Journal of Computational Mathematics, Vol.12, No.1, 1994, 55–60.

ON THE UNIQUE SOLVABILITY OF THE NONLINEAR SYSTEMS IN MDBM*

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Abstract

To obtain the approximate solution of the nonlinear ordinary differential equations requires the solution to systems of nonlinear equations. The authors study the conditions for the existence and uniqueness of the solutions to the algebraic equations in multiderivative block methods.

1. Introduction

Consider the following initial value problem in ordinary differential equations

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{1.1}$$

where $y_0 \in R^s$, f: $R \times R^s$ to R^s, is continuous. The approximate solution to (1.1) can be obtained by the multiderivative block method (MDBM) with second order derivatives:

$$y_{n+i} = y_n + h\sum_{j=1}^{k} a_{ij} f_{n+j} + h^2 \sum_{j=1}^{k} b_{ij} f_{n+j}^{(1)} + h\beta_{li} f_n + h^2 \beta_{2i} f_n^{(1)}, \tag{1.2}$$

where $i = 1, \cdots, k$, $y_n \in R^s$, $f_n := f(t_n, y_n) \in R^s$, and $f_n^{(1)} := df(t_n, y_n)/dt \in R^s$ are known vectors. It is proved that there exist $a_{ij}, b_{ij}, \beta_{li}$ and $\beta_{2i}, i, j = 1, 2, \cdots, k$, such that (1.2) converges with order $p = 2k + 2$ (see [1]), and is A-stable for $k \leq 5$ (see [5]). To compute the approximate solution $y_{n+j} \doteq y(t_{n+j})$ requires the solution of the following nonlinear equations:

$$y_{n+i} = u_n + h\sum_{j=1}^{k} a_{ij} f_{n+j} + h^2 \sum_{j=1}^{k} b_{ij} f_{n+j}^{(1)}, \quad 1 \leq i \leq k, \tag{1.3}$$

where $u_n = y_n + h\beta_{li} f_n + h^2 \beta_{2i} f_n^{(1)}$.

Denote y_{n+j} by y_j, $f(t_{n+j}, y_{n+j})$ by $f_j(y_j)$, and $f^{(1)}(t_{n+j}, y_{n+j})$ by $g_j(y_j)$. Then (1.3) becomes

$$y_i = u_n + h\sum_{j=1}^{k} a_{ij} f_j(y_j) + h^2 \sum_{j=1}^{k} b_{ij} g_j(y_j), \quad 1 \leq i \leq k. \tag{1.4}$$

* Received May 27, 1992.

There exists a unique solution to (1.4) if and only if the following nonlinear equations have a unique solution (see [2]):

$$y_i = h\sum_{j=1}^{k} a_{ij}f_j(y_j) + h^2\sum_{j=1}^{k} b_{ij}g_j(y_j), \quad 1 \le i \le k. \tag{1.5}$$

2. Sufficient conditions for the existence and uniqueness of the solution to (1.5)

In this section, we will present sufficient conditions for the existence and uniqueness of the solution to (1.5).

Let $A = (a_{ij}), \quad C = (c_{ij}) \in R^{k\times k}$. The Kronnecker product of A and C is defined by

$$A \otimes C = \begin{pmatrix} a_{11}C & a_{12}C & \cdots & a_{1k}C \\ a_{21}C & a_{22}C & \cdots & a_{2k}C \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1}C & a_{k2}C & \cdots & a_{kk}C \end{pmatrix}$$

where

$$a_{ij}C = \begin{pmatrix} a_{ij}c_{11} & a_{ij}c_{12} & \cdots & a_{ij}c_{1k} \\ a_{ij}c_{21} & a_{ij}c_{22} & \cdots & a_{ij}c_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{ij}c_{k1} & a_{ij}c_{k2} & \cdots & a_{ij}c_{kk} \end{pmatrix}.$$

Lemma 2.1. *Let L and D be $m \times m$ matrices, and I_s be the $s \times s$ unit matrix. Then $(LD) \otimes I_s = (L \otimes I_s)(D \otimes I_s)$. Furthermore, if $DL + L^T D$ and D are positive definite, then $\left(DL + L^T D\right) \otimes I_s$ and $D \otimes I_s$ are positive definite.*

Proof. This lemma can be proved directly by the theorems in [3].

Let

$$L = \begin{pmatrix} A & B \\ 0 & I_k \end{pmatrix} \qquad \mathcal{L} = L \otimes I_s = \begin{pmatrix} A \otimes I_s & B \otimes I_s \\ 0 & I_k \otimes I_s \end{pmatrix}$$

where

$$A = (a_{ij})_{k\times k}, \quad B = (b_{ij})_{k\times k}.$$

Denote

$$y = \left(y_1^T, y_2^T, \cdots, y_k^T\right)^T, \quad f(y) = \left(f_1(y_1)^T, f_2(y_2)^T, \cdots, f_k(y_k)^T\right)^T,$$

$$g(y) = g\left(g_1(y_1)^T, g_2(y_2)^T, \cdots, g_k(y_k)^T\right)^T, \quad Y_g = \begin{pmatrix} y \\ h^2 g(y) \end{pmatrix},$$

$$F(Y) = \begin{pmatrix} hf(y) \\ h^2 g(y) \end{pmatrix}$$

where

$$Y_g = \left(y_1^T, y_2^T, \cdots, y_k^t, h^2 g_1(y_1)^T, h^2 g_2(y_2)^T, \cdots, h^2 g_k(y_k)^T\right)^T,$$

$$F(Y) = \left(h f_1(y_1)^T, \cdots, h f_k(y_k)^T, h^2 g_1(y_1)^T, \cdots, h^2 g_k(y_k)^T\right)^T,$$

$$y_i = (y_{i1}, y_{i2}, \cdots, y_{is})^T, \quad f_i(y_i) = (f_{i1}(y_i), f_{i2}(y_i), \cdots, f_{is}(y_i))^T,$$

$$g_i(y_i) = (g_{i1}(y_i), g_{i2}(y_i), \cdots, g_{is}(y_i))^T, \qquad 1 \le i \le k.$$

Then (1.5) can be written as

$$Y_g = \mathcal{L} F(Y), \tag{2.1}$$

or

$$\begin{pmatrix} y \\ h^2 g(y) \end{pmatrix} = \begin{pmatrix} A \otimes I_s & B \otimes I_s \\ 0 & I_k \otimes I_s \end{pmatrix} \begin{pmatrix} h f(y) \\ h^2 g(y) \end{pmatrix}.$$

Let

$$\widetilde{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{D} = \widetilde{D} \otimes I_s = \begin{pmatrix} D \otimes I_s & 0 \\ 0 & D \otimes I_s \end{pmatrix},$$

where D is a $k \times k$ matrix. From Lemma 2.1, if $\widetilde{D}$ and $\widetilde{D} L + L^T \widetilde{D}$ are positive definite, then $\mathcal{D}$ and $\mathcal{D}\mathcal{L} + \mathcal{L}^T \mathcal{D}$ are positive definite.

Define the inner-product in R^{2ks} as

$$[X, Y] := \sum_{i=1}^{k} d_i \langle x_i, y_i \rangle + \sum_{i=1}^{k} d_i \langle x_{k+i}, y_{k+i} \rangle = X^T \mathcal{D} Y,$$

where

$$X = (x_1, x_2, \cdots, x_{2k})^T, \ Y = (y_1, y_2, \cdots, y_{2k})^T,$$

$$\widetilde{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} D \otimes I_s & 0 \\ 0 & D \otimes I_s \end{pmatrix},$$

$$D = \text{diag}\{d_1, d_2, \cdots, d_k\}, \quad d_i > 0, \quad 1 \le i \le k.$$

And the corresponding norm is defined by

$$\|X\| = [X, X]^{1/2}.$$

Throughout this paper, we make the following assumptions:

$$\langle f(t,u) - f(t,v), u - v \rangle \le 0, \quad \forall t \in R, \quad \forall u, v \in R^s, \tag{2.2}$$

$$\|f^{(1)}(t,u) - f^{(1)}(t,v)\| \le \sigma \|u - v\|, \quad \forall t \in R, \quad \forall u, v \in R^s. \tag{2.3}$$

Then for $1 \le j \le k$, we have

$$\langle f_j(u) - f_j(v), u - v \rangle \le 0, \quad \forall u, v \in R^s, \tag{2.2'}$$

$$\|g_j(u) - g_j(v)\| \le \sigma \|u - v\|, \quad \forall u, v \in R^s, \quad \sigma > 0. \tag{2,3'}$$

It is well known that assumption (2.2) implies that, for any two solutions U and $\bar{U}$ to (1.1), the norm $\|U(t) - \bar{U}(t)\|$ does not increase when t increases.

Let $w \in R^{2ks}$, $w \neq 0$. Then

$$[\mathcal{L}w, w] = w^T \mathcal{L}^T \mathcal{D} w, \quad [\mathcal{L}w, w] = [w, \mathcal{L}w] = w^T \mathcal{D} \mathcal{L} w.$$

Hence

$$[\mathcal{L}w, w] = \frac{1}{2} w^T \left\{ \mathcal{D}\mathcal{L} + \mathcal{L}^T \mathcal{D} \right\} w.$$

If $\widetilde{D}L + L^T\widetilde{D}$ is positive definite, then so is $\mathcal{D}\mathcal{L} + \mathcal{L}^T\mathcal{D}$ by Lemma 2.1. Therefore,

$$[\mathcal{L}w, w] > 0 \quad \forall w \in R^{2ks}, \quad w \neq 0,$$

and this implies that $\mathcal{L}$ is regular and there exists $\mathcal{L}^{-1}$, and that $[\mathcal{L}^{-1}w, w] > 0$ for all $w \in R^{2ks}$, $w \neq 0$. From the properties of finite dimensional spaces, we see that

$$\min_{\|w\|=1} \left[\mathcal{L}^{-1}w, w\right] = \beta > 0,$$

and then

$$\left[\mathcal{L}^{-1}w, w\right] \geq \beta \|w\|^2, \quad w \in R^{2ks}, w \neq 0. \tag{2.4}$$

Theorem 2.2. *Let* $D = \text{diag}\,\{d_1, d_2, \cdots d_k\}$, $d_i > 0$, $1 \leq i \leq k$, f_j *and* g_j, $1 \leq j \leq k$, *satisfy* (2.2)′ *and* (2.3)′. *If* $\widetilde{D}L + L^T\widetilde{D}$ *is positive definite and*

$$h < \sqrt{\frac{2\beta}{\sigma}} \tag{2.5}$$

is fulfilled, then (1.5) *has a unique solution.*

Proof. Since (2.1) is equivalent to (1.5), we will focus our attention on the existence and uniqueness of the solution to (2.1).

(i) **Existence.** Let

$$G(X) = \mathcal{L}^{-1}X - F(X), \quad X \in R^{2ks}.$$

Then,

$$G(X) = G(0) = \mathcal{L}^{-1}X - F(X) + F(0).$$

Let X be denoted by $(v_1^T,\ v_2^T)^T$ with $v_1 = (x_1, \cdots, x_k)^T, v_2 = (x_{k+1}, \cdots, x_{2k})^T$. It follows from (2.5) and the definition of $[*, *]$ that

$$\begin{aligned}
[G(X) - G(0), X] &= \left[\mathcal{L}^{-1}X, X\right] - [F(X) - F(0), X - 0] \\
&\geq \beta\|X\|^2 - h\sum_{j=1}^{k} d_j \langle f_j(x_j) - f_j(0), x_j - 0\rangle - h^2 \sum_{j=1}^{k} d_j \langle g_j(x_j) - g_j(0), x_{k+j} - 0\rangle \\
&\geq \beta\|X\|^2 - h^2 \sum_{j=1}^{k} d_j \|g_j(x_j) - g_j(0)\| \quad \|x_{k+j} - 0\| \\
&\geq \beta\|X\|^2 - h^2 \sum_{j=1}^{k} d_j (\sigma/2)(\|x_j\|^2 + \|x_{k+j}\|^2) \\
&= (\beta - h^2\sigma/2) \sum_{j=1}^{k} d_j \|x_j\|^2 + (\beta - h^2\sigma/2) \sum_{j=1}^{k} d_j \|x_{k+j}\|^2 \geq \beta_1 \|X\|^2,
\end{aligned} \tag{2.6}$$

where $\beta_1 = \beta - h^2\sigma/2 > 0$. Therefore,

$$[G(X) - G(0), X - 0] \geq \beta_1 \|X\|^2. \tag{2.7}$$

Since $[G(0), X] \geq -\|G(0)\|\|X\|$, then $[G(X), X] \geq \beta_1\|X\|^2 - \|G(0)\|\|X\| = \|X\| \cdot (\beta_1\|X\| - \|G(0)\|)$, and this implies that

$$[G(X), X] \geq 0 \tag{2.8}$$

for all $X \in R^{2ks}$ and $\|X\| \geq \|G(0)\|/\beta_1$.

Let $0 \neq X \in R^{2ks}$, $\lambda > 1$. Define $H(X) = X - G(X)$. Then

$$[\lambda X - H(X), X] = [(\lambda - 1)X + G(X), X] = [(\lambda - 1)X, X] + [G(X), X]$$

$$\geq [(\lambda - 1)X, X] > 0.$$

Hence $H(X) \neq \lambda X$ for all X with $\|X\| \geq \|G(0)\|/\beta_1$.

By Schauder's fixed point theorem [4.6/3/3]. $H(X)$ has a fixed

$$X^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \in R^{2ks}, \quad v_1^*, v_2^* \in R^{ks} \text{ with } \|X^*\| < \|G(0)\|/\beta_1$$

and

$$X^* = H(X^*) = X^* - G(X^*),$$

or

$$G(X^*) = 0.$$

Hence

$$X^* = \mathcal{L}F(X^*),$$

i.e.

$$v_1^* = hA \otimes I_s f(v_1^*) + h^2 B \otimes I_s g(v_1^*) \quad v_2^* = g(v_1^*).$$

(ii) **Uniqueness.** Assume that there exist two solutions X and $\bar{X}$ satisfying (2.1), that is

$$0 = X - LF(X) = \bar{X} - LF(\bar{X}).$$

From (2.4), we see that

$$\beta\|X - \bar{X}\|^2 = \beta \sum_{j=1}^{k} d_j \|x_j - \bar{x}_j\|^2 + \beta \sum_{j=1}^{k} d_j \|x_{k+j} - \bar{x}_{k+j}\|^2$$

$$\leq \left[\mathcal{L}^{-1}(X - \bar{X}), X - \bar{X}\right] = [F(X) - F(\bar{X}), X - \bar{X}]$$

$$= h \sum_{j=1}^{k} d_j \langle f_j(x_j) - f_j(\bar{x}_j), x_j - \bar{x}_j \rangle + h^2 \sum_{j=1}^{k} d_j \langle g_j(x_j) - g_j(\bar{x}_j), x_{k+j} - \bar{x}_{k+j} \rangle$$

$$\leq h^2 \sum_{j=1}^{k} d_j \|g_j(x_j) - g_j(\bar{x}_j)\| \cdot \|x_{k+j} - \bar{x}_{k+j}\|$$

$$\leq h^2 \sum_{j=1}^{k} d_j \sigma \|x_j - \bar{x}_j\| \cdot \|x_{k+j} - \bar{x}_{k+j}\|$$

$$\leq h^2 \sum_{j=1}^{k} d_j(\sigma/2) \left(\|x_j - \bar{x}_j\|^2 + \|x_{k+j} - \bar{x}_{k+j}\|^2\right) = \frac{\sigma}{2} h^2 \|X - \bar{X}\|^2.$$

This, due to (2.5), implies that $x_j = \bar{x}_j$, $j = 1, 2, \cdots, 2k$, and completes the proof of the theorem.

3. An Example

For the simplicity in calculations, we give a simple example to illustrate the efficiency of Theorem 2.2.

Consider the Obrechkov methods (multiderivative block method with second order derivatives, $k = 1$) :

$$y_{n+1} = y_n + (h/2)\left[f_n + f_{n+1}\right] + (h^2/12)\left[f_n^{(1)} - f_{n+1}^{(1)}\right]. \tag{3.1}$$

In this case,

$$A = 1/2, \quad B = -1/12, \quad D = d > 0, \mathcal{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad L = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

$$\mathcal{D}L + L^T\mathcal{D} = \begin{pmatrix} DA + A^TD & DB \\ B^TD & 2D \end{pmatrix} = \begin{pmatrix} d & -d/12 \\ -d/12 & 2d \end{pmatrix}.$$

Obviously, $\mathcal{D}L + L^T\mathcal{D}$ is positive definite. By the well-known results of logarithmic norms of [6], we see that

$$\beta = \min\left\{\lambda : \det\left(\mathcal{D}L^{-1} + L^{-T}\mathcal{D} - 2\lambda\mathcal{D}\right) = 0\right\} = \frac{18 - \sqrt{37}}{12}.$$

From Theorem 2.2, we conclude that the nonlinear equation (2.1) has a unique solution provided that (2.5) is fulfilled.

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