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THE SOLVABILITY CONDITIONS FOR THE INVERSE PROBLEM OF MATRICES POSITIVE SEMIDEFINITE ON A SUBSPACE*1)

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Abstract

This paper studies the following two problems:

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_{s,n}$, such that $AX = B$, where $Ps,n = \{A \in SR^{n\times n} | x^T Ax \geq 0, \forall S^T x = 0$, for given $S \in R_p^{n\times p}\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$, such that $\|A^* - \hat{A}\| = \inf_{A\in S_E} \|A^* - A\|$ where S_E denotes the solution set of Problem I.

The necessary and sufficient conditions for the solvability of Problem I, the expression of the general solution of Problem I and the solution of Problem II are given for two cases. For the general case, the equivalent form of conditions for the solvability of Problem I is given.

Inverse problems for real symmetric matrices and symmetric nonnegative definite matrices have been studied in [1], [2]. The conditions for the existence of a solution the expression of the general solution and optimal approximate solution have been given. This paper studies the inverse problem of one kind of matrices between the above two kinds of matrices — matrices positive semidefinite on a subspace. The conditions for the existence of a solution, the expression of the general solution and the optimal approximate solution are given.

In this paper, $R^{n\times m}$ denotes the set of all real $n \times m$ matrices, $R_r^{n\times m}$ its subset whose elements have rank r, $SR^{n\times n}$ the set of all real $n \times n$ symmetric matrices, and $SR_0^{n\times n}$ the set of all $n \times n$ symmetric nonnegative definite matrices. I_k denotes the $k \times k$ unit matrix, $R(A), N(A), A^+$ denote column space, null space and Moore-Penrose generalized inverse matrix of matrix A respectively. $\|\cdot\|$ is Frobenius norm, and $A \geq 0$ represents that A is a symmetric nonnegative definite matrix.

Let $P_{s,n} = \{A \in SR^{n\times n} | x^T Ax \geq 0, \forall S^T x = 0$, for given $S \in R_p^{n\times p}\}$.

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_{s,n}$ such that $AX = B$.

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Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$, such that $\|A^* - \hat{A}\| = \inf_{A\in S_E} \|A^* - A\|$, where S_E denotes the solution set of Problem I.

We will introduce some Lemmas in **1**, and give the conditions for the solvability, the expression of the general solution of Problem I and optimal approximate solution of Problem II for two cases respectively in **2** and **3**. We will give an equivalent form of conditions for the solvability of Problem I for the general case in **4** and suggest a problem which is worth investigating.

1. Some Lemmas

Suppose $S \in R_p^{n\times n}$. We construct the orthogonal triangular decomposition for S:

$$S = Q^T \begin{pmatrix} 0 \\ L \end{pmatrix} = Q_2^T L, \tag{1.1}$$

where $Q^T = (Q_1^T, Q_2^T)$ is an $n \times n$ orthogonal matrix, and $Q_1^T \in R^{n\times(n-p)}$ L is a $p\times p$ nonsingular lower triangular matrix. Then

$$R(S) = R(Q_2^T), N(S^T) = R(Q_1^T). \tag{1.2}$$

Lemma 1. *Suppose S has factorization in the form* (1.1). *Let*

$$QAQ^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in R^{(n-p)\times(n-p)}. \tag{1.3}$$

Then $A \in Ps,n \Leftrightarrow A_{11} \in SR_0^{(n-p)\times(n-p)}, A_{21} = A_{12}^T, A_{22} = A_{22}^T$.

Proof. **Sufficiency.** Because $A \in P_{s,n}$ it is evident that $A_{22} = A_{22}^T, A_{12} = A_{21}^T, A_{11} = A_{11}^T, \forall y \in R^{(n-p)}$. Then $Q_1^T y \in N(S^T)$. From $A \in P_{s,n}$, we get $y^T A_{11} y = (Q_1^T y)^T A (Q_1^T y) \geq 0$, i.e. $A_{11} \in SR_0^{(n-p)\times(n-p)}$.

Necessity. It is evident that $A = A^T$, and for any given $x \in N(S^T)$, there exists $y \in R^{n-p}$ satisfying $x = Q_1^T y$, i.e. $Qx = \binom{y}{0}$.

Thus we have $x^T Ax = x^T Q^T QAQ^T Qx = y^T A_{11} y \geq 0$, where $A \in P_{s,n}$.

Lemma 2. *Suppose $X, B \in R^{n\times m}$, and S has factorization in the form* (1.1). *Write*

$$QX = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{1.4}$$

$X_1 = Q_1 X$, $B_1 = Q_1 B \in R^{(n-p)\times m}$, $X_2 = Q_2 X$, $B_2 = Q_2 B \in R^{p\times m}$.

If $R(X) \subseteq R(S)$, then we have (i) $X_2^T B_2 = B_2^T X_2 \Leftrightarrow X^T B = B^T X$; (ii) $B_1 X_2^+ X_2 = B_1$ *and* $B_2 X_2^+ X_2 = B_2 \Leftrightarrow BX^+X = B$.

If $R(X) \subseteq N(S^T)$, then we have (iii) $X_1^T B_1 = B_1^T X_1 \geq 0 \Leftrightarrow X^T B = B^T X \geq 0$; (iv) $\text{rank}(X_1^T B_1) = \text{rank}(B_1) \Leftrightarrow \text{rank}(B_1) = \text{rank}(X^T B)$; (v) $B_1 X_1^+ X_1 = B_1, B_2 X_1^+ X_1 = B_2 \leftrightarrow BX^+X = B$.

Proof. Because $X^T B = X^T Q^T QB = X_1^T B_1 + X_2^T B_2$, if $R(X) \subseteq R(S) = R(Q_2^T)$, then we have $X_1 = 0$. Therefore (i) holds. Furthermore, because $X^T X = X^+ Q^T QX =$

$\binom{X_1}{X_2}^+\binom{X_1}{X_2}$, we have $X^+X = X_2^+X_2$ when $X_1 = 0$. Thus (ii) holds. If $R(X) \subseteq N(S^T)$, then we can obtain (iii)–(v) by $X_2 = 0$ and a similar argument.

Lemma 3. *Suppose $C \in SR_0^{n\times n}, Y \in R^{n\times m}$. Then* rank $(Y^TCY) =$ rank (Y^TC).

Proof. Consider the equation with unknown $A \in SR_0^{n\times n}$.

$$AY = CY.$$

It is evident that C is a solution satisfying the conditions. Thus, from Theorem 2 in [2], we have

$$\text{rank}(Y^TCY) = \text{rank}(CY) = \text{rank}((CY)^T) = \text{rank}(Y^TC).$$

2. The Case of $R(X) \subseteq R(S)$

Theorem 1. *Suppose $B, X \in R^{n\times m}, S \in R_p^{n\times p}$, and S has factorization in the form (1.1). X_i, B_i are the same as in (1.4) . If $R(X) \subseteq R(S)$, Problem I has a solution if and only if:*

$$B = BX^+X, X^TB = B^TX, \tag{2.1}$$

and its general solution can be represented as

$$A = Q^T\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} Q \tag{2.2}$$

where

$$A_{11} \in SR_0^{(n-p)\times(n-p)}, A_{21} = A_{12}^T, \tag{2.3}$$

$$A_{12} = B_1X_2^+ + M(I - X_2X_2^+), \forall M \in R^{(n-p)\times p}, \tag{2.4}$$

$$A_{22} = X_2X_2^+B_2X_2^+ + (X_2^+)^TB_2(I - X_2X_2^+) + (1 - X_2X_2^+)B_2X_2^+ + (I - X_2X_2^+)N(I - X_2X_2^+), \quad \forall N \in SR^{p\times p}. \tag{2.5}$$

Furthermore, suppose

$$B^* = (A^* + A^{*T})/2, \qquad C^* = (A^* - A^{*T})/2. \tag{2.6}$$

Let

$$QB^*Q^T \triangleq \begin{pmatrix} B_{11}^* & B_{12}^* \\ B_{21}^* & B_{22}^* \end{pmatrix} \qquad B_{11}^* \in SR^{(n-p)\times(n-p)}. \tag{2.7}$$

If Problem I has a solution, then Problem II has a unique optimal approximate solution which can be represented as

$$\hat{A} = Q^T\begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix} Q \tag{2.8}$$

where

$$\hat{A}_{11} = [B^*]_+, \hat{A}_{21} = \hat{A}_{12}, \tag{2.9}$$

$$\hat{A}_{12} = B_1 X_2^+ + B_{12}^*(I - X_2 X_2^+), \tag{2.10}$$

$$\begin{aligned} \hat{A}_{22} &= X_2 X_2^+ B_2 X_2^+ + (X_2^+)^T B_2 (I - X_2 X_2^+) \\ &\quad + (I - X_2 X_2^+) B_2 X_2^+ + (I - X_2 X_2^+) B_{22}^* (I - X_2 X_2^+), \end{aligned} \tag{2.11}$$

where $[E]_+$ *denotes the unique optimal approximate solution in the set* $SR^{n\times n}$ *of* $n \times n$ *matrix* E *under Frobenius norm.*

Proof. By using matrix Q in (1.1), let

$$Q^T A Q \triangleq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} \in R^{(n-p)\times(n-p)}. \tag{2.12}$$

Then we can easily transform $AX = B$ into

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{2.13}$$

Because $R(X) \subseteq R(S)$, from (1.2) we have $X_1 = Q_1 X = 0$. Now we use Lemma 1. From (2.13) we can see that Problem I is equivalent to

$$A_{11} \in SR_0^{(n-p)\times(n-p)}, A_{21} = A_{12}^T, \tag{2.14}$$

$$A_{12} X_2 = B_1, \tag{2.15}$$

$$A_{22} X_2 = B_2, A_{22} = A_{22}^T. \tag{2.16}$$

According to [3] and [1], we know that the necessary and sufficient conditions for the solvability of (2.15) and (2.16) are respectively

$$B_1 = B_1 X_2 X_2^+ \quad \text{and} \quad B_2 = B_2 X_2 X_2^+, \qquad X_2^T B_2 = B_2^T X_2. \tag{2.17}$$

From (i), (ii) in Lemma 2 we know that (2.17) is equivalent to (2.2). Evidently, (2.14) is the same as (2.3). According to [3] and [1], we know that the general solution of (2.15) and (2.16) can be represented by (2.4) and (2.5) respectively. Moreover, we can get the expression (2.2) of the general solution of Problem I from (2.12).

When Problem I has a solution, from (2.2) we know that the solution set S_E is a closed convex set, and the corresponding Problem II has a unique optimal approximate solution.

By using (2.6), (2,7) (2.12) and orthogonal matrix Q in (1.1), for $A \in SR^{n\times n}$ we can derive

$$\begin{aligned} \|A - A^*\|^2 &= \|QAQ^T - QA^*Q^T\|^2 = \|QAQ^T - QB^*Q^T\|^2 + \|QC^*Q^T\|^2 \\ &= \left\| \begin{pmatrix} A_{11} - B_{11}^* & A_{12} - B_{12}^* \\ A_{21} - B_{21}^* & A_{22} - B_{22}^* \end{pmatrix} \right\|^2 + \|C^*\|^2. \end{aligned}$$

Therefore, from (2.2) we know that $\inf_{A\in S_E} \|A - A^*\|$ is equivalent to

$$\inf \|A_{11} - B_{11}^*\| \quad A_{11} \in SR_0^{(n-p)\times(n-p)}, \tag{2.18}$$

$$\inf \|A_{12} - B_{12}^*\| \quad A_{12} \quad \text{is given by (2.4)}, \tag{2.19}$$

$$\inf \|A_{21} - B_{21}^*\| \quad A_{21} = A_{12}^T, \tag{2.20}$$

$$\inf \|A_{22} - B_{22}^*\| \quad A_{22} \quad \text{is given by (2.5)}. \tag{2.21}$$

According to [4], we know that the solution $\hat{A}_{11}$ of (2.18) can be represented by the first equation of (2.9); by [3], we know that the solution $\hat{A}_{12}$ of (2.19) can be represented by (2.10); by [1], we know that the solution of (2.20) can be represented by (2.11). Because $B_{21}^* = B^{*T}$, the solution of (2.20) satisfies $\hat{A}_{21} = \hat{A}_{12}^T$. At last, by (2.12), we conclude that the optimal approximate solution $\hat{A}$ of Problem II can be represented by (2.8). Thus we complete the proof.

Corollary 1. Suppose $X, B \in R^{n\times m}$. If $X = S$, then Problem I has a solution if and only if

$$X^T B = B^T X. \tag{2.22}$$

If S has factorization in the form (1.1), then the general solution of Problem I can be represented as

$$A = Q^T \begin{pmatrix} A_{11} & B_1 L^{-1} \\ (B_1 L^{-1})^T & B_2 L^{-1} \end{pmatrix} Q, \quad \forall A_{11} \in SR_0^{(n-p)\times(n-p)}, \tag{2.23}$$

where $B_1 = Q_1 B, B_2 = Q_2 B$. If (2.22) holds, then Problem II has a unique optimal approximate solution:

$$\hat{A} = Q^T \begin{pmatrix} [B_{11}^*]_+ & B_1 L^{-1} \\ (B_1 L^{-1})^T & B_2 L^{-1} \end{pmatrix} Q, \tag{2.24}$$

where $[B_{11}^*]_+$ is the same as $[B_{11}^*]_+$ of Theorem 1.

Corollary 2. Suppose $X, B \in R^{n\times m}, S \in R_p^{n\times p}$ If $R(X) = R(S)$. Then, Problem I has a solution if and only if

$$X^T B = B^T X, \quad B = BX^+X. \tag{2.25}$$

If S has factorization in the form (1.1), then the general solution of Problem I can be represented as

$$A = Q^T \begin{pmatrix} A_{11} & B_1 X_2^+ \\ (B_1 X_2^+)^T & B_2 X_2^+ \end{pmatrix} Q, \quad \forall A_{11} \in SR_0^{(n-p)\times(n-p)},$$

where $B_1 = Q_1 B, B_2 = Q_2 B, X_2 = Q_2 X$. If (2.25) holds, then Problem II has a unique optimal approximate solution which can be represented as

$$\hat{A} = Q^T \begin{pmatrix} [B_{11}^*]_+ & B_1 X_2^+ \\ (B_1 X_2^+)^T & B_2 X_2^+ \end{pmatrix} Q,$$

where $[B_{11}^*]_+$ is the same as in Theorem 1.

The proofs of Corollaries 1–2 are omitted.

For $R(X) \subseteq R(S)$, by Theorem 1, we obtain the following numerical procedure for computing the unique optimal approximate solution $\hat{A}$ of Problem II:

1. Construct the orthogonal lower-triangular decomposition for S according to (1.1),
2. Calculate B_1, B_2, X_2 according to (1.4),
3. Compute $B_{11}^*, B_{12}^*, B_{22}^*$ according to (2.6), (2.7),
4. Compute $\hat{A}_{11}, \hat{A}_{12}, \hat{A}_{21}, \hat{A}_{22}$ according to (2.9), (2.10), (2.11),
5. Compute $\hat{A}$ according to (2.8).

3. The Case of $R(X) \subseteq N(S^T)$

Theorem 2. *Suppose $X, B \in R^{n\times m}$, $S \in R_p^{n\times p}$ and S has factorization in the form (1, 1). Introduce the notation in (1.4). If $R(X) \subseteq N(S^T)$, then Problem I has a solution if and only if*

$$B = BX^+X, \quad X^TB = B^TX \geq 0, \quad \operatorname{rank}(X^TB) = \operatorname{rank}(B_1) \tag{3.1}$$

and its general solution can be represented as

$$A = Q^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} Q, \tag{3.2}$$

where

$$A_{11} = B_1X_1^+ + (B_1X_1^+)^T(I - X_1X_1^+) + (I - X_1X_1^+)B_1(X_1^TB_1)^+B_1^T(I - X_1X_1^+)$$
$$+ (I - X_1X_1^+)G((I - X_1X_1^+), \quad \forall G \in SR_0^{(n-p)\times(n-p)}, \tag{3.3}$$

$$A_{21} = B_2X_1^+ + M(I - X_1X_1^+), \quad \forall M \in R^{p\times(n-p)}, \tag{3.4}$$

$$A_{22} = A_{22}^T, \quad A_{12} = A_{21}^T \tag{3.5}$$

Furthermore, if Problem I *has a solution, then Problem II has a unique optimal approximate solution which can be represented as*

$$\hat{A} = Q^T \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix} Q, \tag{3.6}$$

where

$$\hat{A}_{11} = B_1X_1^+ + (B_1X_1^+)^T(I - X_1X_1^+) + (I - X_1X_1^+)B_1(X_1^TB_1)^+B_1^T(I - X_1X_1^+)$$
$$+ U_2[U_2^T(B_{11}^* - B_1(B_1^TX_1)^+B_1^T)U_2] + U_2^T, \tag{3.7}$$

$$\hat{A}_{21} = B_2X_1^+ + B_{21}^*(I - X_1X_1^+), \tag{3.8}$$

$$\hat{A}_{12} = \hat{A}_{21}^T, \quad \hat{A}_{22} = B_{22}^*, \tag{3.9}$$

U_2 is a unit column-orthogonal matrix and $R(U_2) = N(X_1^T), B_{11}^*, B_{21}^*, B_{22}^*$ and $[*]_+$ are the same as in Theorem 1.

Proof. Because $R(X) \subseteq N(S^T)$, from (1.2) we obtain $X_2 = Q_2X = 0$. By the same argument as in the proof of Theorem 1, we know that Problem I is equivalent to

$$\begin{cases} A_{11}X_1 = B_1, \quad \forall A_{11} \in SR_0^{(n-p)\times(n-p)}, & (3.10) \\ A_{21}X_1 = B_2, & (3,11) \\ A_{12} = A_{21}^T, \quad A_{22} = A_{22}^T. & (3.12) \end{cases}$$

By [2] and [3], we know that the necessary and sufficient conditions for the solubility of (3.10) and (3.11) are respectively

$$X_1^TB_1 = B_1^TX_1 \geq 0, \ \mathrm{rank}(X_1^TB_1) = \mathrm{rank}(B_1) \quad \text{and} \quad B_2 = B_2X^+X_1. \tag{3.13}$$

Furthermore, the first two equations of (3.13) imply $B_1 = B_1X_1^+X_1$. Therefore, by (iii) – (v) of Lemma 2, we conclude that (3.13) is equivalent to (3.1).

Now, by using the results of [2] and [3], and the same argument as in the proof of Theorem 1, we can complete the proof of this theorem.

Corollary 3. Suppose $X, B \in R^{n\times m}$. If $R(X) = N(S^T)$, then Problem I has a solution if and only if

$$X^TB = B^TX \geq 0, B = BX^+X. \tag{3.14}$$

The general expression of the solution can be represented as

$$A = Q^T \begin{pmatrix} B_1X^+ & (B_2X_1^+)^T \\ B_2X^+ & A_{22} \end{pmatrix} Q, \forall A_{22} \in SR^{p\times p}, \tag{3.15}$$

where $B_1 = Q_1B, B_2 = Q_2B, X_1 = Q_1X$. If Problem I has a solution, then Problem II has a unique optimal approximate solution which can be represented as

$$\hat{A} = Q^T \begin{pmatrix} B_1X_1^+ & (B_2X_1^+)^T \\ B_2X_1^+ & B_{22}^* \end{pmatrix} Q, \tag{3.16}$$

where B_{22}^* is the same as in Theorem 1.

For $R(X) \subseteq N(S^T)$, by Theorem 2 we can get a numerical procedure for computing the unique optimal approximate solution $\hat{A}$ of Problem II that is similar to the procedure in **2.**

4. The General Case

Theorem 3. *Suppose $X, B \in R^{n\times m}$, and $S \in R_p^{n\times p}$ has factorization in the form (1.1). Introduce the notations in (1.3) and (1.4). Then, the necessary and sufficient condition for the solubility of Problem I is that there is a matrix $A_{12} \in R^{(n-p)\times p}$ satisfying*

$$X_1^TB_1 - X_1^TA_{12}X_2 = B_1^TX_1 - X_2^TA_{12}^TX_1 \geq 0, \tag{4.1}$$

$$\mathrm{rank}(B_1^TX_1 - X_2^TA_{12}^TX_1) = \mathrm{rank}(B_1^T - X_2^TA_{12}^T), \tag{4.2}$$

$$X_2^TB_2 - X_2^TA_{12}^TX_1 = B_2^TX_2 - X_1^TA_{12}X_2, \tag{4.3}$$

$$R(B_2^T - X_1^TA_{12}) \subset R(X_2^T). \tag{4.4}$$

Proof. From $AX = B$ we get

$$QAQ^TQX = QB,$$

which can be transformed into

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{4.5}$$

i.e.

$$A_{11}X_1 + A_{12}X_2 = B_1, \tag{4.6}$$

$$A_{21}X_1 + A_{22}X_2 = B_2. \tag{4.7}$$

By Lemma 1, that Problem I has a solution is equivalent to that (4.6)–(4.7) have the solution

$$A_{11} \in SR_0^{(n-p)\times(n-p)}, \quad A_{12} \in R^{(n-p)\times p}, \quad A_{21} = A_{12}^T, \quad A_{22} \in SR^{p\times p}.$$

We transform (4.6) and (4.7) into

$$A_{11}X_1 = B_1 - A_{12}X_2, \tag{4.8}$$

$$A_{22}X_2 = B_2 - A_{21}X_1 = B_2 - A_{12}^TX_1. \tag{4.9}$$

Thus by [1] and [2], we can obtain the conclusion of this theorem.

Corollary 4. Suppose $X, B \in R^{n\times m}$, and $S \in R_p^{n\times p}$ has factorization in the form (1.1). Introduce the notations in (1.3) and (1.4). Then, the necessary conditions for the solvability of Problem I are

$$X^TB = B^TX, \quad R(B^T) \subset R(X^T). \tag{4.10}$$

Proof. If Problem I has a solution, then there is a matrix A_{12} satisfying (4.1) – (4.4). By (4.1) and (4.3), we have

$$X^TB = X_1^TB_1 + X_2^TB_2 = B_1^TX_1 + B_2^TX_2 = B^TX.$$

By (4.4), there is a matrix $G_1 \in R^{p\times p}$, satisfying

$$B_2^T - X_1^TA_{12} = X_2^TG_1, \ i.e.\ B_2^T = X^TA_{12} + X_2^TG_1. \tag{4.11}$$

From (4.1) and (4.2), we have

$$R(B_1^T - X_2^TA_{12}^T) = R((B_1^T - X_2^TA_{12}^T)X_1) = R(X_1^T(B_1 - A_{12}X_2)) \subset R(X^T).$$

Therefore, there is a matrix $G_2 \in R^{(n-p)\times(n-p)}$, satisfying

$$B_1^T - X_2^TA_{12}^T = X_1^TG_2,$$

i.e.

$$B_1^T = X_2^TA_{12}^T + X_1^TG_2. \tag{4.12}$$

From (4.11) and (4.12), we have

$$B^TQ^T = (B_1^T, B_2^T) = (X_1^T, X_2^T)\begin{pmatrix} G_2 & A_{12} \\ A_{12}^T & G_1 \end{pmatrix}.$$

Let

$$G \triangleq \begin{pmatrix} G_2 & A_{12} \\ A_{12}^T & G_1 \end{pmatrix}.$$

Then $B^TQ^T = (QX)^TG$, i.e. $B^T = (QX)^TGQ = X^TQ^TGQ$,

$$R(B^T) \subset R(X^T).$$

Corollary 5. Suppose $X, B \in R^{n\times m}$, and $S \in R_p^{n\times p}$ has factorization in the form (1.1). Introduce the notations in (1.3) and (1.4). Then the sufficient conditions for the solvability of Problem I are

$$X^TB = B^TX \geq 0, \quad \text{rank}(X^TB) = \text{rank}(B). \tag{4.13}$$

Proof. We now prove that there is a matrix A_{12} satisfying (4.1)–(4.4). Because (4.13) holds, from [2] we know that there is a matrix $G^T \geq 0$ satisfying

$$G^TX = B, \quad i.e.\ B^T = X^TG.$$

Write

$$QGQ^T \triangleq \begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix} \geq 0.$$

Then, $G_2 = G_2^T \geq 0, G_1 = G_1^T \geq 0, G_3 = G_4^T$,

$$B^TQ^T = X^TGQ^T = X^TQ^TQGQ^T = X^TQ^T \begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix}, \tag{4.14}$$

whereas

$$B^TQ^T = (B_1^T, B_2^T), \quad X^TQ^T = (X_1^T, X_2^T).$$

Thus

$$B_1^T = X_1^TG_2 + X_2^TG_4, \tag{4.15}$$

$$B_2^T = X_1^TG_3 + X_2^TG_1. \tag{4.16}$$

Therefore

$$B_1^TX_1 = X_1^TG_2X_1 + X_2^TG_4X_1, \tag{4.17}$$

$$B_2^TX_2 = X_1^TG_3X_2 + X_2^TG_1X_2. \tag{4.18}$$

From these we can obtain

$$B_1^TX_1 - X_2^TG_4X_1 = X_1^TG_2X_1 = X_1^TB_1 - X_1^TG_4^TX_2 \geq 0, \tag{4.19}$$

$$B_2^TX_2 - X_1^TG_3X_2 = X_2^TG_1X_2 = X_2^TB_2 - X_2^TG_3^TX_1 \geq 0. \tag{4.20}$$

By (4.17), Lemma 3 and (4.15), we can get

$$\text{rank}(B_1^TX_1 - X_2^TG_4X_1) = \text{rank}(X_1^TG_2X_1) = \text{rank}(X_1^TG_2) = \text{rank}(B_1^T - X_2^TG_4). \tag{4.21}$$

From (4.16), we know that

$$R(B_2^T - X_1^TG_3) = R(X_2^TG_1) \subset R(X_2^T). \tag{4.22}$$

Take $A_{12} = G_3$, $A_{12}^T = G_4 = G_3^T$, and substitute them into (4.19) – (4,22). Then (4.1) – (4.4) hold.

Therefore, Problem I has a solution.

Note 1. Theorem 3 only gives an equivalent form of conditions for the solvability of Problem I for the general case. But from it we can easily obtain the conditions for the solvability of Problem I for the two cases in **2** and **3**.

Note 2. In the general case, how to give the necessary and sufficient conditions for the solvability is an open problem. The conditions should be stronger than (4.10) but weaker than (4.13).

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