that to prove the convergence of Method 5.1, under the conditions of Theorem 3 we only need to demonstrate that

$$H_{i,p}(\gamma_1, \omega_1; \gamma_2, \omega_2) \ge 0, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0, \tag{19}$$

and there exist a real constant $\overline{\theta}_\varepsilon \in [0, 1)$ and a positive vector $\overline{u}_\varepsilon \in \mathbb{R}^n$ such that

$$H_{i,p}(\gamma_1, \omega_1; \gamma_2, \omega_2)\overline{u}_\varepsilon \le \overline{\theta}_\varepsilon \overline{u}_\varepsilon, \qquad i = 1, 2, \cdots, \alpha, \quad p \in N_0. \tag{20}$$

Because for each $i \in \{1, 2, \cdots, \alpha\}$ and each $p \in N_0$,

$$\begin{cases} \langle B_{i,p}(\gamma_1, \omega_1)\rangle &= \frac{1}{\omega_1}(D - \gamma_1|L_{i,p}|) := \widehat{\mathcal{B}}_{i,p}(\gamma_1, \omega_1) \\ \langle B'_{i,p}(\gamma_2, \omega_2)\rangle &= \frac{1}{\omega_2}(D - \gamma_2|U_{i,p}|) := \widehat{\mathcal{B}}'_{i,p}(\gamma_2, \omega_2) \\ |C_{i,p}(\gamma_1, \omega_1)| &= \frac{1}{\omega_1}[|\omega_1 - 1|D + (\omega_1 - \gamma_1)|L_{i,p}| + \omega_1(|U_{i,p}| + |W_{i,p}|)] := \widehat{\mathcal{C}}_{i,p}(\gamma_1, \omega_1) \\ |C'_{i,p}(\gamma_2, \omega_2)| &= \frac{1}{\omega_2}[|\omega_2 - 1|D + (\omega_2 - \gamma_2)|U_{i,p}| + \omega_2(|L_{i,p}| + |W_{i,p}|)] := \widehat{\mathcal{C}}'_{i,p}(\gamma_2, \omega_2), \end{cases} \tag{21}$$

we see that $\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)$ and $\widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)$ are M-matrices, and $\widehat{\mathcal{C}}_{i,p}(\gamma_1,\omega_1)$ and $\widehat{\mathcal{C}}'_{i,p}(\gamma_2,\omega_2)$ are nonnegative matrices, respectively. Hence, $\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\geq 0$, $\widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)^{-1}\geq 0$, and $H_{i,p}(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is nonnegative. Moreover, it follows from (14) that

$$\left\{\begin{array}{lcl}\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)-\widehat{\mathcal{C}}_{i,p}(\gamma_1,\omega_1) &=& \frac{1}{\omega_1}[(1-|1-\omega_1|)D-\omega_1|B|]:=\widehat{\mathcal{M}}(\gamma_1,\omega_1),\\ \widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)-\widehat{\mathcal{C}}'_{i,p}(\gamma_2,\omega_2) &=& \frac{1}{\omega_2}[(1-|1-\omega_2|)D-\omega_2|B|]:=\widehat{\mathcal{M}}(\gamma_2,\omega_2),\end{array}\right.$$

where

$$\widehat{\mathcal{M}}(\gamma,\omega)=\frac{1}{\omega}[(1-|1-\omega|)D-\omega|B|]. \tag{22}$$

Therefore,

$$\left\{\begin{array}{lcl}\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\widehat{\mathcal{C}}_{i,p}(\gamma_1,\omega_1) &=& I-\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\widehat{\mathcal{M}}(\gamma_1,\omega_1),\\ \widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)^{-1}\widehat{\mathcal{C}}'_{i,p}(\gamma_2,\omega_2) &=& I-\widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)^{-1}\widehat{\mathcal{M}}(\gamma_2,\omega_2).\end{array}\right. \tag{23}$$

Since $M\in\mathbb{R}^{n\times n}$ is an H_+-matrix, from Lemma 1 we have $\rho(D^{-1}|B|)<1$. Therefore, there exists a positive vector $\overline{u}_\varepsilon\in\mathbb{R}^n$ such that

$$J_\varepsilon\overline{u}_\varepsilon=\rho_\varepsilon\overline{u}_\varepsilon, \tag{24}$$

where

$$J_\varepsilon=(D^{-1}|B|+\varepsilon ee^T),\quad 0<\varepsilon\in\mathbb{R}^1,\quad e=(1,1,\cdots,1)^T\in\mathbb{R}^n,\quad \rho_\varepsilon=\rho(D^{-1}|B|+\varepsilon ee^T)<1.$$

Furthermore, by (22) we can immediately get

$$\begin{array}{rcl}\widehat{\mathcal{M}}(\gamma,\omega) &\geq& \frac{1}{\omega}D[(1-|1-\omega|)I-\omega J_\varepsilon]\overline{u}_\varepsilon\\ &=& \frac{1}{\omega}D(1-|1-\omega|-\omega\rho_\varepsilon)\overline{u}_\varepsilon.\end{array} \tag{25}$$

Noticing that

$$\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)\leq\frac{1}{\omega_1}D,\quad \widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)\leq\frac{1}{\omega_2}D,$$

we have

$$\frac{1}{\omega_1}D\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\geq I,\quad \frac{1}{\omega_2}D\widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)^{-1}\geq I. \tag{26}$$

Now, by applying (24)-(26) to the first identity in (23), considering that (15) implies $|1-\omega_j|+\omega_j\rho(D^{-1}|B|)<1(j=1,2)$, and restricting ε small enough such that $|1-\omega_j|+\omega_j\rho_\varepsilon<1(j=1,2)$, we obtain

$$\begin{array}{rcl}\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\widehat{\mathcal{C}}_{i,p}(\gamma_1,\omega_1)\overline{u}_\varepsilon &=& \left(I-\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\widehat{\mathcal{M}}(\gamma_1,\omega_1)\right)\overline{u}_\varepsilon\\ &\leq& \left(I-\widehat{\mathcal{B}}_{i,p}(\gamma_1,\omega_1)^{-1}\left[\frac{1}{\omega_1}D(1-|1-\omega_1|-\omega_1\rho_\varepsilon)\right]\right)\overline{u}_\varepsilon\\ &\leq& (1-(1-|1-\omega_1|-\omega_1\rho_\varepsilon))\overline{u}_\varepsilon\\ &=& (|1-\omega_1|+\omega_1\rho_\varepsilon)\overline{u}_\varepsilon:=\overline{\vartheta}_\varepsilon(\omega_1)\overline{u}_\varepsilon,\end{array} \tag{27}$$

where $\overline{\vartheta}_\varepsilon(\omega)=|1-\omega|+\omega\rho_\varepsilon$. Similarly, by applying (24)-(26) to the second identity in (23) we obtain

$$\widehat{\mathcal{B}}'_{i,p}(\gamma_2,\omega_2)^{-1}\widehat{\mathcal{C}}'_{i,p}(\gamma_2,\omega_2)\overline{u}_\varepsilon\leq\overline{\vartheta}_\varepsilon(\omega_2)\overline{u}_\varepsilon. \tag{28}$$

Define $\overline{\theta}_\varepsilon=\overline{\vartheta}_\varepsilon(\omega_1)\overline{\vartheta}_\varepsilon(\omega_2)$. Then it is clear that $\overline{\theta}_\varepsilon\in[0,1)$. After substituting (21) into (18), and using (27) and (28), we immediately know that

$$H_{i,p}(\gamma_1,\omega_1;\gamma_2,\omega_2)\overline{u}_\varepsilon\leq\overline{\vartheta}_\varepsilon(\omega_2)\overline{\vartheta}_\varepsilon(\omega_1)\overline{u}_\varepsilon=\overline{\theta}_\varepsilon\overline{u}_\varepsilon.$$

Up to now, we have shown that under the conditions of Theorem 3, there exist a real constant $\overline{\theta}_\varepsilon\in[0,1)$ and a positive vector $\overline{u}_\varepsilon\in\mathbb{R}^n$ such that (20) holds. This completes our proof.

Theorem 4. *If the conditions of Theorem 3 are satisfied, then we have*

(a) the asynchronous multisplitting symmetric block AOR method, given by the choices of relaxation parameters $\gamma_1 = \omega_1 = \omega$ *and* $\gamma_2 = \omega_2 = 0$ *in Method 5.1, converges to the unique solution of the LCP(M,q), provided the relaxation parameters* γ *and* ω *satisfy* $0 \le \gamma \le \omega$ *and* $0 < \omega < \frac{2}{1+\rho(D^{-1}|B|)}$;

(b) the asynchronous multisplitting unsymmetric block SOR method, given by the choices of relaxation parameters $\gamma_1 = \omega_1 = \gamma$ *and* $\gamma_2 = \omega_2 = \omega$ *in Method 5.1, converges to the unique solution of the LCP(M,q), provided the relaxation parameters* γ *and* ω *satisfy* $0 < \gamma, \omega < \frac{2}{1+\rho(D^{-1}|B|)}$;

(c) the asynchronous multisplitting symmetric block SOR method, given by the choices of relaxation parameters $\gamma_1 = \omega_1 = \gamma_2 = \omega_2 = \omega$ *in Method 5.1, converges to the unique solution of the LCP(M,q), provided the relaxation parameter* ω *satisfies* $0 < \omega < \frac{2}{1+\rho(D^{-1}|B|)}$; *and*

(d) the asynchronous multisplitting symmetric block Gauss-Seidel method, given by the choices of relaxation parameters $\gamma_1 = \omega_1 = \gamma_2 = \omega_2 = 1$ *in Method 5.1, converges to the unique solution of the LCP(M,q).*

6. Numerical Experiments

We consider the linear complementarity problem LCP(M,q) with the following system matrix $M \in \mathbb{R}^{n\times n}$ corresponding to the Laplacian 5-point finite difference operator and given vector $q \in \mathbb{R}^n$:

$$M = \begin{pmatrix} R & -I & & & \\ -I & R & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & R & -I \\ & & & -I & R \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad q = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{pmatrix} \in \mathbb{R}^n,$$

respectively, where $R = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{\overline{n}\times\overline{n}}$, $I \in \mathbb{R}^{\overline{n}\times\overline{n}}$ is the identity matrix, and $n = \overline{n}^2$. This problem may arise from finite difference discretization at equidistant grid of a free boundary value problem about the flow of water through a porous dam (see [19]). Note that $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix. Therefore, the LCP(M,q) has a unique solution.

The tested methods in our numerical experiments are as follows:

(a) the sequential relaxation methods in [14], [16] and [28]:

Method	γ_1	ω_1	γ_2	ω_2	Description
SOR	ω	ω	0	0	the successive overrelaxation method
SSOR	ω	ω	ω	ω	the symmetric SOR method
USOR	γ	γ	ω	ω	the unsymmetric SOR method
AOR	γ	ω	0	0	the accelerated overrelaxation method
SAOR	γ	ω	γ	ω	the symmetric AOR method

(b) the synchronous multisplitting relaxation methods in [4] and [27]:

Method	γ_1	ω_1	γ_2	ω_2	Description
MSOR	ω	ω	0	0	the multisplitting SOR method
MSSOR	ω	ω	ω	ω	the multisplitting SSOR method
MUSOR	γ	γ	ω	ω	the multisplitting USOR method
MAOR	γ	ω	0	0	the multisplitting AOR method
MSAOR	γ	ω	γ	ω	the multisplitting SAOR method

(c) the new asynchronous multisplitting block relaxation methods:

Method	γ_1	ω_1	γ_2	ω_2	Description
AMSOR	ω	ω	0	0	the asynchronous MSOR method
AMSSOR	ω	ω	ω	ω	the asynchronous MSSOR method
AMUSOR	γ	γ	ω	ω	the asynchronous MUSOR method
AMAOR	γ	ω	0	0	the asynchronous MAOR method
AMSAOR	γ	ω	γ	ω	the asynchronous MSAOR method

We run our programs as PVM applications on an SGI Power Challenge multiprocessor computer. It consists of four 75 MHz TFP 64-bit RISC processors. These CMOS processors each delivers a peak theoretical performance of 0.3 GFLOPS. The data cache size is 16 Kbytes. The code is written by C^{++}-language, and uses neither an explicit compiler directive, nor a automatic compiler parallelization.

In our experiments, we take

$$J_i = \{\overline{n}_{i-1}\overline{n}+1, \overline{n}_{i-1}\overline{n}+2, \cdots, \overline{n}_{i+1}\overline{n}\}, \qquad i = 1, 2, \cdots, \alpha,$$

where $\overline{n}_i = Int\left(\frac{i\overline{n}}{\alpha+1}\right)$, $i = 1, 2, \cdots, \alpha$, and

$$L_{i,p} = (\mathcal{L}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{L}_{kj}^{(i,p)} = \begin{cases} m_{kj}, & \text{for } k,j \in J_i \text{ and } k > j, \\ 0, & \text{otherwise,} \end{cases}$$

$$U_{i,p} = (\mathcal{U}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{U}_{kj}^{(i,p)} = \begin{cases} m_{kj}, & \text{for } k,j \in J_i \text{ and } k < j, \\ 0, & \text{otherwise,} \end{cases}$$

$$W_{i,p} = (\mathcal{W}_{kj}^{(i,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{W}_{kj}^{(i,p)} = \begin{cases} 0, & \text{for } k = j, \\ 0, & \text{for } k,j \in J_i, \\ m_{kj}, & \text{otherwise,} \end{cases}$$

$$E_i = \mathrm{diag}(e_k^{(i)}) \in \mathbb{R}^{n\times n}, \quad e_k^{(i)} = \begin{cases} 1, & \text{for } 1 \le k \le \overline{n}_1\overline{n}, \quad i = 1, \\ 0.5, & \text{for } \overline{n}_{i-1}\overline{n}+1 \le k \le \overline{n}_i\overline{n}, 2 \le i \le \alpha, \\ 0.5, & \text{for } \overline{n}_i\overline{n}+1 \le k \le \overline{n}_{i+1}\overline{n}, 1 \le i \le \alpha-1, \\ 1, & \text{for } \overline{n}_\alpha\overline{n}+1 \le k \le n, \quad i = \alpha. \end{cases}$$

These splittings allow that the i-th processor of the multiprocessor system solves only the variables located in J_i. Hence, the computation of a single iteraton takes on one processor about $\left(\frac{4(\alpha+2)}{3(\alpha+1)^2}T_{seq}\right)$ time, where T_{seq} represents the sequential time of computing the iteration. Furthermore, since the communication overheads add to the execution time, $\mathrm{SP}_{exp} = \frac{3(\alpha+1)^2}{4(\alpha+2)}$ represents an upper bound for the expected speed-up of the implementations (see [7]). When $\alpha = 3$, we have $\mathrm{SP}_{exp} = 2.4$. Here, we have neglected the sparsity of the system matrix $M \in \mathbb{R}^{n\times n}$, and assumed that the related sequential, synchronous and asynchronous parallel relaxation methods have the same convergence speed.

The LCP(M,q) of various sizes are tested with the processor numbers being 2, 3 and 4, respectively. All our computations are started from an initial vector having all components equal to 40.0, and terminated once the current iteration z^p obey

$$\frac{|(z^p)^T(Mz^p+q)|}{|(z^0)^T(Mz^0+q)|} \le 10^{-7}.$$

Table 1: CPUs and SPs for the SOR-like methods

ω		0.6	0.7	0.9	1.1	1.3	1.5
SOR	CPU	91.92	73.14	48.09	32.51	21.38	13.23
SSOR	CPU	45.98	37.03	24.27	16.26	10.72	∞
MSOR	CPU	48.12	37.70	24.92	16.82	11.10	6.97
	SP	1.91	1.94*	1.93	1.93	1.93	1.90
MSSOR	CPU	23.23	19.06	12.55	8.47	5.62	∞
	SP	1.98*	1.94	1.93	1.92	1.91	-
AMSOR	CPU	42.41	33.59	22.09	14.99	10.04	6.34
	SP	2.17	2.18*	2.18*	2.17	2.13	2.09
AMSSOR	CPU	21.30	16.80	11.26	7.70	5.12	∞
	SP	2.16	2.20*	2.16	2.11	2.09	-

This stopping criterion is reasonable because the constructions of the above-described methods ensure that $z^p \geq 0$ and $Mz^p + q \geq 0$ ($\forall p \in N_0$) are satisfied at least approximately.

From the numerical computations we see that in the sense of CPU time and parallel efficiency, asynchronous multisplitting relaxation methods are superior to the corresponding synchronous multisplitting relaxation methods, the multisplitting accelerated overrelaxation methods are superior to the corresponding multisplitting successive overrelaxation methods, and the multisplitting symmetric relaxation methods are superior to the corresponding multisplitting relaxation methods. In particular, the advantages of the AMAOR and AMSAOR methods over the AMSOR, AMSSOR and AMUSOR methods, respectively, are, roughly speaking, that (i) when the latter ones diverge, the former ones can still converge; (ii) when the latter ones converge, the former ones converge faster with higher parallel efficiency; and (iii) the former ones are less sensitive to the relaxation parameters and they have larger convergence domains than the latter ones. Therefore, we can conclude that the new asynchronous multisplitting relaxation methods have better numerical properties than both their corresponding synchronous and sequential alternatives.

Because the experiments about different problem sizes and different processor numbers show a very analogous numerical behaviour, as a representative, only for the case that $n = 4900$ and $\alpha = 3$, some of the numerical results are listed in Tables 1-2. Here, we use CPU to denote the CPU time required for an iteration to reach the above stopping criterion, ∞ to denote that an iteration does not satisfy the stopping criterion after 5000 iterations, SP to denote the speed-up of a parallel execution, which is defined to be the ratio of the CPU times of the sequential to the corresponding parallel methods, and $*$ to denote that the corresponding data is the largest in that row. Again, we point out that for this case, $\mathrm{SP}_{exp} = 2.4$.

We note from Tables 1-2 that the best speed-ups attained by the synchronous multisplitting relaxation methods are less than 1.98, while those attained by the asynchronous multisplitting relaxation methods are greater than 2.17. Even if this, it seems that the asynchronous version improves not too much on the synchronous version. The reason is that asynchronous computation is quite helpful in the case of poor load balancing, however, the regular structure of the system matrix $M \in \mathbb{R}^{n \times n}$ in our experiment allows for good load balancing on the processors. Moreover, this makes the asynchronous multisplitting relaxation method has similar convergence property to the corresponding synchronous one.

Table 2: CPUs and SPs for the AOR-like methods

γ		0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.7	1.8	1.9
ω		1.0	1.0	1.0	1.1	1.1	1.1	1.2	1.2	1.2	1.2
AOR	CPU	63.03	54.90	47.08	35.65	28.53	21.42	13.06	9.79	6.50	3.03
SAOR	CPU	29.69	26.03	22.97	17.79	14.87	11.42	7.21	5.59	3.91	2.04
USOR	CPU	39.31	31.74	25.37	18.41	13.94	9.93	∞	∞	∞	∞
MAOR	CPU	33.17	28.74	24.52	18.59	15.00	11.35	7.04	5.43	3.68	2.00
	SP	1.90	1.91	1.92*	1.92*	1.90	1.89	1.86	1.80	1.77	1.52
MSAOR	CPU	15.14	13.59	11.97	9.48	7.31	6.16	4.01	3.12	2.21	1.29
	SP	1.96	1.92	1.92	1.88	2.03*	1.85	1.80	1.79	1.77	1.58
MUSOR	CPU	20.45	16.53	13.19	9.76	7.46	5.25	∞	∞	∞	∞
	SP	1.92*	1.92*	1.92*	1.89	1.87	1.89	-	-	-	-
AMAOR	CPU	28.80	25.16	21.91	16.53	13.35	10.25	6.30	4.84	3.30	1.82
	SP	2.19*	2.18	2.15	2.16	2.14	2.09	2.07	2.02	1.97	1.66
AMSAOR	CPU	13.64	12.14	10.72	8.40	6.86	5.47	3.52	2.72	1.96	1.19
	SP	2.18*	2.14	2.14	2.12	2.17	2.09	2.08	2.06	1.99	1.71
AMUSOR	CPU	18.15	14.67	11.78	8.73	6.63	4.76	∞	∞	∞	∞
	SP	2.17*	2.16	2.15	2.11	2.10	2.09	-	-	-	-

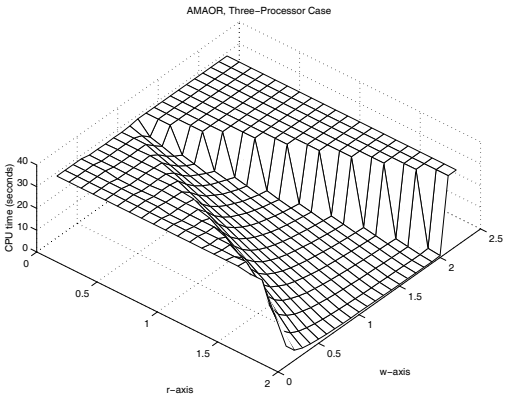

Figure 1: The behaviour of AMAOR method for the problem with $n = 4900$. The divergent points are represented in the graph by CPU time being 30. The optimal relaxation parameters are approximately given by $\gamma = 1.9$ and $\omega = 1.2$.

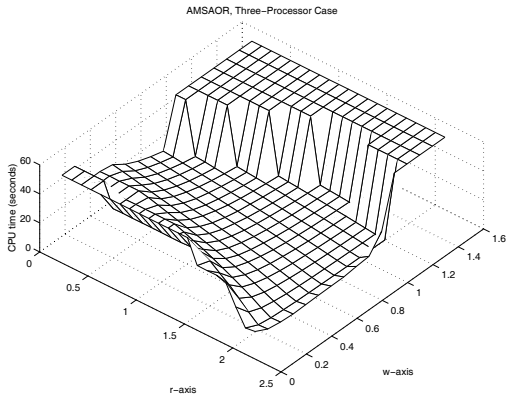

Figure 2: The behaviour of AMSAOR method for the problem with $n = 4900$. The divergent points are represented in the graph by CPU time being 40. The optimal relaxation parameters are approximately given by $\gamma = 1.9$ and $\omega = 1.2$.

The above observations are further confirmed by Figures 1-3, which give the behaviours of the asynchronous multisplitting relaxation methods, namely, AMAOR, AMSAOR and AMUSOR, respectively. The r and w axes in each figure correspond to the γ and ω axes, respectively. It is clearly demonstrated that all these methods have good convergence properties over a wide range of the relaxation parameters.

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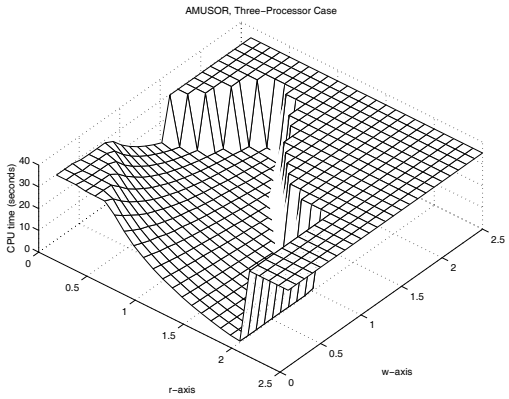

Figure 3: The behaviour of AMUSOR method for the problem with $n = 4900$. The divergent points are represented in the graph by CPU time being 30. The optimal relaxation parameters are approximately given by $\gamma = 1.4$ and $\omega = 1.1$.

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