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A SIMPLE FINITE ELEMENT METHOD FOR THE REISSNER-MINDLIN PLATE*[1)]

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Abstract

A simple finite element method for the Reissner-Mindlin plate model in the primitive variables is presented and analyzed. The method uses conforming linear finite elements for both the transverse displacement and rotation. It is proved that the method converges with optimal order uniformly with respect to thickness.It is simpler and more economical than the Arnold-Falk element[1].

1. Introduction

The Reissner-Mindlin model describes the deformation of a plate subject to a transverse load. This model, as well as its generalization to shells, is frequently used for plates and shells of small to moderate thickness. It is well known that many numerical schemes for this model are satisfactory only when the thickness parameter t is not too small. For a very small t, some bad behaviors (such as the locking phenomenon) might occur. In 1986, F.Brezzi and M.Fortin[2] derived an equivalent formulation of the Reissner-Mindlin plate equations by using the Helmholtz theorem to decompose the shear strain vector. The optimal error estimates for transverse displacement, rotations and shear stresses were obtained uniformly with respect to thickness. Unfortunately their method is not known to be equivalent to any discretization of the original Reissner-Mindlin model.

In [1] Arnold and Falk modified the method in [2] and obtained a finite element method for the Reissner-Mindlin problem in the primitive variables. This so-called Arnold-Falk element may be the only method with the approximate values of displacement and the rotation, together with their first derivatives, all converging at an optimal rate uniformly with respect to thickness. Recently, R.Duran et al. [3] introduced a modification of Arnold-Falk element, with the internal degrees of freedom removed.

In this paper, we present a new finite element method for the Reissner-Mindlin model which is based on a different discrete version of the Helmholtz decomposition.

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The method used here is easier to implement and optimal error estimates for displacement and the rotation are proved uniformly in the plate thickness. It is simpler and more economical than the methods in [1,3].

2. The Reissner-Mindlin Plate Model

We will use the usual L^2-based Sobolev spaces H^s. The space H^{-1} denotes the normed dual of H^1, the closure of C_0^∞ in H^1. We use a circumflex above a function space to denote the subspace of elements with mean value zero. An underline to a space denotes the 2-vector-valued analogue. The underline is also affixed to vector-valued functions and operators, and double underlines are used for matrix-valued objects. The letter C denotes a generic constant, not necessarily the same at each occurrence. Finally, we use various standard differential operators:

$$\underline{\text{grad}}\ r = (\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y})^T, \quad \underline{\text{curl}}\ p = (\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x})^T, \quad \text{div}\,\underline{\psi} = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y},$$

$$\text{rot}\,\underline{\psi} = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}, \quad \underline{\underline{\text{grad}}}\ \underline{\psi} = \begin{pmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{pmatrix}.$$

Let Ω denote the region in R^2 occupied by the midsection of the plate, and denote by ω and $\underline{\phi}$ the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. The Reissner-Mindlin plate model determines ω and $\underline{\phi}$ as the unique solution to the following variational problem.

Problem RM. Find $(\omega, \underline{\phi}) \in H_0^1(\Omega) \times \underline{H}_0^1(\Omega)$ such that

$$a(\underline{\phi}, \underline{\psi}) + \lambda t^{-2}(\underline{\phi} - \underline{\text{grad}}\ \omega, \underline{\psi} - \underline{\text{grad}}\ \mu) = (g, \mu), \tag{2.1}$$

$$\forall(\mu, \underline{\psi}) \in H_0^1(\Omega) \times \underline{H}_0^1(\Omega).$$

Here g is the scaled transverse loading function, t is the plate thickness, $\lambda = Ek/2(1+\nu)$ with E Young's modulus, ν the Poisson ratio, k the shear correction factor, and the parentheses denote the usual L^2 inner product. The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(\underline{\phi}, \underline{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega \Big[\Big(\frac{\partial \phi_1}{\partial x} + \nu \frac{\partial \phi_2}{\partial y}\Big)\frac{\partial \psi_1}{\partial x} + \Big(\nu \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y}\Big)\frac{\partial \psi_2}{\partial y}$$

$$+ \frac{1-\nu}{2}\Big(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}\Big)\Big(\frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x}\Big)\Big].$$

By Korn's inequality, $a(\cdot,\cdot)$ is an inner product on $\underline{H}_0^1(\Omega)$ equivalent to the usual one. For simplicity of notation, we will consider the problem whose weak formulation is given by (2.1) with $\lambda = 1$, and

$$a(\underline{\phi},\ \underline{\psi}) = (\underline{\underline{\text{grad}}}\ \underline{\phi},\ \underline{\underline{\text{grad}}}\ \underline{\psi}).$$

For our analysis we shall also make use of an equivalent formulation of the Reissner-Mindlin plate equations suggested by Brezzi and Fortin[2]. Which is derived from

Problem RM by using the Helmholtz theorem to decompose the shear strain vector

$$\underline{\eta} := t^{-2}(\underline{\text{grad}}\ \omega - \underline{\phi}) = \underline{\text{grad}}\ r + \underline{\text{curl}}\ p \tag{2.2}$$

with $(r,p) \in H_0^1(\Omega) \times \hat{H}^1(\Omega)$. Moreover, since $\underline{\eta} \cdot \underline{\tau} = 0$ on $\partial\Omega$, p satisfies a homogeneous Neumann condition on $\partial\Omega$ in weak sense, where $\underline{\tau}$ is the unit vector tangent to $\partial\Omega$.

Problem BF. Find $(r, \underline{\phi}, p, \omega) \in H_0^1(\Omega) \times \underline{H}_0^1(\Omega) \times \hat{H}^1(\Omega) \times H_0^1(\Omega)$ such that

$$(\underline{\text{grad}}\ r, \underline{\text{grad}}\ \mu) = (g, \mu), \quad \forall \mu \in H_0^1(\Omega), \tag{2.3}$$

$$a(\underline{\phi}, \underline{\psi}) - (\underline{\text{curl}}\ p, \underline{\psi}) = (\underline{\text{grad}}\ r, \underline{\psi}), \quad \forall \underline{\psi} \in \underline{H}_0^1(\Omega), \tag{2.4}$$

$$-(\underline{\phi}, \underline{\text{curl}}\ q) - t^2(\underline{\text{curl}}\ p, \underline{\text{curl}}\ q) = 0, \quad \forall q \in \hat{H}^1(\Omega), \tag{2.5}$$

$$(\underline{\text{grad}}\ \omega, \underline{\text{grad}}\ s) = (\underline{\phi} + t^2 \underline{\text{grad}}\ r, \underline{\text{grad}}\ s), \quad \forall s \in H_0^1(\Omega). \tag{2.6}$$

The following regularity results were proved by Arnold and Falk[1].

Theorem 2.1. *Let Ω be a convex polygon or smoothly bounded domain in the plane. For any $t \in (0,1]$ and any $g \in H^{-1}$, there exists a unique quadruple $(r, \underline{\phi}, p, \omega) \in H_0^1(\Omega) \times \underline{H}_0^1(\Omega) \times \hat{H}^1(\Omega) \times H_0^1(\Omega)$ solving Problem BF. Moreover, $\underline{\phi} \in \underline{H}^2(\Omega)$ and there exists a constant C independent of t and g, such that*

$$\| r \|_1 + \| \underline{\phi} \|_2 + \| p \|_1 + t \| p \|_2 + \| \omega \|_1 \le C \| g \|_{-1}.$$

If $g \in L^2(\Omega)$, then $r, \omega \in H^2(\Omega)$ and

$$\| r \|_2 + \| \omega \|_2 \le C \| g \|_0.$$

3.The Finite Element Method

Let Ω be a convex polygon and $\Im_h$ [4] be a regular triangulation of Ω where as usual h stands for the mesh size. Denoting by $P_k(T)$ the set of functions on T which are the restrictions of polynomials of degree no greater than k, we define the following finite element spaces:

$$V_h^0 = \{v \in H_0^1(\Omega) : v|_T \in P_1(T), \forall T \in \Im_h\}, \quad \underline{V_h^0} = V_h^0 \times V_h^0,$$

$$\underline{\Gamma_h} = \{\underline{\eta} \in [L^2(\Omega)]^2, \forall T \in \Im_h\},$$

$$M_h = \{ v \in L^2(\Omega) : v|_T \in P_1(T), \forall T \in \Im_h \},$$

$$M_h^* = \{ v \in M_h : v \text{ is continuous at midpoints of element edges} \},$$

$$\hat{M}_h = \left\{ v \in M_h^* : \int_\Omega v = 0 \right\}.$$

Our approximation scheme is given in the following problem.

Problem $\mathrm{C_h}$. Find $(\omega_h, \underline{\phi}_h) \in V_h^0 \times \underline{V}_h^0$ such that

$$a(\underline{\phi}_h,\ \underline{\psi}) + \frac{1}{t^2 + \alpha h^2}(\underline{P}_0\underline{\phi}_h - \underline{\mathrm{grad}}\ \omega_h,\ \underline{\psi} - \underline{\mathrm{grad}}\ \mu) = (g, \mu),$$

$$\forall(\mu, \underline{\psi}) \in V_h^0 \times \underline{V}_h^0 \tag{3.1}$$

where $\underline{P}_0 : \underline{L}^2(\Omega) \longrightarrow \underline{\Gamma}_h$ is the orthogonal projection, and α is a given constant independent of h, t.

Lemma 3.1. *Equation* (3.1) *has a unique solution.*

Proof. Let $g = 0, \underline{\psi} = \underline{\phi}_h$ and $\mu = \omega_h$ in (3.1). Then

$$\|\underline{\underline{\mathrm{grad}}}\ \underline{\phi}_h\|_0^2 + (t^2 + \alpha h^2)^{-1}\ \|\ \underline{P}_0\underline{\phi}_h - \underline{\mathrm{grad}}\ \omega_h\ \|_0^2 = 0.$$

This implies first that $\underline{\underline{\mathrm{grad}}}\ \underline{\psi}_h = 0$, so $\underline{\phi}_h = 0$; whence $\underline{\mathrm{grad}}\ \omega_h = 0$ and $\omega_h = 0$. It completes the proof.

4. A New Discrete Version of the Helmhotlz Theorem

The Helmholtz theorem states that any L^2 vector field can be decomposed uniquely into the sum of the gradient of a function r in H_1^0 and the curl of a function p in $\hat{H}^1$; moreover, the two summands are orthogonal in $\underline{L}^2$. It is not true in general that, if the vector field is piecewise constant, then r and p will be continuous piecewise linear functions. However, Arnold and Falk [1] gave a discrete version of the Helmholtz theorem by using a nonconforming element. Here we will give another orthogonal decomposition like that in [1].

Theorem 4.1.

$$\underline{\Gamma}_h = \ \underline{\mathrm{grad}}\ V_h^0\ \oplus\ \underline{\mathrm{curl}}_h\ \hat{M}_h, \tag{4.1}$$

which is an $\underline{L}^2$ *orthogonal decomposition.*

Proof. It is obvious that the summands in (4.1) are piecewise constant functions. Let $r \in V_h^0$ and $p \in \hat{M}_h$. Then

$$(\underline{\mathrm{grad}}\ r,\ \underline{\mathrm{curl}}_h p) = \sum_T \int_T \underline{\mathrm{grad}}\ r \cdot \underline{\mathrm{curl}}\ p\ = -\sum_T \int_{\partial T} p \frac{\partial r}{\partial s_T}.$$

Here s_T is the counterclockwise tangent to ∂T. Now let e be any interior edge of the triangulation, say $e = T_+ \cap T_-$. Let $r_-^+ = r|_{T_-^+}$. Since r is a piecewise linear function, the derivatives $\frac{\partial r_-^+}{\partial s_{T_-^+}}$ are constant on e, and since r is continuous, $\frac{\partial r_+}{\partial s_{T_+}} = -\frac{\partial r_-}{\partial s_{T_-}}$.

Since $p \in \hat{M}_h$, $p_+ - p_-$ is a linear function on e and vanishes at the midpoint. It follows that

$$\int_e p_+ \frac{\partial r_+}{\partial s_{T_+}} + \int_e p_- \frac{\partial r_-}{\partial s_{T_-}} = 0.$$

If e is a boundary edge contained in the triangle T, since $r \in V_h^0$, then $\frac{\partial r}{\partial s_T} = 0$. Adding over all edges, we get

$$(\underline{\text{grad}}\ r,\ \underline{\text{curl}}_h\ p) = 0 \tag{4.2}$$

so that $\underline{\text{grad}}\ r$ is orthogonal to $\underline{\text{curl}}_h\ p$.

It remains to check that

$$\dim \underline{\Gamma}_h = \dim \underline{\text{grad}}\ V_h^0 + \dim \underline{\text{curl}}_h\ \hat{M}_h. \tag{4.3}$$

Let

IS = number of interior sides ,

BS = number of boundary sides ,

T = number of triangles,

IV = number of interior vertices,

BV = number of boundary vertices.

Obviously, $BS = BV$ and

$$\dim \underline{\text{grad}}\ V_h^0 = IV, \quad \dim \underline{\text{curl}}_h\ \hat{M}_h = IS + BS - 1, \quad \dim \underline{\Gamma}_h = 2T.$$

We then use Euler's relation on $\Im_h$, namely

$$T + (IV + BV) - (IS + BS) = 1$$

and

$$3T = BS + 2IS.$$

It is easy to see that $2T = IS + IV + BS - 1$ and this implies equation (4.3).

5. Error Analysis

First we introduce an equivalent formulation of Problem $\mathrm{C_h}$.

Problem $\mathrm{C_h^*}$. Find $(r_h, \underline{\phi}_h, p_h, \omega_h) \in V_h^0 \times \underline{V}_h^0 \times \hat{M}_h \times V_h^0$ such that

$$(\underline{\text{grad}}\ r_h,\ \underline{\text{grad}}\ \mu) = (g, \mu), \quad \forall \mu \in V_h^0, \tag{5.1}$$

$$a(\underline{\phi}_h,\ \underline{\psi}) - (\underline{\text{curl}}_h\ p_h,\ \underline{\psi}) = (\underline{\text{grad}}\ r_h,\ \underline{\psi}), \quad \forall \underline{\psi} \in \underline{V}_h^0, \tag{5.2}$$

$$-(\underline{\phi}_h, \underline{\text{curl}}_h\ q) - (t^2 + \alpha h^2)(\underline{\text{curl}}_h\ p_h,\ \underline{\text{curl}}_h\ q) = 0, \quad \forall q \in \hat{M}_h, \tag{5.3}$$

$$(\underline{\text{grad}}\ \omega_h, \underline{\text{grad}}\ s) = (\underline{\phi}_h + (t^2 + \alpha h^2)\underline{\text{grad}}\ r_h,\ \underline{\text{grad}}\ s), \quad \forall s \in V_h^0. \tag{5.4}$$

Lemma 5.1. *For ang $g \in L^2(\Omega)$ and any $t \in (0,1]$ there exists a unique solution $(r_h, \underline{\phi}_h, p_h, \omega_h)$ to Problem C_h^*. Moreover ,the pair $(\omega_h, \underline{\phi}_h)$ is the unique solution of Problem C_h and*

$$(t^2 + \alpha h^2)^{-1}(\underline{\text{grad}}\ \omega_h - \underline{P}_0 \underline{\phi}_h) = \underline{\text{grad}}\ r_h + \underline{\text{curl}}_h\ p_h. \tag{5.5}$$

Proof. Supose $g = 0$. Choosing $\mu = r_h$ in (5.1) we see that $r_h = 0$. Next, set $\underline{\psi} = \underline{\phi}_h$ in (5.2) and $q = p_h$ in (5.3) and subtract. This shows that $\underline{\phi}_h = 0$ and $p_h = 0$. Finally, taking $s = \omega_h$ in (5.4), we conclude that $\omega_h = 0$. This establishes the existence and uniqueness.

Since $\underline{\mathrm{curl}}_h\ q$ and $\underline{\mathrm{grad}}\ s$ are piecewise constant for $q \in \hat{M}_h,\ s \in V_h^0$, we may replace $\underline{\phi}_h$ by $\underline{P}_0\underline{\phi}_h$ in (5.3)–(5.4). Using the orthogonality proved in Theorem 4.1, we deduce that

$$(\underline{\mathrm{grad}}\ \omega_h - \underline{P}_0\underline{\phi}_h,\ \underline{\mathrm{curl}}_h\ q) = (t^2 + \alpha h^2)(\underline{\mathrm{grad}}\ r_h + \underline{\mathrm{curl}}_h\ p_h,\ \underline{\mathrm{curl}}_h\ q), \quad \forall q \in \hat{M}_h,$$

$$(\underline{\mathrm{grad}}\ \omega_h - \underline{P}_0\underline{\phi}_h,\ \underline{\mathrm{grad}}\ s) = (t^2 + \alpha h^2)(\underline{\mathrm{grad}}\ r_h + \underline{\mathrm{curl}}_h\ p_h,\ \underline{\mathrm{grad}}\ s), \quad \forall s \in V_h^0.$$

By Theorem 4.1, these two equations are equivalent to the single equation

$$(\underline{\mathrm{grad}}\ \omega_h - \underline{P}_0\underline{\phi}_h, \underline{\eta}) = (t^2 + \alpha h^2)(\underline{\mathrm{grad}}\ r_h + \underline{\mathrm{curl}}_h\ p_h, \underline{\eta}), \ \forall \underline{\eta} \in \underline{\Gamma}_h$$

from which (5.5) follows.

From (5.2), (5.1) and a similar application of orthogonality, we have

$$a(\underline{\phi}_h, \underline{\psi}) - (\underline{\mathrm{grad}}\ r_h + \underline{\mathrm{curl}}_h\ p_h,\ \underline{\psi} - \underline{\mathrm{grad}}\ \mu) = (g, \mu),$$
$$\forall (\mu, \underline{\psi}) \in V_h^0 \times \underline{V}_h^0. \tag{5.6}$$

By (5.5), equation (5.6) implies that $(\omega_h, \underline{\phi}_h)$ is the unique solution of Problem $\mathrm{C_h}$.

We now give the energy estimates for our method, the proof of which is similar to the proof in [1,2,3]. Because of the term αh^2, we do not need the Babuska-Brezzi inequality.

Theorem 5.2.. *Let $(r, \underline{\phi}, p, \omega)$ and $(r_h, \underline{\phi}_h, p_h, \omega_h)$ be the solutions of Problem BF and Problem C_h^*, respectively. For some $g \in L^2(\Omega)$ and some $t \in (0, 1]$,*

$$\| r - r_h \|_1 + \| \underline{\phi} - \underline{\phi}_h \|_1 + (t + \alpha h) \| \underline{\mathrm{curl}}_h (p - p_h) \|_0 + \| \omega - \omega_h \|_1 \le Ch \| g \|_0$$

where the constant C is independent of g, h and t.

Proof. Let us start with the error $r - r_h$, where r_h is the usual conforming approximation to the solution of the problem

$$-\Delta r = g \quad \text{in } \Omega, \quad r = 0 \quad \text{on } \partial\Omega.$$

It is well known that (see [4])

$$\| r - r_h \|_0 + h \ \| r - r_h \|_1 \le Ch^2 \ \| g \|_0 . \tag{5.7}$$

We now derive an estimate for the errors $\underline{\phi} - \underline{\phi}_h$ and $p - p_h$. Let $\underline{\phi}^I \in \underline{V}_h^0$ and $p^I \in \hat{M}_h$ be the interpolation of $\underline{\phi}$ and p, respectively. Then from the standard interpolation theory[4], we have

$$\| \underline{\phi} - \underline{\phi}^I \|_1 \le \ Ch \ \| \underline{\phi} \|_2, \tag{5.8}$$

$$\| p - p^I \|_0 \le \ Ch \| p \|_1, \quad \| \underline{\mathrm{curl}}_h (p - p^I) \|_0 \le \ Ch \| p \|_2 . \tag{5.9}$$

From (2.4) and (5.2) it follows that

$$\| \underline{\underline{\mathrm{grad}}}(\underline{\phi}_h - \underline{\phi}^I) \|_0^2 = (\underline{\underline{\mathrm{grad}}}(\underline{\phi} - \underline{\phi}^I),\ \underline{\underline{\mathrm{grad}}}(\underline{\phi}_h - \underline{\phi}^I)) + (\underline{\mathrm{curl}}_h(p_h - p),\ \underline{\phi}_h - \underline{\phi}^I) + (\underline{\mathrm{grad}}(r_h - r),\ \underline{\phi}_h - \underline{\phi}^I).$$

From (2.5) and (5.3) it follows that

$$t^2 \| \underline{\mathrm{curl}}_h(p_h - p^I) \|_0^2 = t^2(\underline{\mathrm{curl}}_h(p - p^I),\ \underline{\mathrm{curl}}_h(p_h - p^I)) - (\underline{\phi}_h - \underline{\phi},\ \underline{\mathrm{curl}}_h(p_h - p^I)) - \alpha h^2(\underline{\mathrm{curl}}_h p_h,\ \underline{\mathrm{curl}}_h(p_h - p^I)) + E_1(\underline{\phi}, p, p_h - p^I)$$

where

$$E_1(\underline{\phi}, p, p_h - p^I) = -\sum_K \int_{\partial K} (\underline{\phi} + t^2 \underline{\mathrm{curl}}\ p)\underline{\tau}(p_h - p^I). \tag{5.10}$$

Here $\underline{\tau}$ is the counterclockwise tangent to ∂K. Adding these two equations we get

$$\begin{aligned}
&\| \underline{\underline{\mathrm{grad}}}(\underline{\phi}_h - \underline{\phi}^I) \|_0^2 + (t^2 + \alpha h^2) \| \underline{\mathrm{curl}}_h(p_h - p^I) \|_0^2 \\
&= (\underline{\underline{\mathrm{grad}}}(\underline{\phi} - \underline{\phi}^I),\ \underline{\underline{\mathrm{grad}}}(\underline{\phi}_h - \underline{\phi}^I)) + t^2(\underline{\mathrm{curl}}_h(p - p^I),\ \underline{\mathrm{curl}}_h(p_h - p^I)) \\
&+ \alpha h^2(\underline{\mathrm{curl}}_h p^I,\ \underline{\mathrm{curl}}_h(p_h - p^I)) + (\underline{\mathrm{curl}}_h(p^I - p),\ \underline{\phi}_h - \underline{\phi}^I) \\
&+ (\underline{\phi} - \underline{\phi}^I,\ \underline{\mathrm{curl}}_h(p_h - p^I)) + (\underline{\mathrm{grad}}(r_h - r),\ \underline{\phi}_h - \underline{\phi}^I) + E_1(\underline{\phi}, p, p_h - p^I) \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

By (5.8)–(5.9), then

$$\begin{aligned}
I_1 &\le Ch \| \underline{\phi} \|_2 \cdot \| \underline{\underline{\mathrm{grad}}}(\underline{\phi}_h - \underline{\phi}^I) \|_0, \\
I_2 &\le Cht^2 \| p \|_2 \cdot \| \underline{\mathrm{curl}}_h(p_h - p^I) \|_0, \\
I_3 &\le \alpha h^2 \| p \|_1 \cdot \| \underline{\mathrm{curl}}_h(p_h - p^I) \|_0 .
\end{aligned}$$

In order to bound I_4, we integrate by parts

$$\begin{aligned}
I_4 &\le |(p^I - p, \mathrm{rot}\,(\underline{\phi}_h - \underline{\phi}^I)| + |E_2(P^I - p, \underline{\phi}_h - \underline{\phi}^I)| \\
&\le Ch \| p \|_1 \| \underline{\phi}_h - \underline{\phi}^I) \|_0 + |E_2(p^I - p, \underline{\phi}_h - \underline{\phi}^I)|
\end{aligned}$$

where

$$E_2(p^I - p, \underline{\phi}_h - \underline{\phi}^I) = |\sum_K \int_{\partial K} (\underline{\phi}_h - \underline{\phi}^I)\underline{\tau}(p^I - p)\ ds|.$$

And

$$|I_5 + I_6| \le Ch^2 \| \underline{\phi} \|_2 \cdot \| \underline{\mathrm{curl}}_h(p_h - p^I) \|_0 + Ch \| r \|_2 \cdot \| (\underline{\phi}_h - \underline{\phi}^I) \|_1 .$$

It remains to bound $E_1(\underline{\phi}, p, p_h - p^I)$ and $E_2(p^I - p, \underline{\phi}_h - \underline{\phi}^I)$. Define

$$\underline{I\Gamma}_h = \{\underline{v} : \underline{v}|_\Gamma \in [P_0(\Gamma)]^2, \Gamma \text{ is the interior side of triangulation}\}$$

Using the standard analysis of nonconforming methods [5,6] and the fact $\underline{\phi} = 0$ on $\partial\Omega$ and $\underline{\text{curl}}\ p \cdot \underline{\tau} = \frac{\partial p}{\partial n} = 0$ on $\partial\Omega$, we get

$$E_1(\underline{\phi}, p, p_h - p^I) = \inf \Big| \sum_K \int_{\partial K} (\underline{\phi} + t^2 \underline{\text{curl}}\ p - \underline{s}_h) \underline{\tau} (p_h - p^I) ds \Big| (\forall \underline{s}_h \in \underline{I\Gamma}_h)$$

$$\leq Ch \parallel \underline{\phi} + t^2 \underline{\text{curl}}\ p \parallel_1 \cdot \parallel \underline{\text{curl}}_h (p_h - p^I) \parallel_0 .$$

By the trace theorem, we obtain

$$E_2(p^I - p, \underline{\phi}_h - \underline{\phi}^I) \leq \ Ch \parallel p \parallel_1 \cdot \parallel \underline{\underline{\text{grad}}}(\underline{\phi}_h - \underline{\phi}^I) \parallel_0 .$$

Applying the Schwartz inequality and the arithmetic-geometric mean inequality, we can obtain

$$\parallel \underline{\underline{\text{grad}}}(\underline{\phi}_h - \underline{\phi}^I) \parallel_0^2 + (t^2 + \alpha h^2) \parallel \underline{\text{curl}}_h (p_h - p^I) \parallel_0^2$$

$$\leq \ Ch^2 (\parallel \underline{\phi} \parallel_2 + \parallel p \parallel_1 + t^2 \parallel p \parallel_2 + \parallel r \parallel_2)^2 .$$

Then, using the regularity result (Lemma 2.1), we get

$$\parallel \underline{\phi}_h - \underline{\phi}^I \parallel_1 + (t + \alpha h) \parallel \underline{\text{curl}}_h (p_h - p^I) \parallel_0 \leq \ Ch \parallel g \parallel_0 .$$

By the triangle inequality and (5.8)–(5.9), we derive the estimate for the errors $\underline{\phi}_h - \underline{\phi}$ and $p - p_h$. Finally, let $\omega^I \in V_h^0$ be the interpolation of ω. Then

$$\parallel \underline{\text{grad}}(\omega_h - \omega^I) \parallel_0^2 = (\underline{\text{grad}}(\omega - \omega^I),\ \underline{\text{grad}}(\omega_h - \omega^I)) + (\underline{\text{grad}}(\omega_h - \omega),\ \underline{\text{grad}}(\omega_h - \omega^I))$$

$$= (\underline{\text{grad}}(\omega - \omega^I),\ \underline{\text{grad}}(\omega_h - \omega^I)) + (\underline{\phi}_h - \underline{\phi},\ \underline{\text{grad}}(\omega_h - \omega^I)$$

$$+ t^2 (\underline{\text{grad}}(r_h - r),\ \underline{\text{grad}}(\omega_h - \omega^I)) + \alpha h^2 (\underline{\text{grad}}\ r_h,\ \underline{\text{grad}}(\omega_h - \omega^I))$$

$$\leq \ Ch \parallel g \parallel_0 \cdot \parallel \underline{\text{grad}}(\omega_h - \omega^I) \parallel_0 .$$

Since

$$\parallel \omega - \omega^I \parallel_1 \leq \ Ch \parallel \omega \parallel_2 \leq \ Ch \parallel g \parallel_0 ,$$

using the triangle inequality, we can obtain the estimate $\omega - \omega_h$ easily. Applying a variant of the usual Aubin-Nitsche duality argument, we estimate $\underline{\phi} - \underline{\phi}_h$ and $\omega - \omega_h$ in L^2 norm.

Theorem 5.3. *Under the hypotheses of Theorem* 5.2,

$$\parallel \underline{\phi} - \underline{\phi}_h \parallel_0 + \parallel \omega - \omega_h \parallel_0 \leq \ Ch^2 \parallel g \parallel_0 ,$$

where C is independent of h, g, t.

The proof of this theorem is similar to the proof of Theorem 6.1 in [1], so we omit the details here.

Remark. It is easy to see that we can obtain the quasi-optimal pointwise error estimates and the multigrid method for our element; see the method in [8,9].

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