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A PRECONDITIONER DETERMINED BY A SUBDOMAIN COVERING THE INTERFACE*1)

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Abstract

A method for construction of a preconditioner for the capacitance matrix on the interface is described. The preconditioner is determined by a subdomain covering the interface, and the condition number of the preconditioned matrix is dependent on the width of the covering subdomain and independent of the discrete mesh size and discontinuity of the coefficients of the differential operator. Some applications of our theory are presented at last.

1. Introduction

Let $\Omega \subset R^2$ be a polygonal region, and

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial u}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + cu.$$

be an elliptic operator defined on it; here, $(a_{i,j})_{i,j=1,2}$ is symmetric positive definite and bounded from above and below on Ω, $c \geq 0$.

$$\begin{cases} a(u,v) = (f,v), \quad v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

is the variational form of the boundary value problem, with the bilinear form

$$a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + cuv\Big].$$

For convenience we discuss only the homogeneous Dirichlet boundary value problem here. The norm in $H_0^1(\Omega)$ introduced by $a(\cdot,\cdot)$ is equivalent to the original one. $H_0^1(\Omega)$ will be treated as a Hilbert space with inner product $a(\cdot,\cdot)$ in the following.

$(1,1)$ is discretized by the finite element method. Triangulation and linear continuous element will be discussed. The triangulation is supposed to be local and regular. The discrete form of (1.1) is

$$\begin{cases} a(u,v) = (f,v), \quad v \in S_0^h(\Omega), \\ u \in S_0^h(\Omega). \end{cases} \tag{1.2}$$

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The domain Ω is decomposed into two non-overlapping subdomains Ω_1 and Ω_2 by the interface Γ which coincides with the finite element triangulation. $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2, \bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma, \Omega_1 \cap \Omega_2 = \emptyset$.

$\hat{\Omega}$ represents the set of finite element node points in Ω, $\hat{\Omega}_1 = \hat{\Omega} \cap \Omega_1, \hat{\Omega}_2 = \hat{\Omega} \cap \Omega_2, \hat{\Gamma} = \Gamma \cap \hat{\Omega}$. (1.2) may be written in matrix -vector form

$$\begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \tag{1.3}$$

where x_0, x_1 and x_2 are vectors corresponding to restrictions of the finite element function on $\hat{\Gamma}, \hat{\Omega}_1$ and $\hat{\Omega}_2$. (1.3) can be attributed to a small scale problem on $\hat{\Gamma}$ by Gauss elimlnation

$$Cx_0 = d \tag{1.4}$$

where

$$C = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20},$$

$$d = b_0 - A_{01}A_{11}^{-1}b_1 - A_{02}A_{22}^{-1}b_2,$$

and C is the capacitance matrix or Schur complement.

When (1.4) is solved, (1.3) will become two isolated Dirichlet boundary value problems. The iterative method is often used to solve (1.4). Since Cond $(C) = O(h^{-1})$, a proper preconditioner is necessary. There are many preconditioners constructed in recent years ([1–4] and the probing technique in [4]), but those preconditioners are proved or verified numerically to be spectrally equivalent to the capacitance matrix, but it should be noted that the condition number of the preconditioned matrix by these preconditioners will depend on the shape, size of Ω, Ω_1 and Ω_2 and the coefficients of the differential operator.

We will construct a new preconditioner in this paper, which is determined by a subdomain covering the interface. The condition number of the preconditioned matrix is independent of the subdomains and the interface $(\Omega_1, \Omega_2, \Gamma)$ and discontinuity of the coefficients of the differential operator, which is determined by the width of the covering subdomain.

2. A Preconditioner Determined by a Subdomain Covering the Interface

Ω_0 is a subdomain of Ω covering Γ, the boundary of which coincides with the finite element mesh line. Ω_0 has a uniform overlap with Ω_1 and Ω_2; the width of Ω_0 is of order $O(\delta)$. $\Omega_{01} = \Omega_0 \cap \Omega_1$, $\Omega_{02} = \Omega_0 \cap \Omega_2$.

$\{\phi_i, i \in \hat{\Omega}\}$ is the set of the usual finite element basis functions. The element of the stiffness matrix is

$$a_{ij} = a(\phi_i, \phi_j), \quad i, j \in \hat{\Omega};$$

the element of the capacitance matrix is

$$c_{ij} = a(\tilde{\phi}_i, \tilde{\phi}_j), \quad i, j \in \hat{\Gamma},$$

i.e. , the capacitance matrix is

$$C = (a(\tilde{\phi}_i, \tilde{\phi}_j))_{i,j \in \hat{\Gamma}} \tag{2.1}$$

where $\tilde{\phi}_i$ is a finite element function, $\tilde{\phi}_i\Big|_{\Gamma} = \phi_i|_{\Gamma}$, $\tilde{\phi}_i\Big|_{\partial\Omega} = 0$ and $\tilde{\phi}_i$ is discrete harmonic in Ω_1 and Ω_2. The right-hand side of (1.4) is $d = ((f, \tilde{\phi}_i))_{i \in \hat{\Gamma}}$.

Our preconditioner Q is defined as

$$Q = (q_{ij}) = (a(\hat{\phi}_i, \hat{\phi}_j))_{i,j \in \hat{\Gamma}} \tag{2.2}$$

where $\hat{\phi}_i$ is a finite element function, $\hat{\phi}_i\Big|_{\Gamma} = \phi_i|_{\Gamma}$, $\hat{\phi}_i\Big|_{\Omega - \Omega_0} = 0$ and $\hat{\phi}_i$ is discrete harmonic in Ω_{01} and Ω_{02}.

3. Estimation of the Condition Number

It is obvious that $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, i.e., $\{\Omega_0, \Omega_1, \Omega_2\}$ amounts to an open cover on Ω. There must be a unit decomposition $\{\psi_0, \psi_1, \psi_2\}$ belong to this open cover, $0 \le \psi_i \le 1$, Supp $(\psi_i) \subset \bar{\Omega}_i, i = 0, 1, 2$, and $\psi_0 + \psi_1 + \psi_2 = 1$ on Ω. We suppose these unit decomposition functions are Lipschitz continuous and L is the maximum of these three Lipschitz constants. We have

Theorem 3.1. *There exists a constant C independent of the finite element triangulation and the shape of the domain, subdomain and splitting line, so that*

$$\text{Cond}\,(Q^{-1}C) \le C(1 + L^2).$$

Before we prove this theorem, we introduce some lemmas. $S_0^h(\Omega_0)$, $S_0^h(\Omega_1)$ and $S_0^h(\Omega_2)$ are subspaces of $S_0^h(\Omega)$, $S_0^h(\Omega_i) = \{u \in S_0^h(\Omega), \text{Supp } (u) \subset \bar{\Omega}_i\}$, and P_i is the orthogonal projection operator from $S_0^h(\Omega)$ to $S_0^h(\Omega_i)$ under the inner product $a(\cdot, \cdot)$, $i = 0, 1, 2$.

Lemma 3.1.[6] *If there exists a constant C, and if for any $u \in S_0^h(\Omega)$ there exist $u_i \in S_0^h(\Omega_i)$ so that $u = u_0 + u_1 + u_2$ and $\sum_{i=0}^2 ||u_i||^2 \le C||u||^2$, we have*

$$a(u, u) \le Ca\Big(\sum_{i=0}^{2} P_i u, u\Big).$$

Lemma 3.2. *There exists a constant C independent of the finite element triangulation so that for any $u \in S_0^h(\Omega)$*

$$\frac{1}{C(1 + L^2)} \le \frac{a(\sum_{i=0}^2 P_i u, u)}{a(u, u)} \le 2. \tag{3.1}$$

Proof.

$$a\Big(\sum_{i=0}^{2} P_i u, u\Big) = \sum_{i=0}^{2} a(P_i u, P_i u) \le \sum_{i=0}^{2} a_{\Omega_i}(u, u)$$

where

$$a_{\Omega_i}(u,v)=\int_{\Omega_i}\Big[\sum_{i,j=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+cu\dot{v}\Big].$$

It is obvious that[8]

$$a\Big(\sum_{i=0}^{2}P_iu,u\Big)\le 2a(u,u).$$

For any $u\in S_0^h(\Omega)$, we have $u=\sum_{i=0}^{2}(\psi_iu)$, where ψ_iu is a continuous function and $\psi_iu\in H_0^1(\Omega_i)$, $i=0,1,2$. We use I_i to represent the interpolation operator from $C_0(\Omega_i)$ to $S_0^h(\Omega_i)$. Then

$$u=\sum_{i=0}^{2}I_i(\psi_iu).$$

It is obvious that $I_i(\psi_iu)\in S_0^h(\Omega_i)$. We have got a decomposition of u, by the Poincare inequality

$$||I_i(\psi_iu)||^2\le C|I_i(\psi_iu)|_{1,\Omega_i}^2=C\sum_{T\subset\Omega_i}|I_i(\psi_iu)|_{1,T}^2.$$

On any element T of the triangulation,

$$|I_i(\psi_iu)|_{H^1(T)}^2=|I_i[(\psi_i-\bar{\psi}_i+\bar{\psi}_i)u]|_{H^1(T)}^2$$

(where $\bar{\psi}_i$ is the average value of ψ_i on T)

$$\le 2|I_i(\bar{\psi}_iu)|_{H^1(T)}^2+2|I_i[(\psi_i-\bar{\psi}_i)u]|_{H^1(T)}^2$$

(by the inverse inequality of the finite element space and $0\le\psi_i\le 1$)

$$\le 2|I_iu|_{H^1(T)}^2+2Ch^{-2}||I_i[(\psi_i-\bar{\psi}_i)u]||_{L^2(T)}^2$$

$$\Big(||I_i[(\psi_i-\bar{\psi}_i)u]||_{L^2(T)}^2=\int_T|I_i[(\psi_i-\bar{\psi}_i)u]|^2$$

$$\le|T|\cdot\max_T|I_i[(\psi_i-\bar{\psi}_i)u]|^2\le L^2h^2|T|max_T|u|^2$$

$$\le CL^2h^2[\frac{\int_T|u|}{|T|}]^2|T|\le CL^2h^2[\frac{\int_T|u|^2}{|T|}]|T|\Big)$$

$$\le 2|u|_{H^1(T)}^2+2CL^2||u||_{L^2(T)}^2\le C(1+L^2)||u||_{H^1(T)}^2.$$

By Lemma 3.1, Lemma 3.2 is proved.

Proof of Theorem 3.1. Cond $(Q^{-1}C)$ may be estimated by the ratio of the upper and lower bounds of the generalized Rayleigh quotient

$$\frac{(CQ^{-1}Cx,x)}{(Cx,x)}. \tag{3.2}$$

For $x\in R^{|\hat{\Gamma}|}$, let $\tilde{u}_x=\sum_{i\in\hat{\Gamma}}x_i\tilde{\phi}_i$, $\hat{u}_x=\sum_{i\in\hat{\Gamma}}x_i\hat{\phi}_i$. It is obvious that $\tilde{u}_x=0|_{\partial\Omega}$ and $\hat{u}_x=0|_{\partial\Omega_0}$, $\tilde{u}_x$ is discrete harmonic in Ω_1 and Ω_2, $\hat{u}_x$ is discrete harmonic in Ω_{01} and

Ω_{02},

$$\frac{(CQ^{-1}Cx,x)}{(Cx,x)} = \frac{a(\tilde{u}_{Q^{-1}Cx},\tilde{u}_x)}{a(\tilde{u}_x,\tilde{u}_x)} = \frac{a(\hat{u}_{Q^{-1}Cx},\tilde{u}_x)}{a(\tilde{u}_x,\tilde{u}_x)} = \frac{a(P_0\tilde{u}_x,\tilde{u}_x)}{a(\tilde{u}_x,\tilde{u}_x)}.$$

In the last step, we have used

$$\hat{u}_{Q^{-1}Cx} = P_0\tilde{u}_x.$$

Let $P_0\tilde{u}_x|_{\hat{\Gamma}} = y$. We need only to show $y = Q^{-1}Cx$. Obviously,

$$Qy = a(P_0\tilde{u}_x,\hat{\phi}_i) = a(P_0\tilde{u}_x,\phi_i) = a(\tilde{u}_x,\phi_i) = a(\tilde{u}_x,\tilde{\phi}_i) = Cx.$$

Since

$$P_1\tilde{u}_x = P_2\tilde{u}_x = 0,$$

we have

$$\frac{(CQ^{-1}Cx,x)}{(Cx,x)} = \frac{a(\sum_{i=0}^{2} P_i\tilde{u}_x,\tilde{u}_x)}{a(\tilde{u}_x,\tilde{u}_x)}.$$

By (3.1) we obtain

$$\frac{1}{C(1+L^2)} \le \frac{(CQ^{-1}Cx,x)}{(Cx,x)} \le 1.$$

The theorem is proved.

4. The Condition Number is Independent of the Discontinuity of the Coefficients of the Differential Operator

We will discuss only the model problem

$$Lu = -a\Delta u = f, \quad u \in H_0^1(\Omega) \tag{4.1}$$

where $a = a_1$ on Ω_1, $a = a_2$ on Ω_2, and a_1, a_2 are two positive constants, and $a_1 < a_2$.

Proposition 4.1. *The condition number of the finite element capacitance matrix of* (4.1) *is*

$$\mathrm{Cond}\,(C) = O\Big(\frac{a_2}{a_1}h^{-1}\Big).$$

Proof.

$$\frac{(Cx,x)}{(x,x)} = \frac{a(\tilde{u}_x,\tilde{u}_x)}{(x,x)} = \frac{a_1|\tilde{u}_x|^2_{1,\Omega_1} + a_2|\tilde{u}_x|^2_{1,\Omega_2}}{(x,x)}.$$

Hence

$$a_1 \le \frac{a_1|\tilde{u}_x|^2_{1,\Omega_1}}{(x,x)} \le \frac{(Cx,x)}{(x,x)} \le \frac{a_2|\tilde{u}_x|^2_{1,\Omega_2}}{(x,x)} \le a_2h^{-1}.$$

We get

$$\mathrm{Cond}\,(C) = O\Big(\frac{a_2}{a_1}h^{-1}\Big).$$

Theorem 4.1. *There exists a constant C independent of the discontinuity and the shape of the domains and subdomains so that*

$$\mathrm{Cond}\,(Q^{-1}C) \le C(1+L^2),$$

where the preconditioner Q is defined by (2.2).

Proof. We need only to show that the constant C in Lemma 3.2 is independent of the discontinuity of the coefficients, i.e., we need to show

$$\sum_{i=0}^{2} a(I_i(\psi_i u), I_i(\psi_i u)) \leq C(1+L^2)a(u,u) \tag{4.2}$$

where C is independent of the discontinuity of the coefficients.

$$\begin{aligned} a(I_1(\psi_1 u), I_1(\psi_1 u)) &= a_1|I_1(\psi_1 u)|^2_{1,\Omega_1} \leq Ca_1(1+L^2)\|u\|^2_{1,\Omega_1} \\ &= C(1+L^2)a_{\Omega_1}(u,u). \end{aligned} \tag{4.3}$$

Similarly,

$$a(I_2(\psi_2 u), I_2(\psi_2 u)) \leq C(1+L^2)a_{\Omega_2}(u,u), \tag{4.4}$$

$$\begin{aligned} a(I_0(\psi_0 u), I_0(\psi_0 u)) &= a_1|I_0(\psi_0 u)|^2_{1,\Omega_{01}} + a_2|I_0(\psi_0 u)|^2_{1,\Omega_{02}} \\ &\leq C(1+L^2)(a_1\|u\|^2_{1,\Omega_{01}} + a_2\|u\|^2_{1,\Omega_{02}}) \leq C(1+L^2)a_{\Omega_0}(u,u). \end{aligned} \tag{4.5}$$

From (4.3),(4.4) and (4.5) we get (4.2), and the theorem is proved.

When the covering subdomain Ω_0 is a strip of width 2δ, the Lipschitz constant will be $L=\frac{1}{\delta}$, and the condition number is Cond $(Q^{-1}C)=O(1+\frac{1}{\delta^2})$. The wider the covering subdomain, the smaller the condition number, and vice versa. But the increace of the width of the covering subdomain will increase the cost of preconditioning.

The resolution of $Qx=d$ is equivalent to the resolution of a Dirichlet problem on Ω_0:

$$A_0 \begin{pmatrix} x \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}, \tag{4.6}$$

$$A_0 = (a(\phi_i,\phi_j)) - i, \quad j \in \hat{\Omega}_0.$$

The capacitance matrix is a full matrix in general. When the width of the covering subdomain is small, the preconditioner may have a band structure; for example, when the triangulation is uniform and $\delta=h$ (the mesh size), Q will be a tri-diagonal matrix, and when $\delta=2h$, Q will be a seven-diagonal matrix. When the covering subdomain is fixed and the triangulation is refined, the bandwith of Q will increase, but the relative bandwith (bandwith/order of the matrix) of Q remains unchanged.

5. Some Applications

There is a great freedom in the shape and width of the covering subdomain. Some preconditioners constructed in other papers may be deduced from the covering subdomain preconditioner when the covering subdomain is properly selected.

If a rectangular region is decomposed into many strips by parallel interfaces, the capacitance matrix will be a block tri-diagonal matrix. If we select a strip to cover every interface, and these covering strips do not intersect each other, the covering subdomain preconditioner will be a block diagonal matrix. If the minimum width of these covering subdomains is of order $O(\delta)$, the condition number will be Cond $(Q^{-1}C) = O(1+\frac{1}{\delta^2})$. This result is equivalent to that of [7].

If the domain is a rectangular and the operator is Laplacian, the domain is decomposed into two smaller rectangles by a straight line. $C = S_1 + S_2$ ([2]). It was proved that C is spectrally equivalent to S_1 and S_2. If Ω_1 is wider than Ω_2, we may select a subdomain Ω_1' of Ω_1 so that Ω_1' and Ω_2 are symmetric about the interface and the covering subdomain is $\Omega_0 = \Omega_1' \cup \Omega_2 \cup \Gamma$, and the covering subdomain preconditioner will be $Q = 2S_2$. From Theorem 3.1 we know Cond $(S_2^{-1}C) = O(1+\frac{1}{\delta^2})$, where δ is the width of Ω_2.

We may select a fictitious domain Ω_2' containing Ω_2 so that Ω_2' and Ω_1 are symmetric about the interface. Ω may be treated as a covering subdomain of $\Omega_2 \cup \Omega_2' \cup \Gamma$, and from Theorem 3.1 we know Cond $(S_1^{-1}C) =$ Cond $(C^{-1}S_1) = O(1+\frac{1}{\delta^2})$, where δ is the width of Ω_2. We will discuss the fictitious technique further in another paper.

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