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THE MULTIGRID METHOD OF NONCONFORMING FINITE ELEMENTS FOR SOLVING THE BIHARMONIC EQUATION

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Abstract

An optimal order of the multigrid method is given in energy-norm for the nonconforming finite element for solving the biharmonic equation, by using the nodal interpolation operator as the transfer operator between grids.

1. Introduction

Several aspects of the nonconforming finite element method have been discussed in [1-5]. In this paper, we will introduce the multigrid method and prove that the multigrid method of nonconforming finite elements can attain the same optimal convergence order as the nonconforming finite element method in energy-norm.

The multigrid method of conforming finite elements has been studied[7–8]. For the multigrid method of nonconforming finite elements, because the finite element spaces associated with the nets are not nest ($V_{k-1} \not\subset V_K$), it is difficult to define the transfer operator or the prolongation operator , especially when the nonconforming element interpolation order is greater than 2, as in Fraeijs de Veubeke triangular elements, the Adini rectangular element, the Zienkiewicz triangular element, etc.[1,4]. For the Morley elements, the transfer operators between grids can be defined e.g. [10–12]. At present we try to use the nodal interpolation operator as the intergrid transfer operator. The error estimate of an optimal order convergence property of the multigrid method is given in energy-norm for the multigrid method of nonconforming finite elements such as the Morley element, Fraeijs de Veubeke triangular elements, the Adini rectangle element and the Zienkiewicz triangular element. Furthermore, the method presented in this paper is effective for other high order conforming or nonconforming element interpolations.

The remainder of the paper is organized as follows. In Section 2, the nonconforming finite elements and their properties of approximating the biharmonic equation as well as properties of the nodal interpolation operator as the intergrid transfer operator are stated. In Section 3, the multigrid method is given. In Section 4, the convergence

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properties of the multigrid method of nonconforming finite elements are analyzed in energy-norm.

2. The Biharmonic Equation and Nonconforming Finte Element

We consider the biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ U = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded polygonal domain, $f \in H^{-l}$, $l = 0, 1$.

The boundary problem (2.1) has a unique solution $u \in H^{4-l}(\Omega) \cap H_0^2(\Omega)$, which satisfies the elliptic regularity[6]:

$$\| u \|_{H^{4-l}(\Omega)} \leq c(\Omega) \| f \|_{H^{-l}(\Omega)} . \tag{2.2}$$

Let $V = H_0^2(\Omega)$. A variational form of equation (2.1) can be written as: Find $u \in V$ such that

$$a(u, v) = \int_\Omega f v dx, \quad \forall v \in V \tag{2.3}$$

where $a(u.v) = \int_\Omega D^2 u D^2 v dx = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial y_j} \frac{\partial^2 v}{\partial x_i \partial y_j} dx.$

Let Γ_1 be an initial triangulation and satisfy the quasi-uniformity condition, namely, for $\forall \tau \in \Gamma_1$,

$$\frac{h(\tau)}{\rho(\tau)} \leq \lambda \tag{2.4}$$

where $h(\tau)$ denotes the diameter of triangle τ, $\rho(\tau)$ denotes the diameter of the inscribed circle for triangle τ, and λ is a constant independent of $h(\tau)$. If the following finite element space is the Zienkiewicz finite element space, then the above initial triangulation must be in such a way that the three sides of every triangle in Γ_1 are parallel to three given directions[1].

We now construct Γ_{k+1} for $k \geq 1$, inductively. For each $\tau \in \Gamma_k$, four triangles in Γ_{k+1} are obtained by connecting the midpoints of the edges of triangle τ. Thus Γ_{k+1} satisfies the quasi-uniform condition (2.4). For the Zienkiewicz finite element, three sides of all $\tau \in \Gamma_{k+1}$ are also parallel to three given directions. We denotes $h_k = \max\{ \operatorname{diam} \tau, \tau \in \Gamma_k\}$. Then $h_{k+1} = \frac{1}{2} h_k$.

Let V_k be the nonconforming finite element space associated with triangulation Γ_k. Then $V_k \not\subset V$ and V_k is affine (such as the Adini element and the Zienkiewicz element) or almost-affine (such as the Morley element and Fraeijs de Veubeke elements). Thus the interpolation polynomials of nonconforming elements satisfy the inverse inequality[4], namely, there exists a constant c independent of h_k, such that

$$| v_k |_{H^2(\tau)} \leq c h_k^{-2} \| v_k \|_{L^2(\tau)}, \quad for\ \forall v_k \in V_k, \tau \in \Gamma_k, \tag{2.5}$$

where $|v_k|_{H^2(\tau)} = \int_\tau D^2 v_k D^2 v_k dx$.

The finite element approximation of equation (2.3) is as follows : Find $u_k \in V_k$ such that

$$a_k(u_k, v_k) = \int_\Omega f v_k dx, \quad \forall v_k \in V_k \tag{2.6}$$

where the bilinear form $a_k(u_k, v_k) = \sum_{\tau \in \Gamma_k} \int_\tau D^2 u_k D^2 v_k dx$.

We define a matrix operator A_k associated with the bilinear form $a_k(\cdot\ ,\ \cdot)$ by

$$a_k(u, v) = (A_k u, v), \quad \forall u, v \in V_k$$

and the mesh-dependent energy norm by

$$\| u \|_k = \sqrt{a_k(u,u)}, \quad \forall u \in V_k.$$

Let I_k denote a nodal interpolation operator of the nonconforming element from Sobolev space $H^3(\Omega) \cap H_0^1(\Omega)$ onto V_k. Then according to [1–4], we have the interpolaton approximate property as follows:

Lemma 1. *Assume that the triangulation $\Gamma_k (k \geq 1)$ is quasi-uniformity. Then for $\forall \tau \in \Gamma_k$, there exists a constant c independent of h_k such that: for all $u \in H^3(\tau)$*

$$| u - I_k u |_{m,\tau} \leq c h_k^{3-m} | u |_{3,\tau}, \quad 0 \leq m \leq 3. \tag{2.7}$$

If I_k is used as the transfer operator between grids in the following multigrid method, then we have

Corollary 1. For $\forall \tau \in \Gamma_k$ and $u_k \in V_{k-1}$, there exists a constant c independent of h_k, such that

$$| u_k - I_k u_k |_{m,\tau} \leq c h_k^{2-m} | u_k |_{2,\tau}, \quad 0 \leq m \leq 2. \tag{2.8}$$

Proof. By the inverse inequality (2.5) and the interpolation approximate property (2.7), (2.8) can be obtained immediately.

Corollary 2. For $\forall v \in V_{k-1}$ and $\tau \in \Gamma_k$, we have

$$| v^I - (I_k v)^I |_{1,\tau} \leq c h_k | v |_{2,\tau} \tag{2.9}$$

where c is a constant independent of h_k, and v^I denotes a linear interpolation of v.

Proof. Since v^I*and* $(I_k v)^I$ are linear interpolation functions of v and $I_k v$ respectively at the $k-1$ level and the k level, by the interpolation approximation theorem[4] and (2.8) for $m = 1$, we have

$$\begin{aligned} | v^I - (I_k v)^I |_{1,\tau} &\leq | v^I - v |_{1,\tau} + | v - I_k v |_{1,\tau} + | I_k v - (I_k v)^I |_{1,\tau} \\ &\leq c h_{k-1} | v |_{2,\tau} + c h_k | v |_{2,\tau} + c h_k | I_k v |_{2,\tau} \\ &\leq c h_k | v |_{2,\tau} + c h_k | I_k v - v |_{2,\tau} + c h_k | v |_{2,\tau} \\ &\leq c h_k | v |_{2,\tau} \end{aligned}$$

where we used $h_k = \frac{1}{2} h_{k-1}$.

Corollary 3. For $\forall v \in V_{k-1}$, we have

$$\| (I - I_k)v \|_k \leq c \| v \|_{k-1} \cdot \tag{2.10}$$

Proof. In virtue of Corollary 1, $m = 2$, summing over all $\tau \in \Gamma_k$, (2.10) is obtained.

Corollary 4. For $\forall v \in V_{k-1}$, we have

$$\| I_k v \|_k \leq \|v\|_{k-1}. \tag{2.11}$$

Proof. Due to (2.10) of Corollary 3 and the triangle inequality, we can get (2.11).

3. The Multigrid Algorithm

We now introduce the multigrid algorithm of the nonconforming finite element for solving the biharmonic equation (2.1).

From the spectral theorem, there exist eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n_k}$ and eigenfunctions $\psi_1, \psi_2, \cdots \psi_{n_k} \in V_k$ at the k level, such that

$$a_k(\psi_i, v) = \lambda_i(\psi_i, v), \quad \forall v \in V_k$$

where $(\psi_i, \psi_j) = \delta_{ij}$ (the Kronecher delta). According to the inverse inequality (2.5), there exists a constant $c > 0$, such that

$$\lambda_{n_k} \leq c h_k^{-4}.$$

The nodal interpolation operator is used as the intergrid transfer operator and the k level algorithm is defined as follows.

Assume z_0 is the initial guess value of the solution. $MG(k, z_0, G)$ is an approximate solution to the following problem : Find $z \in V_k$ such that

$$a_k(z, v) = G(v), \quad \forall v \in V_k, G \in V_k' \tag{3.1}$$

where V_k' denotes the conjugate space of $V_k, G(v) = \int_\Omega f v dx$.

For $k = 1$, $MG(1, z_0, G)$ is the solution obtained from a direct method.

For $k > 1$, there are three steps.

1) Correction step. Let $\bar{G} \in V_{k-1}'$ be defined by

$$\bar{G}(v) = G(I_k v) - a_k(z_0, I_k v) = a_k(z - z_0, I_k v), \quad \forall v \in V_{k-1}.$$

Let $q_i \in V_{k-1} (0 \leq i \leq p,\ p = 2 \text{ or } 3)$ be defined recursively by

$$q_0 = 0, \quad q_i = MG(k-1, q_{i-1}, \bar{G}).$$

Let $z_1 = z_0 + I_k q_k$.

2) Smoothing step. The weighted-Jacobi iteration method is used and $z_i \in V_k$ $(2 \leq i \leq m+1)$ is defined as

$$(z_i - z_{i-1}, v) = \frac{1}{\Lambda_k}(G(v) - a_k(z_{i-1}, v)), \quad \forall v \in V_k \tag{3.2}$$

where $\Lambda_k = ch_k^{-4}$, and m is a positive integer to be fixed.

Let $z_{m+1} = MG(k, z_0, G)$.

3) Stepsize control step. The final approximation is determined by

$$z_{m+2} = z_1 + \alpha_{\min}(z_{m+1} - z_1).$$

Here $\alpha_{\min} = \frac{(f-A_k z_1)(z_{m+1}-z_1)}{(z_{m+1}-z_1)^T A_k (z_{m+1}-z_1)}$ is chosen such that

$$\| z - z_{m+2} \|_k = \min_{\alpha \in \mathcal{R}} \| z - z_1 - \alpha(z_{m+2} - z_1) \|_k .$$

The parameter $\alpha_{\min}$ ensures that the energy-norm of the error decreases in any case[12]. In the analysis of the algorithm, however, we will only estimate $\| z - z_{m+1} \|_k$.

The nested iteration fully multigrid algorithm is defined as

Let $\widehat{u}_1$ be the approximate solution of (2.6) which is obtained by the direct method. $\widehat{u}_k$ is the approximate solution of (2.6) ($k > 1$), which is obtained recursively by the following multigrid algorithm:

$$u_0^k = I_k \widehat{u}_{k-1},$$

$$u_l^k = MG(k, u_{l-1}^k, G), \quad 1 \le l \le r, \quad G(v) = \int_\Omega f v dx,$$

$$\widehat{u}_k = u_r^k$$

where r is a positive integer to be determined.

4.The Convergence Analysis

Before the convergence property of the multigrid algorithm is given, we first analyze the convergence property of the k level.

Given $v \in V_k$, then $v = \sum_{i=1}^{n_k} c_i \psi_i$. Defining the norm as follows:

$$||| v |||_{s,k} = \Big(\sum_{i=1}^{n_k} c_i^2 \lambda_i^{\frac{s}{2}} \Big)^{\frac{1}{2}}.$$

Note that $||| v |||_{2,k} = \| v \|_k$ and $||| v |||_{0,k} = \| v \|_{L^2}$. From [11], we have

Lemma 2. *For $\forall v_k \in V_k$, there has*

$$||| v_k |||_{1,k} \le (\| v_k^I \|_{H^1(\Omega)} + h_k \| v_k \|_k) \tag{4.1}$$

where v^I is a linear interpolation of v.

Let z be the exact solution of the k level, $e_0 = z - z_0, e_1 = z - z_1 = e_0 - I_k q_p, \cdots, e_{m+1} = z - z_{m+1}$.

Let $q \in V_{k-1}$ satisfy

$$a_{k-1}(q, v) = \bar{G}(v) = a_k(e_0, I_k v), \quad \forall v \in V_{k-1}. \tag{4.2}$$

Let $\widetilde{z}_1 = z_0 + I_k q$. The fuction $\widetilde{z}_i \in V_k \quad (2 \le i \le m+1)$ is defined recursively by

$$(\widetilde{z}_i - \widetilde{z}_{i-1}, v) = \frac{1}{\Lambda_k}(G(v) - a_k(\widetilde{z}_{i-1}, v)), \quad \forall v \in V_k. \tag{4.3}$$

Finally, let $\tilde{e}_i = z - \tilde{z}_i \quad (1 \le i \le m+1)$. Obviously, $\tilde{e}_{m+1}$ denotes the error of the k level iteration.

Now we will give the effect of the smoothing step, the proof of which can be found in [11].

Lemma 3 (Smoothing Property).

$$\| \tilde{e}_{m+1} \|_k \le c h_k^{-1} \frac{1}{\sqrt[4]{4m+1}} \| \tilde{e}_1 \|_{1,k} \,. \tag{4.4}$$

The next lemma will give the correction iteration error estimate of the Adini element and the Zienkiewicz element. The proof for the Morley element and Fraeijs de Veubeke elements is similar to our following proof. See also Brenner [10] as well as Peisker and Braess [11].

Lemma 4.

$$| \tilde{e}_1^I |_{H^1(\Omega)} \le c h_k \| e_0 \|_k \tag{4.5}$$

where $\tilde{e}_1^I$ is a linear interpolation of $\tilde{e}_1$.

Proof. By the triangle inequality and linear interpolation approximate property, for $\forall \tau \in \Gamma_k$, we have

$$\begin{aligned} | \tilde{e}_1^I |_{H^1(\Omega)} &= | e_0^I - (I_k q)^I |_{H^1(\Omega)} \\ &\le | e_0^I - e_0 |_{H^1(\Omega)} + | \tilde{e}_1 |_{H^1(\Omega)} + | I_k q - (I_k q)^I |_{H^1(\Omega)} \\ &\le c h_k \| e_0 \|_k + | \tilde{e}_1 |_{H^1(\Omega)} + c h_k \| I_k q \|_k \,. \end{aligned}$$

In (4.2), taking $v = q$, we get

$$\| q \|_{k-1} \le c \| e_0 \|_k \,. \tag{4.6}$$

By (2.11) and (4.6), we have

$$\begin{aligned} | \tilde{e}_1^I |_{H^1(\Omega)} &\le c h_k \| e_0 \|_k + | \tilde{e}_1 |_{H^1(\Omega)} + c h_k \| q \|_{k-1} \\ &\le c h_k \| e_0 \|_k + | \tilde{e}_1 |_{H^1(\Omega)} \,. \end{aligned} \tag{4.7}$$

To estimate $|\tilde{e}_1|_{H^1}$, we apply the duality technique. Since for the Zienkiewicz element and the Adini element, $V_k \subset C^0(\bar{\Omega}) \cap H_0^1$, then let $\phi = -\Delta \tilde{e}_1$ be the right term of the following biharmonic equation:

$$\begin{cases} \Delta^2 \xi = \phi, & \text{in } \Omega\,, \\ \xi = \dfrac{\partial \xi}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

whose variational equation makes sense for $\phi \in H^{-1}(\Omega)$. Ω is also a convex domain. Thus the elliptic regularity (2.2) implies that

$$\| \xi \|_{H^3(\Omega)} \le c \| \phi \|_{H^{-1}(\Omega)} = c | \tilde{e}_1 |_{H^1(\Omega)} \,.$$

$|\tilde{e}_1|_{H^1(\Omega)}$ may be written in the form

$$|\tilde{e}_1|^2_{H^1(\Omega)} = (\phi, \tilde{e}_1) = \Big\{(\phi, \tilde{e}_1) - \sum_{\tau\in\Gamma_k}\int_\tau D^2\xi D^2(e_0 - q)dx\Big\}$$
$$+ \Big\{\sum_{\tau\in\Gamma_k}\int_\tau D^2\xi D^2(e_0 - q)dx\Big\} = I_1 + I_2.$$

For I_1, since

$$I_1 = \Big\{(\phi, e_0) - \sum_{\tau\in\Gamma_k}\int_\tau D^2\xi D^2 e_0 dx\Big\} + \Big\{(\phi, q) - \sum_{\tau\in\Gamma_k}\int_\tau D^2\xi D^2 q dx\Big\} + (\phi, q - I_k q)$$

from the consistency error estimate[1,4],

$$\sup_{\substack{v\in V_k\\ v\neq 0}} \frac{|(\phi, v) - a_k(\xi, v)|}{\| v \|_k} \le ch_k |\xi|_{H^3(\Omega)}\,. \tag{4.8}$$

We obtain

$$\Big|\Big\{(\phi, e_0) - \sum_{\tau\in\Gamma_k}\int_\tau D^2\xi D^2 e_0 dx\Big\}\Big| \le ch_k |\xi|_{H^3(\Omega)} \| e_0 \|_k$$
$$\le ch_k |\tilde{e}_1|_{H^1(\Omega)} \| e_0 \|_k,$$
$$\Big|\Big\{(\phi, q) - \sum_{\tau\in\Gamma_{k-1}}\int_\tau D^2\xi D^2 q dx\Big\}\Big| \le ch_{k-1} |\xi|_{H^3(\Omega)} \| q \|_{k-1}$$
$$\le ch_{k-1} |\tilde{e}_1|_{H^1(\Omega)} \| q \|_{k-1}\,.$$

Applying the Schwarz inequality and Corollary 3 of Lemma 1, we have

$$|(\phi, q - I_k q)| \le |\tilde{e}_1|_{H^1(\Omega)} |q - I_k q|_{H^1(\Omega)} \le ch_k |\tilde{e}_1|_{H^1(\Omega)} \| q \|_{k-1}\,.$$

Hence by $h_k = 1/2h_{k-1}$ and (4.6), we get

$$|I_1| \le ch_k |\tilde{e}_1|_{H^1(\Omega)} \| e_0 \|_k + ch_{k-1} |\tilde{e}_1|_{H^1(\Omega)} \| q \|_{k-1}$$
$$+ ch_k |\tilde{e}_1|_{H^1(\Omega)} \| q \|_{k-1} \le ch_k |\tilde{e}_1|_{H^1(\Omega)} \| e_0 \|_k\,.$$

For I_2, since

$$I_2 = \sum_{\tau\in\Gamma_k}\int_\tau D^2(\xi - I_k\xi) D^2 e_0 dx + \sum_{\tau\in\Gamma_k}\int_\tau D^2(I_k\xi - I_k(I_{k-1}\xi)) D^2 e_0 dx$$
$$+ \sum_{\tau\in\Gamma_k}\int_\tau D^2(I_k(I_{k-1}\xi)) D^2 e_0 dx + \sum_{\tau\in\Gamma_{k-1}}\int_\tau D^2\xi D^2 q dx$$
$$= a_k(\xi - I_k\xi, e_0) + a_k(I_k\xi - I_k(I_{k-1}\xi), e_0)$$
$$+ a_k(I_k(I_{k-1}\xi), e_0) - a_{k-1}(\xi, q)$$

from Lemma 1, Corollary 3 of Lemma 1 and (4.6), we get

$$|I_2| \le \| \xi - I_k\xi \|_{H^2(\Omega)} \| e_0 \|_k + \| I_k\xi - I_k(I_{k-1}\xi) \|_k \| e_0 \|_k + |a_{k-1}(I_{k-1}\xi - \xi, q)|$$

$$\leq ch_k \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k + \| (\xi - I_{k-1}\xi) - I_k(\xi - I_{k-1}\xi) \|_k \| e_0 \|_k$$
$$+ \| \xi - I_{k-1}\xi \|_k \| e_0 \|_k + \| \xi - I_{k-1}\xi \|_{k-1} \| q \|_{k-1}$$
$$\leq ch_k \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k + ch_k \mid \xi - I_{k-1}\xi \mid_{H^3(\Omega)} \| e_0 \|_k$$
$$+ ch_{k-1} \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k + ch_{k-1} \mid \xi \mid_{H^3(\Omega)} \| q \|_{k-1}$$
$$\leq ch_k \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k + ch_{k-1} \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k .$$

Hence by $h_k = 1/2h_{k-1}$, we have

$$\mid I_2 \mid \leq ch_k \mid \xi \mid_{H^3(\Omega)} \| e_0 \|_k \leq ch_k \mid \tilde{e}_1 \mid_{H^1(\Omega)} \| e_0 \|_k .$$

Therefore

$$\mid \tilde{e}_1 \mid^2_{H^1(\Omega)} \leq \mid I_1 \mid + \mid I_2 \mid \leq ch_k \mid \tilde{e}_1 \mid_{H^1(\Omega)} \| e_0 \|_k . \tag{4.9}$$

By (4.7) and (4.9), we get (4.5) of Lemma 4.

In fact, for the Morley element and Fraeijs de Veubeke elements, (4.5) can be proved directly. (2.6) can be modified to

$$a_k(u_k, v_k) = \int_\Omega f v_k^I dx, \quad \forall v_k \in V_k$$

where v_k^I is a linear interpolation of v_k. Since $v_k^I \in H_1^0$, the above variational equation makes sense for $f \in H^{-1}(\Omega)$. By the consistency error estimte[2],

$$\sup_{\substack{v \in V_k \\ v \neq 0}} \frac{\mid (\phi, v) - a_k(\xi, v) \mid}{\| v \|_k} \leq ch_k \mid \xi \mid_{H^3(\Omega)}$$

where $\phi = -\Delta \tilde{e}_1^I \in H^{-1}(\Omega)$. Using a similar proof, we can get (4.5).

Lemma 5 (Approximation Properity).

$$\||| \tilde{e}_1 \|||_{1,k} \leq ch_k \| e_0 \|_k . \tag{4.10}$$

Proof. By Lemma 2 and Poincare's inequality, we have

$$\||| \tilde{e}_1 \|||_{1,k} \leq c(\| \tilde{e}_1^I \|_{H^1} + h_k \| \tilde{e}_1 \|_k) \leq c(\mid \tilde{e}_1^I \mid_{H^1} + h_k \| \tilde{e}_1 \|_k).$$

By Corollary 4 of Lemma 1 and (4.6), we have

$$\| \tilde{e}_1 \|_k \leq \| e_0 \|_k + \| I_k q \|_k \leq \| e_0 \|_k + c \| q \|_{k-1} \leq c \| e_0 \|_k .$$

Hence by Lemma 4, we get (4.10).

After having the smoothing property and the approximation property, we consider the error reduction in a couple cycle.

Theorem 1. *There exists a constant* $0 < \gamma < 1$ *and a positive integer* m *independent of* h_k *such that if*

$$\| q - q_p \|_{k-1} \leq \gamma^p \| q \|_{k-1}, \tag{4.11}$$

then

$$\| e_{m+1} \|_k \leq \gamma \| e_0 \|_k . \tag{4.12}$$

Proof. By the triangle inequality,

$$\| e_{m+1} \|_k \le \| e_{m+1} - \tilde{e}_{m+1} \|_k + \| \tilde{e}_{m+1} \|_k .$$

And $\| e_{m+1} - \tilde{e}_{m+1} \|_k$ satisfying

$$(e_i - \tilde{e}_i, v) = \frac{1}{\Lambda_k}(G(v) - a_k(e_{i-1} - \tilde{e}_{i-1}, v)), \quad \forall v \in V_k.$$

Hence, from the fact that in each relaxation step $||| \cdot |||_{s.k}$ is not increased, (2.11), (4.11) and (4.6), we have

$$\| e_{m+1} - \tilde{e}_{m+1} \|_k \le \| \tilde{e}_1 - e_1 \|_k = \| I_k(q - q_p) \|_k$$

$$\le c \| q - q_p \|_{k-1} \le c\gamma^p \| q \|_{k-1} \le c\gamma^p \| e_0 \|_k .$$

According to Lemma 3 and Lemma 5, we have

$$\| \tilde{e}_{m+1} \|_k \le ch_k^{-1} \frac{1}{\sqrt[4]{4m+1}} \| \tilde{e}_1 \|_{1,k} \le \frac{c}{\sqrt[4]{4m+1}} \| \tilde{e}_0 \|_k .$$

If $\gamma \in (0,1)$ is small enough , then $c\gamma^p < \frac{\gamma}{2}$. If m is large enough, then $\frac{c}{\sqrt[4]{4m+1}} < \frac{\gamma}{2}$. For such a choice, we have the result of Theorem 1.

In [1, 4], the Adini element and the Zienkiewicz element have the following error estimate.

Lemma 6. *Let $u \in H^3(\Omega)$ and $u_k \in V_k$ be the solutions of* (2.3) *and* (2.6), *respectively. Then there has*

$$\| u - u_k \|_k \le ch_k | u |_{H^3(\Omega)} . \tag{4.13}$$

For the Morley element and Fraeijs de Veubeke elements, (4.13) needs to be modified as follows:

$$\| u - u_k \|_k \le ch_k(| u |_{H^3(\Omega)} + h_k \| f \|_{L^2(\Omega)}). \tag{4.14}$$

Without loss of generality, we only use (4.13) to prove the following multigrid algorithm convergence. (4.14) is also used similarly.

Theorem 2. *If the parameter r is chosen large enough, then there exists a constant c independent of h_k such that*

$$\| u - \hat{u}_k \|_k \le ch_k | u |_{H^3(\Omega)} . \tag{4.15}$$

Proof. By (4.13) and (4.12), we have

$$\| u - \hat{u}_k \|_k \le \| u - u_k \|_k + \| u_k - \hat{u}_k \|_k \le ch_k | u |_{H^3(\Omega)} + c\gamma^r \| u_k - I_k\hat{u}_{k-1} \|_k . \tag{4.16}$$

Applying (2.7) and Corollary 4 of Lemma 1, we get

$$\| u_k - I_k\hat{u}_{k-1} \|_k \le \| u_k - u \|_k + \| u - I_k u_{k-1} \|_k + \| I_k u_{k-1} - I_k\hat{u}_{k-1} \|_k$$

$$\le ch_k | u |_{H^3(\Omega)} + \| u - I_k u \|_k + \| I_k(u - u_{k-1}) \|_k$$

$$+ \| I_k(u_{k-1} - \hat{u}_{k-1}) \|_k \le ch_k | u |_{H^3(\Omega)} + \| (u - u_{k-1}) - I_k(u - u_{k-1}) \|_k$$

$$+ \| u - u_{k-1} \|_k + c \| u_{k-1} - \hat{u}_{k-1} \|_{k-1} \leq ch_k | u |_{H^3(\Omega)} + ch_k | u - u_{k-1} |_{H^3(\Omega)}$$

$$+ ch_{k-1} | u |_{H^3(\Omega)} + c \| u_{k-1} - \hat{u}_{k-1}) \|_{k-1}$$

$$\leq ch_k | u |_{H^3(\Omega)} + ch_{k-1} | u |_{H^3(\Omega)} + c \| u_{k-1} - \hat{u}_{k-1} \|_{k-1} . \tag{4.17}$$

By virtue of $h_{k-1} = 2h_k$, (4.16) and (4.17), inductively, we have

$$\| u - \hat{u}_k \|_k \leq ch_k | u |_{H^3(\Omega)} + ch_k \gamma^r | u |_{H^3(\Omega)} + c\gamma^r \| u_{k-1} - \hat{u}_{k-1} \|_{k-1}$$

$$\leq ch_k | u |_{H^3(\Omega)} + ch_k \gamma^r | u |_{H^3(\Omega)} + c^2 h_{k-1} \gamma^{2r} | u |_{H^3(\Omega)}$$

$$+ \cdots + c^k h_1 \gamma^{kr} | u |_{H^3(\Omega)}$$

$$\leq ch_k | u |_{H^3(\Omega)} + \frac{ch_k \gamma^r}{1 - 2c\gamma^r} | u |_{H^3(\Omega)} .$$

Choosing a positive integer r so that $2c\gamma^r < 1$, we obtain

$$\| u - \hat{u}_k \|_k \leq ch_k | u |_{H^3(\Omega)} .$$

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