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A SPECTRAL METHOD FOR A CLASS OF NONLINEAR QUASI-PARABOLIC EQUATIONS*1)

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Abstract

In this paper, we consider the numerical solution of quasi-parabolic equations of higher order by a spectral method, and propose a computational formula. We give an error estimate of approximate solutions, and prove the convergence of the approximate method and numerical stability on initial values. Under certain conditions, which are much weaker than the conditions in [6], we gain the same convergence rate as in [6].

§1. Introduction

This paper considers the following nonlinear quasi-parabolic equations of higher order with periodic boundary conditions:

$$\frac{\partial}{\partial t}u+(-1)^M A\frac{\partial^{2M+1u}}{\partial x^{2M}\partial t}=f\Big(u,\frac{\partial}{\partial x}u,\frac{\partial^2}{\partial x^2}u,\cdots,\frac{\partial^{2M}u}{\partial x^{2M}}\Big), \tag{1}$$

$$u(x-\pi,t)=u(x+\pi,t),\quad x\in I\!R,\quad 0\le t\le T, \tag{2}$$

$$u(x,0)=\hat{u}_0(x)=\phi(x),\quad x\in I\!R. \tag{3}$$

Here, $u(x,t)$ is a vector function with dimension $J, u(x,t)=(u_1(x,t),\ldots,u_J(x,t)), A=(a_{i,j})_{i,j=1}^J$ is a symmetric and positive definite matrix, and a_{ij} are real constants, i.e. $a_{i,j}=a_{j,i};\ \sum_{i,j=1}^J a_{ij}\xi_i\xi_j\ge a_0\sum_{j=1}^J\xi_j^2, a_0>0,\forall\xi_j\in I\!R$. Suppose $f\Big(u,\frac{\partial}{\partial x}u,\ldots,\frac{\partial^{2M}u}{\partial x^{2M}}\Big)$ in (1) takes the following form:

$$f_j=\sum_{m=1}^M(-1)^m D_x^{m+1}\frac{\partial F}{\partial P_{j,m-1}}+\sum_{m=1}^M(-1)^m D_x^m\frac{\partial G}{\partial P_{j,m-1}}+h_j(u)(j=1,\cdots,J), \tag{4}$$

$$f=(f_1,\cdots,f_J).$$

were $h_j(u)$ is a function of the vector u, the Jacobi matrix of $h=(h_j)_{j=1}^J$ is semi-bounded, i.e. there exists a constant b such that

$$\Big(\xi,\frac{\partial h}{\partial u}\xi\Big)\le b(\xi,\xi),\qquad\forall\xi,\ u\in I\!R,$$

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$F = F(P_0, \ldots, P_{M-1}), G = G(P_0, \ldots, P_{M-1})$ are smooth functions, $P_M = (P_{1,M}, \ldots, P_{J,M}), P_{j,M} = D_x^m u_j, D_x = \frac{\partial}{\partial x}, m = 0, 1, \ldots, M-1; j = 1, \ldots, J$, and $\hat{u}_0(x)$ is a known vector function with period 2π.

Equations (1) contain many equations which arise in physics and mechanics. For example, when we consider the long wave problems in a nonlinear dispersion system, the BBM equation $u_t + (\phi(u))_x = u_{xxt}$ arises. Also, the nonlinear advective equation of Sobolev-Galpern type is contained in (1).

Zhou and Fu [1] proved that, under some conditions, system (1)—(3) has a generalized solution $u(x,t) \in C^{k+M-1}(I\!R) \cap W_2^{k+M}(-\pi,\pi)(k \geq 1)$. When $k \geq M+1$, there exists a smooth global classical solution, i.e. $D_t D_x^{2M} u(x,t) \in C([0,T] \times I\!R)$.

Since the solution possesses higher smoothness, we can use the spectral method to obtain the numerical solutions of (1)-(3).

We introduce the usual symbols: the inner product of vectors u, v are

$$(u,v) = \int_{-\pi}^{\pi} \sum_{j=1}^{J} u_j v_j dx, \quad \text{set}\Omega = [-\pi,\pi],$$

$$\|u\|_{L_2(\Omega)}^2 = \int_{-\pi}^{\pi} \sum_{j=1}^{J} u_j^2 dx.$$

$L_\infty[0,T; H^S(\Omega)]$ denotes: when $u(x,t)$ as function of x, which belongs to $H^S(\Omega)$ for fixed $0 \leq t \leq T$, and $\sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{H^S} < \infty$; $\|u\|_{L_\infty}[0,T;H^S] = \sup_{0 \leq t \leq T} \|u\|_{H^S}, \| \cdot \|H^S$ denotes Sobolev norm, i.e. $\|u\|_{H^S(\Omega)}^2 = \|u\|_{W_2^S(\Omega)}^2 = \sum_{0 \leq |\alpha| \leq S} \left(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial x^\alpha}\right)$; $| \cdot |_{H^S(\Omega)} = \sum_{|\alpha|=S} \left(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial x^\alpha}\right)$, define $\|u\|_{L_\infty(\Omega)} = \text{ess} \sup_{\substack{x \in \Omega \\ 0 \leq j \leq J}} |u_j(x,t)|$. Suppose τ is the time step. We define the difference quotient

$$u_{n+1}, \bar{t} = \frac{1}{\tau}(u(x,(n+1)\tau) - u(x,n\tau)).$$

If there exists a constant b, such that

$$\sum_{j,l=1}^{J} \sum_{m,s=1}^{M} \frac{\partial^2 F}{\partial P_{j,m-1} \partial P_{l,s-1}} \xi_{j,m} \xi_{s,l} \leq b \sum_{j=1}^{J} \sum_{m=1}^{M} \xi_{j,m}^2,$$

then the Hessian matrix of function F is said to be semi-bounded.

2. The Spectral Method of System (1)–(3) and the Priori Integral Estimates of Solutions

We introduce $V_N = \text{span}\{1, \cos x, \sin x, \cdots, \cos Nx, \sin Nx\}$ as a space. P_N denotes the projective operator from L_1 to V_N, i.e.

$$P_N f = 1/2\pi(f,1) + 1/\pi \sum_{l=1}^{N}((f,\cos lx)\cos lx + (f,\sin lx)\sin lx), \quad \forall f \in L_1$$

L_1 is a space which consists of the solutions of (1)–(3). Thus, we can take

$$u_{i,n}^N(x) = u_i^N(x,n\tau) = \alpha_{i,0}^n + \sum_{l=1}^{N}(\alpha_{i,l}^n \cos lx + \beta_{i,l}^n \sin lx) \tag{4'}$$

as the approximate solutions of (1)–(3). Here, $\alpha_{i,l}^n = \alpha_{i,l}(n\tau)$, $\beta_{i,l}^n = \beta_{i,l}(n\tau)$, $i = 1,2,\dots,J$. Also, we can write the vector form $u^N = u^N(x,n\tau) = \alpha_0^n + \sum_{l=1}^{N}(\alpha_l^n \cos lx + \beta_l^n \sin lx)$, where $\alpha_l^n = \alpha_l(n\tau)$, $\beta_l^n = \beta_l(n\tau)$ are all vectors in J dimensions. By virtue of the spectral method[4], $\alpha_{j,l}^n, \beta_{j,l}^n$ should satisfy the following integral relations:

$$\Big(u_{j,n+1,\bar{t}}^N + \sum_{i=1}^{J}(-1)^M a_{ji} D_x^{2M} u_{i,n+1,\bar{t}}^N - f_j(u_{j,n+1}^N, D_x u_{j,n+1}^N, \cdots,$$

$$D_x^{2M} u_{j,n+1}^N), v\Big) = 0, \quad n = 0,1,\cdots, \quad j = 1,2,\cdots,J, \quad \forall v \in V_N, \tag{5}$$

$$u_0^N = \hat{u}_0^N = P_N \hat{u}_0(x). \tag{6}$$

Henceforth, we shall use the integral estimation method and Sobolev's functional interpolational formula to make some priori estimates for the approximate solution $u_{n+1}^N = (u_{1,n+1}^N, \dots, u_{J,n+1}^N)^T$.

Now, we use $C_i (i = 1,2,\dots)$ to denote a constant.

Lemma 1 (Sobolev's inequality)[1,2]. *Suppose $u \in H^n(\Omega)$. Then, for given $\varepsilon > 0$, n, there exists a constant $C(\varepsilon, n)$, such that*

$$\|D_x^k u\|_2 \le C\|u\|_2 + \varepsilon\|D_x^n u\|_2, \quad k \le n,$$

$$\|D_x^k u\|_{L_\infty} \le C\|u\|_2 + \varepsilon\|D_x^n u\|_2, \quad k < n.$$

Lemma 2 (Growall's inequality of discrete operator)[3]. *If for functions $u(t), v(t)$; $0 \le v(t) \le M$, there is a positive constant C_1, such that*

$$u(t) \le C_1 + \sum_{l=0}^{[T/\tau]} u(t_l)v(t_l)\cdot\tau, \qquad 0 \le t \le T,$$

then

$$u(t) \le C_1 e^{MT} = C_2, \qquad 0 \le t \le T.$$

Lemma 3. [5] *For any real $0 \le \mu \le \sigma$, there exists a constant C such that*

$$\|u - P_N u\|_\mu \le C N^{\mu-\sigma}\cdot |u|_\sigma, \qquad \forall u \in H_{(p)}^\sigma(\tilde{\Omega}).$$

Here, $H^{\sigma}_{(p)}(\tilde{\Omega})$=*closure of* $C^{\sigma}_{(p)}(\tilde{\Omega})$ *in* $H^{\sigma}(\tilde{\Omega})$*,* $C^{\sigma}_{(p)}(\tilde{\Omega}) = \{v = V_{|\tilde{\Omega}|}V; R^d \to C(\tilde{\Omega})$ *is infinitely differentiable and* 2π*–periodic in each variable*$\}$*, and* d *is the dimension.*

In this paper, $\mu = 2, d = J$. Then

$$\|u - P_N u\|_2 \le CN^{2-\sigma}|u|_\sigma \le CN^{2-\sigma}, \qquad \forall u \in H^{\sigma}_{(p)}(\tilde{\Omega}).$$

Lemma 4. *Suppose the matrix* A *is symmetric and positive definite, and* $F \in C^{M+2}, G \in C^{M+1}, h \in C^1, \phi(x) \in W_2^M(-\pi,\pi)$*. If the Hessian matrix of* F *and the Jacobi matrix of* $h(u)$ *are semi-bounded, then* $\|u_L^N\|_2 \le C_3; \|u_L^N\|_{L_\infty} \le C_3$*. Here, the constant* C_3 *is independent of* $N, u_L^N = u^N(x, L\tau)$*.*

Proof. Set $v = \cos lx$, and multiply (5) by $\alpha_{j,l}^{n+1}$. Then, take $v = \sin lx$, and multiply (5) by $\beta_{j,l}^{n+1}(0 \le l \le N)$. Summing up for l from $0 \to N$, we obtain

$$\Big(u^N_{j,n+1,\bar{t}} + \sum_{i=1}^{J}(-1)^M a_{ji} D_x^{2M} u^N_{i,n+1,\bar{t}} - f_j(u^N_{j,n+1},\dots,D_x^{2M}u^N_{j,n+1}), u^N_{j,n+1}\Big) = 0,$$

$$j = 1,\cdots,J. \quad (7)$$

Notice that

$$(u^N_{j,n+1,\bar{t}}, u^N_{j,n+1}) = \Big(u^N_{j,n+1,\bar{t}}, \tau/2(u^N_{j,n+1,\bar{t}}) - (\tau/2(u^N_{j,n+1,\bar{t}}) - u^N_{j,n+1})\Big)$$

$$= \tau/2\|u^N_{j,n+1,\bar{t}}\|_2^2 + 1/2(\|u^N_{j,n+1}\|_2^2)_{\bar{t}}\Big(\sum_{i=1}^{J}(-1)^M a_{ji} D_x^{2M} u^N_{i,n+1,\bar{t}}, u^N_{j,n+1}\Big)$$

$$= \sum_{i=1}^{J}(a_{ji}D_x^M u^N_{i,n+1,\bar{t}}, D_x^M u^N_{j,n+1}) = \frac{\tau}{2}\sum_{i=1}^{J}\Big(a_{j,i}(D_x^M u^N_{i,n+1})_{\bar{t}}, (D_x^M u^N_{j,n+1})_{\bar{t}}\Big)$$

$$+ \frac{1}{2}\sum_{i=1}^{J}(a_{ji}D_x^M u^N_{i,n+1}, D_x^M u^N_{j,n+1})_{\bar{t}}((f_j(u^N_{j,n+1},\dots,D_x^{2M}u^N_{j,n+1}), u^N_{j,n+1})$$

$$= \sum_{m=1}^{M}\Big(D_x\frac{\partial F}{\partial p_{j,m-1}}, D_x^m u^N_{j,n+1}\Big) + \sum_{m=1}^{M}\Big(\frac{\partial G}{\partial P_{j,m-1}}, D_x^m u^N_{j,n+1}\Big) + (h_j(u^N_{n+1}), u^N_{j,n+1}).$$

Since $D_x\dfrac{\partial F}{\partial P_{j,m-1}} = \displaystyle\sum_{l=1}^{J}\sum_{s=1}^{M}\frac{\partial^2 F}{\partial P_{j,m-1}\sigma P_{l,s-1}}D_x^s u^N_{l,n+1}$, (7) becomes

$$\frac{1}{2}(\|u^N_{j,n+1}\|_2^2)_{\bar{t}} + \frac{1}{2}\sum_{i=1}^{J}(a_{ji}D_x^M u^N_{i,n+1}, D_x^M u^N_{j,n+1})_{\bar{t}}$$

$$\le \sum_{l=1}^{J}\sum_{s,m=1}^{M}\Big(\frac{\partial^2 F}{\partial P_{j,m-1}\partial P_{l,s-1}}D_x^s u^N_{l,n+1}, D_x^m u^N_{j,n+1}\Big)$$

$$+ \sum_{m=1}^{M}\Big(\frac{\partial G}{\partial P_{j,m-1}}, D_x^m u^N_{j,n+1}\Big) + (h_j(u^N_{n+1}), u^N_{j,n+1}).$$

Summing up the above formulas for j from 1 to J, we obtain

$$\begin{aligned}&\frac{1}{2}(\|u_{n+1}^N\|_2^2)_{\bar{t}}+\frac{1}{2}\sum_{j,i=1}^{J}(a_{j,i}D_x^M u_{i,n+1}^N,D_x^M u_{j,n+1}^N)_{\bar{t}}\\&\le b\sum_{j=1}^{J}\sum_{m=1}^{M}(D_x^m u_{j,n+1}^N,D_x^m u_{j,n+1}^N)+\int_{-\pi}^{\pi}\frac{dG}{dx}dx+(h(u_{n+1}^N),u_{n+1}^N).\end{aligned}\tag{8}$$

By virtue of $G\in C^{M+1}$, $\int_{-\pi}^{\pi}\frac{dG}{dx}dx\le C_2'/2$. From the Taylor expansion of $h(u_{n+1}^N)$, we have $(h(u_{n+1}^N),u_{n+1}^N)=(h(0),u_{n+1}^N)+(h'(\bar{u}_{n+1}^N)u_{n+1}^N,u_{n+1}^N)$.

We utilize the semi-boundedness of the Jacobi an $\partial h/\partial u$ of h, and get

$$\begin{aligned}&(h'(\bar{u}_{n+1}^N)u_{n+1}^N,u_{n+1}^N)\le b\|u_{n+1}^N\|_2^2,\\&(h(0),u_{n+1}^N)=\int_{-\pi}^{\pi}h(0)u_{n+1}^N dx\le\pi|h(0)|^2+\|u_{n+1}^N\|_2^2.\end{aligned}$$

Therefore, we get

$$(h(u_{n+1}^N),u_{n+1}^N)\le\pi|h(0)|^2+\|u_{n+1}^N\|_2^2+b\|u_{n+1}^N\|_2^2\le C_1\|u_{n+1}^N\|_2^2+C_2'/2.$$

From (8), we have

$$\begin{aligned}&(\|u_{n+1}^N\|_2^2)_{\bar{t}}+\sum_{j,i=1}^{J}(a_{j,i}D_x^M u_{i,n+1}^N,D_x^M u_{j,n+1}^N)_{\bar{t}}\le 2b\sum_{m=1}^{M}\|D_x^m u_{n+1}^N\|_2^2\\&\quad+2C_1\|u_{n+1}^N\|_2^2+2C_2'\le b_1(\|u_{n+1}^N\|_2^2+\|D_x^M u_{n+1}^N\|_2^2)+C_2.\end{aligned}$$

where b_1, C_2 are constants independent of N. The last inequality comes from Lemma 1 (Sobolev's inequality).

Summing up the above formulas for n from 0 to $L-1(L=[T/\tau])$, we get

$$\begin{aligned}\|u_L^N\|_2^2+a_0\|D_x^M u_L^N\|_2^2&\le\|\hat{u}_0^N\|_2^2+\sum_{j,i=1}^{J}\left(a_{j,i}D_x^M u_{i,0}^N,D_x^M u_{j,0}^N\right)\\&\quad+C_2+b_1'\tau\sum_{n=0}^{L-1}\left(\|u_{n+1}^N\|_2^2+\|D_x^M u_{n+1}^N\|_2^2\right).\end{aligned}$$

By virtue of Lemma 2 (discrete Gronwall's inequality), we get

$$\|u_L^N\|_2^2+a_0\|D_x^M u_L^N\|_2^2\le\left(\|\hat{u}_0^N\|_2^2+C_2+\sum_{j,i=1}^{J}(a_{j,i}D_x^M u_{i,0}^N,D_x^M u_{j,0}^N)\right)e^{b_1'T}=C_3.$$

where C_3 is a constant independent of N. Thus,

$$\|u_L^N\|_2\le C_3,\qquad\|D_x^M u_L^N\|_2\le C_3.$$

By Sobolev's inequality (Lemma 1), we obtain $\|u_L^N\|_{L_\infty}\le C_3$. Then, Lemma 4 is proved.

3. Error Estimates, Convergence and Stability for Initial Values

Suppose that the solution of (1)–(3) is $u(x, n\tau)$, and the solution of (5)–(6) is $u_n^N(x)$. Then the error vector between the solution of the spectral method and the solutions of differential equations is $\varepsilon_n^N = u(x, n\tau) - u_n^N(x)$.

Define the projective error vector as follows:

$$R_n^N = u(x, n\tau) - P_N u(x, n\tau).$$

From the j–th equation of (1) and (5), we get

$$\Big(\frac{\partial}{\partial t} u_j(x,(n+1)\tau) - u_{j,n+1,\bar{t}}^N + \sum_{i=1}^{J}(-1)^M a_{j,i} D_t D_x^{2M} u_i(x,(n+1)\tau)$$

$$- \sum_{i=1}^{J}(-1)^M a_{j,i} D_x^{2M} u_{i,n+1,\bar{t}}^N - f_j(u_j(x,(n+1)\tau), \cdots, D_x^{2M} u_j(x,(n+1)\tau)$$

$$+ f_j(u_{j,n+1}^N, \cdots, D_x^{2M} u_{j,n+1}^N), v\Big) = 0, \quad j = 1, \cdots, J, \quad \forall v \in V_n. \tag{9}$$

From [1], we know that the solutions of problem (1)–(3) satisfy $\left|\frac{\partial^2}{\partial x^2} u(x,t)\right| \le C_{12}$; $\left|\frac{\partial^{2M+1}}{\partial x^{2M}\partial t} u(x,t)\right| \le C_{12}$.

Then, from the Taylor expansion,

$$(u_j(x,(n+1)\tau))_t - u_{j,n+1,\bar{t}}^N = \varepsilon_{j,n+1,\bar{t}}^N + \theta_{1,j,n+1},$$

$$D_t D_x^{2M} u_i(x,(n+1)\tau) - D_x^{2M} u_{i,n+1,\bar{t}}^N = D_x^{2M} \varepsilon_{i,n+1,\bar{t}}^N + \theta_{2,i,n+1}.$$

Here, $|\theta_{1,j,n+1}| \le C_6 \tau$; $|\theta_{2,i,n+1}| \le C_6 \tau$. Take

$$\mathcal{F} = f_j(u_j(x,(n+1)\tau), \cdots, D_x^{2M} u_j(x,(n+1)\tau)) - f_j(u_{j,n+1}^N, \cdots, D_x^{2M} u_{j,n+1}^N)$$

$$= \sum_{m=1}^{M} (-1)^m D_x^{m+1} \frac{\partial F(u_j, \cdots, D_x^{2M} u_j)}{\partial P_{j,m-1}} + \sum_{m=1}^{M} (-1)^m D_x^m \frac{\partial G(u_j, \cdots, D_x^{2M} u_j)}{\partial P_{j,m-1}}$$

$$+ h_j(u) - \sum_{m=1}^{M} (-1)^m D_x^{m+1} \frac{\partial F(u_{j,n+1}^N, \cdots, D_x^{2M} u_{j,n+1}^N)}{\partial P_{j,m-1}}$$

$$- \sum_{m=1}^{M} (-1)^m D_x^m \frac{\partial G(u_{j,n+1}^N, \cdots, D_x^{2M} u_{j,n+1}^N)}{\partial P_{j,m-1}} + h_j(u_{n+1}^N).$$

Set $v = P_N u_j(x,(n+1)\tau) - u_{j,n+1}^N = \varepsilon_{j,n+1}^N - R_{j,n+1}^N$. Then, from (9) we obtain

$$\Big(\varepsilon_{j,n+1,\bar{t}}^N + \theta_{1,j,n+1} + \sum_{i=1}^{J}(-1)^M a_{j,i}(D_x^{2M} \varepsilon_{i,n+1,\bar{t}}^N + \theta_{2,i,n+1}) - \mathcal{F}, \varepsilon_{j,n+1}^N - R_{j,n+1}^N\Big) = 0,$$

$$j = 1, \cdots, J, \tag{10}$$

i.e.

$$\Big(\varepsilon^N_{j,n+1,\bar t}+\sum_{i=1}^{J}(-1)^M a_{j,i}D_x^{2M}\varepsilon^N_{j,n+1,\bar t},\varepsilon^N_{j,n+1}\Big)-(\varepsilon^N_{j,n+1,\bar t},R^N_{j,n+1})$$

$$=-(\theta_{1,j,n+1},\varepsilon^N_{j,n+1}-R^N_{j,n+1})-\Big(\sum_{i=1}^{J}(-1)^M a_{j,i}\theta_{2,i,n+1},\varepsilon^N_{j,n+1}-R^N_{j,n+1}\Big)$$

$$+\Big(\sum_{i=1}^{J}(-1)^M a_{j,i}D_x^{2M}\varepsilon^N_{i,n+1},R^N_{j,n+1}\Big)+(\mathcal{F},\varepsilon^N_{j,n+1}-R^N_{j,n+1}),\ j=1,\cdots,J. \quad (11)$$

Since

$$(\varepsilon^N_{j,n+1,\bar t},\varepsilon^N_{j,n+1})=\tau/2\|\varepsilon^N_{j,n+1,\bar t}\|_2^2+\frac{1}{2}(\|\varepsilon^N_{j,n+1}\|_2^2)_{\bar t},$$

$$\Big(\sum_{i=1}^{J}(-1)^M a_{j,i}D_x^{2M}\varepsilon^N_{i,n+1,\bar t},\varepsilon^N_{j,n+1}\Big)=\frac{\tau}{2}\Big(\sum_{i=1}^{J}a_{j,i}D_x^M\varepsilon^N_{i,n+1,\bar t},D_x^M\varepsilon^N_{j,n+1,\bar t}\Big)$$

$$+\frac{1}{2}\Big(\sum_{i=1}^{J}a_{j,i}D_x^M\varepsilon^N_{i,n+1},D_x^M\varepsilon^N_{j,n+1}\Big)_{\bar t},$$

$$(\varepsilon^N_{j,n+1,\bar t},R^N_{j,n+1})=\frac{1}{2}(\varepsilon^N_{j,n+1},R^N_{j,n+1})_{\bar t}+\frac{1}{2}(\varepsilon^N_{j,n+1},R^N_{j,n+1,\bar t})-\frac{1}{2}(\varepsilon^N_{j,n},R^N_{j,n+1,\bar t}),$$

$$\Big(\sum_{i=1}^{J}(-1)^M a_{j,i}D_x^{2M}\varepsilon^N_{i,n+1,\bar t},R^N_{j,n+1}\Big)=1/\tau\Big(\sum_{i=1}^{J}(-1)^M a_{j,i}(\varepsilon^N_{i,n+1}-\varepsilon^N_{i,n}),D_x^{2M}R^N_{j,n+1}\Big),$$

from Lemma 4, we know

$$(\mathcal{F},\varepsilon^N_{j,n+1}-R^N_{j,n+1})\le 2b_1(\|\varepsilon^N_{j,n+1}\|_2^2+\|D_x^M\varepsilon^N_{j,n+1}\|_2^2+\|R^N_{j,n+1}\|_2^2+\|D_x^M R^N_{j,n+1}\|_2^2).$$

where b_1 is a constant just as in lemma 4.

Substituting the above formulas into (11), we get

$$\frac{1}{2}\Big(\|\varepsilon^N_{j,n+1}\|_2^2)_{\bar t}+\frac{1}{2}\sum_{i=1}^{J}(a_{j,i}D_x^M\varepsilon^N_{i,n+1},D_x^M\varepsilon^N_{j,n+1})_{\bar t}-\frac{1}{2}(\varepsilon^N_{j,n+1},R^N_{j,n+1})_{\bar t}$$

$$\le\frac{1}{2}\|\theta_{1,j,n+1}\|_2^2+\frac{3}{4}\|\varepsilon^N_{j,n+1}\|_2^2+\frac{3}{4}\|R^N_{j,n+1}\|_2^2+\frac{1}{2}\|R^N_{j,n+1,\bar t}\|_2^2$$

$$+\frac{1}{2}\Big(\sum_{i=1}^{J}a_{j,i}\Big)^2\Big(\|\theta_{2,i,n+1}\|_2^2+\frac{1}{2}\|\varepsilon^N_{j,n+1}\|_2^2+\frac{1}{2}\|R^N_{j,n+1}\|_2^2\Big)$$

$$+\frac{1}{2\tau}\Big(\sum_{i=1}^{J}a_{j,i}\Big)^2(\|\varepsilon^N_{i,n+1}\|_2^2+\|\varepsilon^N_{i,n}\|_2^2+\|D_x^{2M}R^N_{j,n+1}\|_2^2)$$

$$+C_9(\|\varepsilon^N_{j,n+1}\|_2^2+\|D_x^M\varepsilon^N_{j,n+1}\|_2^2+\|R^N_{j,n+1}\|_2^2+\|D_x^M R^N_{j,n+1}\|_2^2),$$

$$j=1,\cdots,J. \quad (12)$$

Summing up (12) for j from 1 to J, and for n from 0 to $L-1$ ($L=[T/\tau]$), we obtain

$$\begin{aligned}&\|\varepsilon_L^N\|_2^2+\sum_{i,j=1}^{J}(a_{j,i}D_x^M\varepsilon_{i,L}^N,D_x^M\varepsilon_{j,L}^N)-\sum_{j=1}^{J}(\varepsilon_{j,L}^N,R_{j,L}^N)\\&\quad\le\|\varepsilon_0^N\|_2^2+\sum_{i,j=1}^{J}(a_{j,i}D_x^M\varepsilon_{i,0}^N,D_x^M\varepsilon_{j,0}^N)-\sum_{j=1}^{J}(\varepsilon_{j,0}^N,R_{j,0}^N)+C_7\tau\sum_{n=0}^{L-1}(\|\theta_{1,n+1}\|_2^2\\&\quad+\|\theta_{2,n+1}\|_2^2+\|\varepsilon_{n+1}^N\|_2^2+\|D_x^M\varepsilon_{n+1}^N\|_2^2+\|D_x^{2M}R_{n+1}^N\|_2^2+\|R_{n+1}^N\|_2^2\\&\quad+\|D_x^MR_{n+1}^N\|_2^2+\|R_{n+1,\bar t}^N\|_2^2)+C_7\tau\|\varepsilon_0^N\|_2^2.\end{aligned}\tag{13}$$

By

$$(\varepsilon_{j,L}^N,R_{j,L}^N)\le\frac{1}{2}(\|\varepsilon_{j,L}^N\|_2^2+\|R_{j,L}^N\|_2^2),\sum_{j=1}^{J}(\varepsilon_{j,L}^N,R_{j,L}^N)\le\frac{1}{2}(\|\varepsilon_L^N\|_2^2+\|R_L^N\|_2^2),$$

$$\sum_{j=1}^{J}(\varepsilon_{j,0}^N,R_{j,0}^N)\le\frac{1}{2}(\|\varepsilon_0^N\|_2^2+\|R_0^N\|_2^2),$$

(13) becomes

$$\|\varepsilon_L^N\|_2^2+a_0\|D_x^M\varepsilon_L^N\|_2^2\le C_8R+C_7\tau\sum_{n=0}^{L-1}(\|\varepsilon_{n+1}^N\|_2^2+\|D_x^M\varepsilon_{n+1}^N\|_2^2).$$

Here,

$$\begin{aligned}R=&\|\varepsilon_0^N\|_2^2+\|R_L^N\|_2^2-\|R_0^N\|_2^2+\sum_{j,i=1}^{J}(a_{j,i}D_x^M\varepsilon_{i,0}^N,D_x^M\varepsilon_{j,0}^N)\\&+C_7\tau\sum_{n=0}^{L-1}(\|\theta_{1,n+1}\|_2^2+\|\theta_{2,n+1}\|_2^2+\|R_{n+1}^N\|_2^2+\|D_x^MR_{n+1}^N\|_2^2\\&+\|\varepsilon_0^N\|_2^2+\|D_x^{2M}R_{n+1}^N\|_2^2+\|R_{n+1,\bar t}^N\|_2^2).\end{aligned}$$

From discrete Gronwall's inequality (Lemma 2), we obtain $\|\varepsilon_L^N\|_2^2+a_0\|D_x^M\varepsilon_L^N\|_2^2\le C_{11}R$, i.e. $\|\varepsilon_L^N\|_2\le C_{11}R^{\frac{1}{2}}$; $\|D_x^M\varepsilon_L^N\|_2\le C_{11}R^{\frac{1}{2}}$.

From Lemma 1, we obtain $\|\varepsilon_L^N\|_{L_\infty}\le C_{11}R^{\frac{1}{2}}$. Therefore,

$$\|\varepsilon_L^N\|_2\le C_{11}R^{\frac{1}{2}},\quad\|D_x^M\varepsilon_L^N\|_2\le C_{11}R^{\frac{1}{2}},\quad\|\varepsilon_L^N\|_{L_\infty}\le C_{11}R^{\frac{1}{2}}.\tag{14}$$

Theorem 1. *Suppose that the conditions of Lemma 4 are satisfied, and $\hat{u}_0(x)$, $u(x,t)$, $u_t(x,t)$ are all bounded and have continuous $s+1$-th partial derivatives ($s\ge M+1$) with respect to x. Then,*

$$\|\varepsilon_L^N\|_2\le C_{14}(\tau+1/N^{s-1}),\quad\|\varepsilon_L^N\|_{L_\infty}\le C_{14}(\tau+1/N^{s-1}),$$

$$\|D_x^M\varepsilon_L^N\|_2\le C_{14}(\tau+1/N^{s-1}).$$

Proof. From the conditions of Theorem 1, we know that $\hat{u}_0(x)$, $u(x,t)$, $u_t(x,t)$ all belong to $H^{s+1}_{(p)}(\tilde{\Omega})$. In the experssion of R,

$$\|\varepsilon_0^N\|_2^2 = \|u(x,0) - u_0^N(x)\|_2^2 = \|\hat{u}_0(x) - P_N\hat{u}_0(x)\|_2^2 \le C_{13}(1/N^{s-1})^2,$$

$$\Big\| \sum_{j,i=1}^{J} (a_{j,i} D_x^M \varepsilon_{i,0}^N, D_x^M \varepsilon_{j,0}^N)\Big\|_2^2 \le \max_{1\le i,j\le J} |a_{j,i}|^2 \|D_x^M(\hat{u}_0(x) - P_N\hat{u}_0(x))\|_2^2$$
$$\le C_{13}(1/N^{s-1})^2,$$

$$\|R_0^N\|_2^2 = \|u(x,0) - P_N u(x,0)\|_2^2 \le C_{13}(1/N^{s-1})^2,$$

$$\|R_L^N\|_2^2 = \|u(x,L\tau) - P_N u(x,L\tau)\|_2^2 \le C_{13}(1/N^{s-1})^2,$$

$$\|\theta_{1,n+1}\|_2^2 \le C_{13}\tau^2, \quad \|\theta_{2,n+1}\|_2^2 \le C_{13}\tau^2,$$

$$\|R_{n+1}^N\|_2^2 = \|u(x,(n+1)\tau) - P_N u(x,(n+1)\tau)\|_2^2 \le C_{13}(1/N^{s-1})^2,$$

$$\|D_x^M R_{n+1}^N\|_2^2 = \|D_x^M u(x,(n+1)\tau) - P_N D_x^M u(x,(n+1)\tau)\|_2^2 \le C_{13}(1/N^{s-1})^2,$$

$$\|D_x^{2M} R_{n+1}^N\|_2^2 = \|D_x^{2M} u(x,(n+1)\tau) - P_N D_x^{2M} u(x,(n+1)\tau)\|_2^2$$
$$\le C_{13}(1/N^{s-1})^2,$$

$$\|R_{n+1,\bar{t}}^N\|_2^2 \le \|u_t(x,(n+1)\tau) - P_N u_t(x,(n+1)\tau) + \rho_{n+1}\|_2^2$$
$$\le C_{13}(\tau^2 + (1/N^{s-1})^2),$$

where, $|\rho_{n+1}| < C_{14}\tau$.

Substituting the above estimates into the expression of R, we get $R \le C_{13}(\tau^2 + (1/N^{s-1})^2)$.

From (14), we get the conclusion of Theorem 1.

From Theorem 1, we directly get

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied. Then, when $\tau \to 0$, $n \to \infty$, the solutions of the spectral method (5)–(6) converge to the solutions of problem (1)–(3) in L_∞ norm. The convergence rate is $0(\tau + 1/N^{s-1})$.*

Theorem 3. *Suppose that the conditions of Lemma 4 are satisfied. Then, the formulas (5)–(6) of the spectral method are stable with respect to initial values.*

Proof. Suppose that $\hat{u}_0^N$ has disturbance v_0^N. From (5) and (6) we set up equations which are satisfied by v_n^N (u_n^N's disturbance). Similarly, we obtain analogous estimates of v_n^N. By these estimates, the computational formulas (5)–(6) of the spectral method are stable with respect to initial values.

References

[1] Zhou Yu-lin and Fu Hong-yuan, *Scientia Sinica,* ser. A, 25 (1982), 1021–1031. (in Chinese)

[2] A.Adams, Sobolev Spaces, Academic Press, New york, 1975.
[3] Guo Bo-ling, *Mathematica Numerica Sinica,* **3** (1981), 211–233. (in Chinese)
[4] D. Gottlieb and S.A. Orszag, Numerical analysis of spectral methods, CBMS–NSF, Regional Conference Series in Applied Math., 26 (1977).
[5] C. Canto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev space, *Math. Comp.,* 38 (1982), 67–86.
[6] Zhou Zhen-zhong, Spectral method for nonlinear quasi-parabolic equations of higher order, *Hunan Annals of Mathematics,* **7** : 2 (1987), 57–66. (in Chinese)