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THE APPLICATION OF INTEGRAL EQUATIONS TO THE NUMERICAL SOLUTION OF NONLINEAR SINGULAR PERTURBATION PROBLEMS*

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Abstract

The nonlinear singular perturbation problem is solved numerically on non-equidistant meshes which are dense in the boundary layers. The method presented is based on the numerical solution of integral equations [1]. The fourth order uniform accuracy of the scheme is proved. A numerical experiment demonstrates the effectiveness of the method.

1. A Continuous Problem

We consider the following singularly perturbed boundary value problem:

$$\varepsilon^2 \frac{d^2u}{dx^2} = f(x,y), \quad x \in I = [0,1], \quad u(0) = u(1) = 0, \tag{1}$$

where ε is a small positive parameter. We assume that

$$\begin{aligned} &f \in C^4(I \times R), \quad g(x) \le f_u(x,u) \le G(x), \quad (x,u) \in I \times R, \\ &\min\{5g(x) - 2G(x) : x \in I\} > 0, \ 0 < r^2 < g(x), \ |g'(x)| \le L, \\ &|G'(x)| \le L, \ x \in I. \end{aligned} \tag{2}$$

According to [2], we can prove

Lemma 1. *Suppose that condition* (2) *is satisfied. There exists a unique solution* $u \in C^6(I)$ *to problem* (1), *and the following representation holds:* $u(x) = u_0(x)+V_0(x)+V_1(x)$, *where* $V_0(x) = M\exp\left(-\gamma\frac{x}{\varepsilon}\right)$, $V_1(x) = M\exp\left(-\gamma\frac{(1-x)}{\varepsilon}\right)$, *and* $|u_0^{(i)}(x)| \le M$, $i = 0, 1, \cdots, 6$, $x \in I$ (*Throughout the paper M denotes any constant independent of* ε).

The proof of the following lemma is based on the monotonicity of (1), and can be found in [3,4].

Lemma 2. *Let* (2) *be satisfied. Then, for the solution* $u \in C^6(I)$ *to problem* (1) *there holds, for* $i = 0, 1, \cdots, 6$,

$$|u^{(i)}(x)| \le \begin{cases} M\left(1 + \varepsilon^{-i}\exp\left(\gamma\frac{x}{\varepsilon}\right)\right), & 0 \le x \le \frac{1}{2}, \\ M\left(1 + \varepsilon^{-i}\exp\left(\gamma\frac{(1-x)}{\varepsilon}\right)\right), & \frac{1}{2} \le x \le 1. \end{cases}$$

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2. An Equivalent Integral Equation Problem

Introduce the non-equidistant mesh $I_h = \{x_i\}$, $0 = x_0 < x_1 < \ldots < x_n = 1$, $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. In the subinterval $[x_{i-1}, x_{i+1}]$, we consider

$$\varepsilon^2 \frac{d^2u}{dx^2} = f(x, u), \quad u(x_{i-1}) = A, \quad \frac{du(x_{i-1})}{dx} = B.$$

Integrating once, we have

$$\varepsilon^2 u' = \varepsilon^2 B + \int_{x_{i-1}}^{x} f(t, u(t))dt;$$

hence

$$\begin{aligned}\varepsilon^2 u(x) &= \varepsilon^2 A + \varepsilon^2 B(x - x_{i-1}) + \int_{x_{i-1}}^{x} dt \int_{x_{i-1}}^{t} f(s, u(s))ds \\ &= \varepsilon^2 A + \varepsilon^2 B(x - x_{i-1}) + \int_{x_{i-1}}^{x} (x - t) f(t, u(t))dt.\end{aligned}$$

Now if we require that $u(x_{i-1}) = u_{i-1}$, $u(x_{i+1}) = u_{i+1}$, we have $u_{i-1} = u(x_{i-1}) = A$, $u_{i+1} = u(x_{i+1}) = A + B(x_{i+1} - x_{i-1}) + \frac{1}{\varepsilon^2} \int_{x_{i-1}}^{x_{i+1}} (x_{i+1} - t) f(t, u(t))dt$. Solving for A and B, we find that $u(x)$ satisfies the integral equation

$$\begin{aligned}\varepsilon^2 u(x) = \varepsilon^2 u_{i-1} + \varepsilon^2 (x - x_{i-1}) \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} + \int_{x_{i-1}}^{x} (x - t) f(t, u(t))dt \\ - \frac{x - x_{i-1}}{x_{i+1} - x_{i-1}} \int_{x_{i-1}}^{x_{i+1}} (x_{i+1} - t) f(t, u(t))dt.\end{aligned}$$

which can be rewritten in the form

$$\varepsilon^2 u(x) = \varepsilon^2 \frac{x_{i+1} - x}{x_{i+1} - x_{i-1}} u(x_{i-1}) + \varepsilon^2 \frac{x - x_{i-1}}{x_{i+1} - x_{i-1}} u(x_{i+1}) - \int_{x_{i-1}}^{x_{i+1}} K(x, t) f(t, u(t))dt, \tag{3}$$

where

$$K(x, t) = \begin{cases} (t - x_{i-1}) \dfrac{x_{i+1} - x}{x_{i+1} - x_{i-1}}, & x_{i-1} \le t \le x, \\ (x - x_{i-1}) \dfrac{x_{i+1} - t}{x_{i+1} - x_{i-1}}, & x \le t \le x_{i+1}. \end{cases}$$

The kernel is then Green's function for the problem, in the notation of classical mechanics.

3. Discretization

Letting $x = x_i$ in (3), we obtain an exact three-point difference scheme:

$$\varepsilon^2 u(x_i) + \varepsilon^2 \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} u(x_{i-1}) + \varepsilon^2 \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} u(x_{i+1}) - \int_{x_{i-1}}^{x_{i+1}} K(x_i, t) f(t, u(t))dt.$$

We denote

$$Nu(x_i) \equiv \varepsilon^2 [A_{i-1} u(x_{i-1}) + A_i u(x_i) + A_{i+1} u(x_{i+1})] + \int_{x_{i-1}}^{x_{i+1}} K(x_i, t) f(t, u(t))dt = 0, \tag{4}$$

where
$$A_{i-1} = \frac{-h_{i+1}}{h_i + h_{i+1}},\ A_i = 1,\ A_{i+1} = \frac{-h_i}{h_i + h_{i+1}}. \tag{5}$$

Let $If = \int_{x_{i-1}}^{x_{i+1}} K(x_i,t)f(t,u(t))dt$, and P_2 be the quadric algebra interpolation of $f(t,u(t))$,

$$P_2(t) = \frac{(t-x_i)(t-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}f_{i-1} + \frac{(t-x_{i-1})(t-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}f_i + \frac{(t-x_{i-1})(t-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}f_{i+1},$$

where $f_j = f(x_j, u(x_j))$, $j = i-1, i, i+1$.

In If, approximating $f(t,u(t))$ by $P_2(t)$, we get Qf, the approximation of If:

$$\begin{aligned} Qf &= f_{i-1}\int_{x_{i-1}}^{x_{i+1}} K(x_i,t)\frac{(t-x_i)(t-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}dt \\ &+ f_i\int_{x_{i-1}}^{x_{i+1}} K(x_i,t)\frac{(t-x_{i-1})(t-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}dt \\ &+ f_{i+1}\int_{x_{i-1}}^{x_{i+1}} K(x_i,t)\frac{(t-x_{i-1})(t-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}dt \\ &= B_{i-1}f_{i-1} + B_if_i + B_{i+1}f_{i+1}, \end{aligned}$$

where

$$\left.\begin{aligned} B_{i-1} &= \int_{x_{i-1}}^{x_{i+1}} K(x_i,t)\frac{(t-x_i)(t-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}dt \\ &= \int_{x_{i-1}}^{x_i} \frac{x_{i+1}-x_i}{x_{i+1}-x_{i-1}}(t-x_{i-1})\frac{(t-x_i)(t-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}dt \\ &+ \int_{x_i}^{x_{i+1}} \frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}(x_{i+1}-t)\frac{(t-x_i)(t-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}dt \\ &= \frac{h_{i+1}}{h_i(h_i+h_{i+1})}\left(\frac{h_i^4}{12} + \frac{h_i^3h_{i+1}}{6}\right) - \frac{1}{12}\frac{h_{i+1}^4}{(h_i+h_{i+1})^2} \\ &= \frac{h_{i+1}(h_i^2 - h_{i+1}^2 + h_ih_{i+1})}{12(h_i+h_{i+1})}. \\ \text{Similarly, } B_i &= \frac{1}{12}(h_i^2 + h_{i+1}^2 + 3h_ih_{i+1}),\ B_{i+1} = \frac{h_i(h_{i+1}^2 - h_i^2 + h_ih_{i+1})}{12(h_i+h_{i+1})}. \end{aligned}\right\} \tag{6}$$

Thus, in mesh I_h, we get the difference scheme of problem (1):

$$\begin{cases} N_hu_i = \varepsilon^2(A_{i-1}u_{i-1} + A_iu_i + A_{i+1}u_{i+1} + B_{i-1}f_{i-1} + B_if_i + B_{i+1}f_{i+1} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\ldots,n-1, \\ N_hu_0 \equiv u_0 = 0, \quad N_hu_n \equiv 0. \end{cases} \tag{7}$$

In order to get the uniform fourth order difference scheme for problem (1), we take $x_i = \lambda(t_i)$, $t_i = ih$, $i = 0, 1, \ldots, n$, $h = \dfrac{1}{n}$, $n = 2m$, $m \in N$.

$$\lambda(t) = \begin{cases} \omega(t) = \dfrac{\alpha\varepsilon t}{q-t}, & t \in [0, t_k], \\ \omega'(t_k)(t - t_k) + \omega(t_k), & t \in [t_k, \frac{1}{2}] \text{ for some } k \in \{1, 2, \ldots, m-1\}, \\ 1 - \lambda(1-t), & t \in [\frac{1}{2}, 1], \end{cases} \tag{8}$$

where $q = t_k + \sqrt{\varepsilon}$, $\alpha = \dfrac{1}{2}[q(\dfrac{1}{2} - t_k) + \sqrt{\varepsilon}t_k]^{-1}$. It is easy to see $\lambda(\dfrac{1}{2}) = \dfrac{1}{2}$.

We have $\lambda : I \to I$, $\lambda \in C(I)$, $\lambda \in C^1[1, \frac{1}{2}]$, $\lambda \in C^\infty[0, t_k]$, $\lambda \in C^\infty[t_k, \frac{1}{2}]$.

Because of symmetry, we consider only $\lambda(t)$ in $[0, \frac{1}{2}]$. When $t \in [t_k, \frac{1}{2}]$, $\lambda(t)$ is a linear function, so an equidistant mesh is developed, i.e. $h_i = x_i - x_{i-1} = \omega'(t_k)h = \tilde{h}$, and $A_{i-1} = A_{i+1} = -\frac{1}{2}$, $A_i = 1$, $B_{i-1} = B_{i+1} = \frac{1}{24}\tilde{h}^2$, $B_i = \frac{5}{12}\tilde{h}^2$. When $t \in [0, t_k]$, we can prove that $B_i \geq \frac{5h_ih_{i+1}}{12}$, $B_{i+1} \geq B_{i-1}$, $\frac{h_ih_{i+1}}{24} \leq B_{i+1} \leq \frac{h_ih_{i+1}}{12}$, $B_{i-1} \geq -\frac{h_ih_{i+1}}{12}$.

Lemma 3. *If $x_i = \lambda(t_i)$, $t_i = ih$, $i = 0, 1, \ldots, m$, $h = \frac{1}{n}$, $n = 2m$, A_{i-1}, A_i, A_{i+1} are given by* (5), *B_{i-1}, B_i, B_{i+1} are given by* (6), *then*

(i) $A_{i-1} + A_i + A_{i+1} = 0$, $B_{i-1} + B_i + B_{i+1} = \dfrac{h_ih_{i+1}}{2}$,

(ii) $B_i \geq \dfrac{5h_ih_{i+1}}{12}$, $B_{i+1} \geq B_{i-1}$, $\dfrac{h_ih_{i+1}}{24} \leq B_{i+1} \leq \dfrac{h_ih_{i+1}}{12}$,

(iii) $B_{i-1} \geq -\dfrac{h_ih_{i+1}}{12}$.

Proof. (i) It is easy to verify the results directly. (ii) Since $\lambda'(t) \geq 0$, $h_{i+1} \geq h_i > 0$, so $B_{i+1} - B_{i-1} = \dfrac{1}{12(h_i + h_{i+1})}(h_{i+1}^3 - h_i^3) \geq 0$. Thus, $B_{i+1} \geq B_{i-1}$. It is not hard to see that

$$B_i = \frac{1}{12}(h_i^2 + h_{i+1}^2 + 3h_ih_{i+1}) = \frac{h_ih_{i+1}}{12}\left(\frac{h_i^2 + h_{i+1}^2}{h_ih_{i+1}} + 3\right) \geq \frac{5}{12}h_ih_{i+1},$$

$$B_{i+1} = \frac{h_i(h_{i+1}^2 - h_i^2 + h_ih_{i+1})}{12(h_i + h_{i+1})} = \frac{h_ih_{i+1}}{12}\left[1 - \frac{h_i^2}{h_{i+1}(h_i + h_{i+1})}\right]$$

$$= \frac{h_ih_{i+1}}{12}\left[1 - \frac{1}{\dfrac{h_{i+1}}{h_i}\left(1 + \dfrac{h_{i+1}}{h_i}\right)}\right] \geq \frac{h_ih_{i+1}}{12}\left[1 - \frac{1}{2}\right] = \frac{h_ih_{i+1}}{12},$$

and $$B_{i+1} = \frac{h_ih_{i+1}}{12}\left[1 - \frac{h_i^2}{h_i(h_i + h_{i+1})}\right] \leq \frac{h_ih_{i+1}}{12}.$$

(iii) $$B_{i-1} = \frac{h_{i+1}(h_i^2 - h_{i+1}^2 + h_ih_{i+1})}{12(h_i + h_{i+1})} = \frac{h_ih_{i+1}}{12}\left[1 - \frac{h_{i+1}^2}{h_i(h_i + h_{i+1})}\right].$$

It is easy to see that $B_{i-1} \geq \dfrac{h_ih_{i+1}}{12}$ is equivalent to $1 - \dfrac{h_{i+1}^2}{h_i(h_i + h_{i+1})} \geq -1$, i.e.

$\left(\frac{h_{i+1}}{h_i}\right)^2 - 2\frac{h_{i+1}}{h_i} - 2 \le 0$, i.e. $\frac{h_{i+1}}{h_i} \le 1 + \sqrt{3} = \beta^2$. Let $s = \frac{\beta - 1}{\beta}$, $\beta = \sqrt{1 + \sqrt{3}} \approx 1.653$, $s \approx 0.395$. Let $P = t_k + s\sqrt{\varepsilon} - \frac{2h}{s} > 0$. If h is sufficiently small, then $P \ge t_k$.

Let us now consider the case $i < k+1$. We have $t_{i-1} \le P$, $t_{i+1} = t_{i-1} + 2h \le P + 2h = t_k + s\sqrt{\varepsilon} - \frac{2h}{s} + 2h = t_k + s\sqrt{\varepsilon} - 2h\left(\frac{1}{s} - 1\right) < t_k + \sqrt{\varepsilon} = q$. In this case, $t_{i-1} \le t_k$, $\frac{\lambda'(t_{i+1})}{\lambda'(t_{i-1})} \le \frac{\omega'(t_{i+1})}{\omega'(t_{i-1})} = \left(\frac{q - t_{i-1}}{q - t_{i+1}}\right)^2 = \left(\frac{q - t_{i+1} + 2h}{q - t_{i+1}}\right)^2 = \left(1 + \frac{2h}{q - t_{i+1}}\right)^2$. Because $q - t_{i+1} = q - t_{i-1} - 2h \ge q - P - 2h = t_k + \sqrt{\varepsilon} - t_k - s\sqrt{\varepsilon} + \frac{2h}{s} - 2h = (1-s)\sqrt{\varepsilon} + \frac{2h}{s}(1-s) \ge \frac{2h}{s}(1-s)$, so $\frac{\omega'(t_{i+1})}{\omega'(t_{i-1})} \le \left[1 + \frac{2h}{(2h/s)(1-s)}\right]^2 = \left(1 - \frac{s}{1-s}\right)^2 = \left(\frac{1}{1-s}\right)^2 = \beta^2$.

Thus, $\frac{h_{i+1}}{h_i} = \frac{\lambda'(\xi_{i+1})h}{\lambda'(\xi_i)h} \le \frac{\lambda'(t_{i+1})}{\lambda'(t_{i-1})}$, where $t_i < \xi_{i+1} < t_{i+1}$, $t_{i-1} < \xi_i < t_i$, $\lambda'(t_{i+1}) \ge \lambda'(\xi_{i+1})$, $\lambda'(t_{i-1}) \le \lambda'(\xi_i)$. This implies that $\frac{h_{i+1}}{h_i} \le \beta^2 = 1 + \sqrt{3}$.

In the case $i \ge k+1$, i.e. $t_{i-1} \ge t_k$, $x_i = \lambda(t_i)$ is an equidistant mesh. The result holds obviously.

Theorem 1. *Let condition* (2) *hold and let the discrete problem* (7) *be given on the mesh* (8) *with sufficiently large* n. *Then problem* (7) *has a unique solution* $V_h = [u_0, u_1, \ldots, u_n]^T$, *which is a point of attraction of SOR–Newton and Newton–SOR method for the relaxation parameter in* $(0, 1]$. *Moreover, for any* V_h^1, $V_h^2 \in R^{n+1}$, *the following stability inequality holds:* $\|V_h^1 - V_h^2\|_\infty \le \sigma^{-1}\|N_h V_h^1 - N_h V_h^2\|_\infty$, *where* σ *is a positive constant, independent of* ε.

Proof. The Frechet derivative $N_h^1(V)$ of N_h for arbitrary $V = [V_0, V_1, \cdots, V_n]^T$ is a tridiagonal matrix

$$N_h^1(V) = \begin{bmatrix} 1 & 0 & 0 & \\ a_1 & b_1 & c_1 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & a_n & b_n & c_n \\ & 0 & 0 & 1 \end{bmatrix}$$

where $a_i = \varepsilon^2 A_{i-1} + B_{i-1} f_u(x_{i-1}, V_{i-1})$, $b_i = \varepsilon^2 A_i + B_i f_u(x_i, V_i)$, $c_i = \varepsilon^2 A_{i+1} + B_{i+1} f_u(x_{i+1}, V_{i+1})$.

Let $\sigma = \min_{1\le i\le n} \sigma_i$, $\sigma_i = |b_i| - |a_i| - |c_i|$. Note that $B_{i-1} \ge -\frac{h_i h_{i+1}}{12}$. We now consider two cases: $B_{i-1} < 0$ and $B_{i-1} \ge 0$.

I. $B_{i-1} < 0$, $a_i < 0$, $b_i > 0$.

i) $c_i \le 0$, $\sigma_i = |b_i| - |a_i| - |c_i| = b_i + a_i + c_i = \varepsilon^2(A_{i-1} + A_i + A_{i+1}) + B_{i-1} f_u(x_{i-1}, V_{i-1}) + B_i f_u(x_i, V_i) + B_{i+1} f_u(x_{i+1}, V_{i+1}) \ge B_i g(x_i) + B_{i+1} g(x_{i+1}) + B_{i-1} G(x_{i-1}) = (B_i + B_{i+1}) g(x_i) + B_{i+1}(g(x_{i+1}) - g(x_i)) + B_{i-1} G(x_i) + B_{i-1}(G(x_{i-1}) - G(x_i))$.

According to Lemma 3, for the coefficients A_j, B_j $(j = i-1, i, i+1)$ we have $A_{i-1} + A_i + A_{i+1} = 0$, $B_{i-1} + B_i + B_{i+1} = \frac{h_i h_{i+1}}{2}$, $B_{i-1} \ge -\frac{h_i h_{i+1}}{12}$, $\frac{h_i h_{i+1}}{24} \le B_{i+1} \le$

$\frac{h_i h_{i+1}}{12}$. We also have $h_{i+1} \geq h_i > 0$, $|g'(x)| \leq L$, $|G'(x)| \leq L$, $5g(x) - 2G(x) \geq \delta > 0$. So $\frac{2}{h_i h_{i+1}}\sigma_i = \left(1 - \frac{2}{h_i h_{i+1}}B_{i-1}\right) g(x_i) + [B_{i+1}g'(\theta_{i+1})h_{i+1} + B_{i-1}G(x_i) + B_{i-1}G'(\theta_{i-1})h_i]\frac{2}{h_i h_{i+1}}$, where, $x_i < \theta_{i+1} < x_{i+1}$, $x_{i-1} < \theta_i < x_i$. Thus, $\frac{12}{h_i h_{i+1}}\sigma_i \geq 7g(x_i) - G(x_i) - |g'(\theta_{i+1})|h_{i+1} - |G'(\theta_i)|h_i \geq 5g(x_i) - 2G(x_i) - 2Lh_{i+1} \geq \delta - 2Lh_{i+1}$.

ii) $c_i \geq 0$, $\sigma_i = b_i + a_i - c_i = \varepsilon^2(A_{i-1} + A_i + A_{i+1}) + B_{i-1}f_u(x_{i-1}, V_{i-1}) + B_i f_u(x_i, V_i) - B_{i+1}f_u(x_{i+1}, V_{i+1}) \geq B_i g(x_i) + B_{i-1}G(x_{i-1}) - B_{i+1}G(x_{i+1}) = B_i g(x_i) + (B_{i-1} - B_{i+1})G(x_i) + B_{i-1}[G(x_{i-1}) - G(x_i)] + B_{i+1}[G(x_i) - G(x_{i+1})] = B_i g(x_i) + (B_{i-1} - B_{i+1})G(x_i) - B_{i-1}|G'(\theta_i)|h_i - B_{i+1}|G'(\theta_{i+1})|h_{i+1}$, where $x_{i-1} < \theta_i < x_i$, $x_i < \theta_{i+1} < x_{i+1}$.

Thus $\frac{12\sigma_i}{h_i h_{i+1}} \geq 5g(x_i) - 2G(x_i) - |G'(\theta_i)|h_i - |G'(\theta_{i+1})|h_{i+1} \geq \delta - 2Lh_{i+1}$.

II. $B_{i-1} \geq 0$.

i) $a_i \leq 0$, $c_i \leq 0$, $\sigma_i = b_i + a_i + c_i = \varepsilon^2(A_{i-1} + A_i + A_{i+1}) + B_{i-1}f_u(x_{i-1}, V_{i-1}) + B_i f_u(x_i, V_i) - B_{i+1}f_u(x_{i+1}, V_{i+1}) \geq (B_{i-1} + B_i + B_{i+1})r^2 \geq \frac{h_i h_{i+1}}{2}r^2 > 0$.

Similarly, we can prove that

ii) $a_i \leq 0, c_i \geq 0$. $\frac{2}{h_i h_{i+1}}\sigma_i \geq \frac{\delta}{6} - \frac{Lh_{i+1}}{6}$. iii) $a_i \geq 0, c_i \leq 0$. $\frac{2}{h_{ki+1}}\sigma_i \geq \frac{\delta}{6} - \frac{Lh_{i+1}}{6}$.

iv) $a_i \geq 0$, $c_i \geq 0$. $\frac{2}{h_i h_{i+1}}\sigma_i \geq \frac{\delta}{6} - \frac{Lh_{i+1}}{3}$.

Summarizing all the cases above, we have

$$\frac{2}{h_i h_{i+1}}\sigma_i \geq \min\left\{r^2, \frac{\delta}{6} - \frac{Lh_{i+1}}{3}\right\}, \qquad i = 1, 2, \ldots, n.$$

For sufficiently large n, when $t_i < t_k$, we take $n > \frac{4L\lambda'(t_k)}{\delta}$. Then $h_{i+1} = \lambda(t_{i+1}) - \lambda(t_i) = \lambda'(\theta)h$, $\theta \in (t_i, t_{i+1})$, $h = \frac{1}{n} < \frac{\delta}{4L\lambda'(t_k)}$, $\frac{\delta}{6} - \frac{Lh_{i+1}}{3} \geq \frac{\delta}{6} - \frac{\lambda'(\theta)\delta}{12\lambda'(t_k)} \geq \frac{\delta}{12}$. Thus $\sigma_i \geq \min\left\{\frac{h_i h_{i+1}}{2}\gamma^2, \frac{h_i h_{i+1}}{24}\delta\right\} > 0$, $i = 1, 2, \ldots, n$. So $\sigma = \min_{1\leq i\leq n} \sigma_i > 0$.

In the case $t_k \leq t_i \leq \frac{1}{2}$, $\lambda(t)$ is a linear function. $\{x_i\}$ is an equidistant mesh. $h_i = \omega'(t_k)h = \tilde{h}$. $B_{i-1} = \frac{\tilde{h}^2}{24}$, $B_i = \frac{5\tilde{h}^2}{12}$, $B_{i+1} = \frac{\tilde{h}^2}{24}$, $A_{i-1} = A_{i+1} = -\frac{1}{2}$, $A_i = 1$. According to case I, it is easy to see that

$$\sigma = \min_{1\leq i\leq n} \sigma_i \geq \min\left\{\gamma^2\frac{\tilde{h}^2}{2}, \delta\frac{\tilde{h}^2}{24}\right\} > 0.$$

From the results above, we have

$$\|N_h'(V)^{-1}\|_\infty \leq \frac{1}{\sigma}. \tag{9}$$

Now by Hadamard's theorem [5], (7) has a unique solution $V_h = [u_0, u_1, \ldots, u_n]^T$. The matrix $N_h^1(V_h)$ is a strictly diagonally dominant matrix and the convergence of Newton–SOR and SOR–Newton iterative methods for the relaxation parameter in $(0,1]$ follows by the well known theorem from [5].

Since $N_hV_h^1 - N_hV_h^2 = N_h^1(V)(V_h^1 - V_h^2)$, from (9) it follows that

$$\|V_h^1 - V_h^2\|_\infty \le \sigma^{-1}\|N_hV_h^1 - N_hV_h^2\|_\infty.$$

Thus, we have proven the theorem.

4. Error Analysis

Let us consider the consistency error

$$r_h = N_hu_h - N_hV_h = N_hu_h$$

where $V_h = [u_0, u_1, \ldots, u_n]^T$ is the solution of problem (7), and $u_h = [u(x_0), u(x_1), \ldots, u(x_n)]^T \in R^{n+1}$ is the restriction of solution $u(x)$ to problem (1) on the mesh I_h. The components of the vector r_h are $r_0 = r_n = 0$, and for $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} r_i = N_hu(x) - N_hu_i = N_hu(x_i) &= \varepsilon^2(A_{i-1}u(x_{i-1}) + A_iu(x_i) + A_{i+1}u(x_{i+1})) \\ &+ B_{i-1}f(x_{i-1}, u(x_{i-1})) + B_if(x_i, u(x_i)) + B_{i+1}f(x_{i+1}, u(x_{i+1})) = \varepsilon^2(A_{i-1}u(x_{i-1}) \\ &+ A_iu(x_i) + A_{i+1}u(x_{i+1})) + \varepsilon^2(B_{i-1}u''(x_i) + B_{i+1}u''(x_{i+1})). \end{aligned} \tag{10}$$

By using the Taylor expansion we have

$$\begin{aligned} u(x_{i-1}) &= u(x_i) - u'(x_i)h_i + \frac{1}{2}u''(x_i)h_i^2 - \frac{1}{3!}u'''(x_i)h_i^3 + \frac{1}{4!}u^{(4)}(x_i)h_i^4 \\ &\quad - \frac{1}{5!}u^{(5)}(x_i)h_i^5 + \frac{1}{6!}u^{(6)}(\xi_i^1)h_i^6, \\ u(x_{i+1}) &= u(x_i) + u'(x_i)h_{i+1} + \frac{1}{2}u''(x_i)h_{i+1}^2 + \frac{1}{3!}u'''(x_i)h_{i+1}^3 + \frac{1}{4!}u^{(4)}(x_i)h_{i+1}^4 \\ &\quad + \frac{1}{5!}u^{(5)}(x_i)h_{i+1}^5 + \frac{1}{6!}u^{(6)}(\xi_i^2)h_{i+1}^6, \\ u''(x_{i-1}) &= u''(x_i) - u'''(x_i)h_i + \frac{1}{2}u^{(4)}(x_i)h_i^2 - \frac{1}{3!}u^{(5)}(x_i)h_i^3 + \frac{1}{4!}u^{(6)}(\xi_i^3)h_i^4, \\ u''(x_{i+1}) &= u''(x_i) + u'''(x_i)h_{i+1} + \frac{1}{2}u^{(4)}(x_i)h_{i+1}^2 + \frac{1}{3!}u^{(5)}(x_i)h_{i+1}^3 + \frac{1}{4!}u^{(6)}(\xi_i^4)h_{i+1}^4. \end{aligned}$$

Let α_s denote the coefficient of $u^{(5)}(x_i)$ in (10), $s = 0, 1, 2, 3, 4, 5$, and let α_6 denote the sum of the terms of $u^{(6)}(\xi_i^j)$, $j = 1, 2, 3, 4$. We get

$$\begin{aligned} \alpha_0 &= A_{i-1} + A_i + A_{i+1} = 0, \\ \alpha_1 &= -h_iA_{i-1} + h_{i+1}A_{i+1} = \frac{h_ih_{i+1}}{h_i + h_{i+1}} - \frac{h_ih_{i+1}}{h_i + h_{i+1}} = 0, \\ \alpha_2 &= \frac{1}{2}h_i^2A_{i-1} + \frac{1}{2}h_{i+1}^2A_{i+1} + B_{i-1} + B_i + B_{i+1} = \frac{-h_i^2h_{i+1}}{2(h_i + h_{i+1})} + \frac{-h_ih_{i+1}^2}{2(h_i + h_{i+1})} \\ &\quad + \frac{h_ih_{i+1}}{2} = 0, \end{aligned}$$

$$\alpha_3 = \frac{-1}{3!}h_i^3A_{i-1} + \frac{1}{3!}h_{i+1}^3A_{i+1} - h_iB_{i-1} + h_{i+1}B_{i+1} = 0,$$
$$\alpha_4 = \frac{1}{4!}h_i^4A_{i-1} + \frac{1}{4!}h_{i+1}^4A_{i+1} + \frac{1}{2}h_i^2B_{i-1} + \frac{1}{2}h_{i+1}^2B_{i+1} = 0,$$
$$\alpha_5 = \frac{-1}{5!}h_i^5A_{i-1} + \frac{1}{5!}h_{i+1}^5A_{i+1} - \frac{1}{3!}h_i^3B_{i-1} + \frac{1}{3!}h_{i+1}^3B_{i+1} = 0,$$
$$\alpha_6 = \frac{1}{6!}u^{(6)}(\xi_i^1)h_i^6A_{i-1} + \frac{1}{6!}u^{(6)}(\xi_i^2)h_{i+1}^6A_{i+1} + \frac{1}{4!}u^{(6)}(\xi_i^3)h_i^4B_{i-1} + \frac{1}{4!}u^{(6)}(\xi_i^4)h_{i+1}^4B_{i+1}$$
$$= \frac{-h_ih_{i+1}}{6!(h_i+h_{i+1})}[u^{(6)}(\xi_i^1)h_i^5 + u^{(6)}(\xi_i^2)h_{i+1}^5] + \frac{u^{(6)}(\xi_i^3)h_i^4h_{i+1}(h_i^2 - h_{i+1}^2 + h_ih_{i+1})}{12\times 4!(h_i+h_{i+1})}$$
$$+ \frac{u^{(6)}(\xi_i^4)h_ih_{i+1}^4(h_{i+1}^2 - h_i^2 + h_ih_{i+1})}{12\times 4!(h_i+h_{i+1})},$$

where $\xi_i^j \in (x_{i-1}, x_{i+1})$, $j = 1,2,3,4$.

Thus, we obtain

$$r_i = \varepsilon^2\alpha_6.$$

According to Lemma 2,

$$|u^{(i)}(x)| \le \begin{cases} M[1 + \varepsilon^{-i}\exp(-\gamma\frac{x}{\varepsilon})], & 0 \le x \le \frac{1}{2}, \\ M[1 + \varepsilon^{-i}\exp(-\gamma\frac{(1-x)}{\varepsilon})], & \frac{1}{2} \le x \le 1. \end{cases}$$

If $t \in [0, t_k]$, $\exp(-\gamma\lambda(t)/\varepsilon) = \exp(-\gamma\frac{\alpha\varepsilon t}{(q-t)\varepsilon}) = \exp(\gamma\alpha - \frac{\gamma\alpha q}{q-t}) \le M\exp(-\frac{M}{q-t})$; thus, $|u^{(6)}(\xi_i^j)| \le M[1 + \varepsilon^{-6}M\exp(-\frac{M}{q-t_{i-1}})]$, $j = 1,2,3,4$.

Note that

$$h_i = \lambda(t_i) - \lambda(t_{i-1}) = \lambda'(t_{i-1} + \theta_1 h)h = \frac{\alpha\varepsilon qh}{(q - t_{i-1} - \theta_1 h)^2} \le \frac{\alpha\varepsilon q}{(q - t_{i+1})^2}h,$$
$$h_{i+1} = \lambda(t_{i+1}) - \lambda(t_i) = \lambda'(t_i + \theta_2 h)h = \frac{\alpha\varepsilon qh}{(q - t_i - \theta_2 h)^2} \le \frac{\alpha\varepsilon q}{(q - t_{i+1})^2}h,$$
$$0 < \theta_1, \theta_2 < 1, \quad t_{i-1} < t_k \le p = t_k + s\sqrt{\varepsilon} - \frac{2h}{s} < q - 3h, \ (s = 0.395),$$
$$\frac{q - t_{i+1}}{q - t_{i-1}} = \frac{q - t_{i-1} - 2h}{q - t_{i-1}} = 1 - \frac{2h}{q - t_{i-1}} \ge 1 - \frac{2h}{3h} = \frac{1}{3}, \quad \frac{1}{q - t_{i+1}} \le \frac{3}{q - t_{i-1}}.$$

Thus, $|r_i| = |N_hu(x_i) - N_hu_i| \le M\varepsilon^2\left(\frac{\alpha\varepsilon q}{(q - t_{i+1})^2}\right)^6 h^6[1 + \varepsilon^{-6}M\exp(-M/(q - t_{i-1}))] \le M\varepsilon^2\left(\frac{9\alpha q\varepsilon}{(q - t_{i-1})^2}\right)^6 h^6[1 + \varepsilon^{-6}M\exp(-M/(q - t_{i-1}))] \le M\varepsilon^2h^6(q - t_{i-1})^{-12}$ $\exp(-M/(q - t_{i-1})) \le Mh^6$ (M, a constant independent of ε and h, may take different values).

In the case of $t \in [t_k, \frac{1}{2}]$, the mesh is equidistant and is far from the boundary layers. Note that $x \ge \lambda(t_k) = \omega(t_k) = \alpha\sqrt{\varepsilon}t_k$, $\varepsilon^{-i}\exp(-\gamma\frac{x}{\varepsilon}) \le \varepsilon^{-i}\exp(-\gamma\frac{\alpha t_k}{\sqrt{\varepsilon}}) \le M$. Analogously, it is easy to prove $|r_i| \le Mh^6$. Thus we get

Theorem 2. *Let condition* (2) *hold. On the mesh* (8), *we have, for* $i = 1, 2, \ldots, n$, $|r_i| = |N_h u(x_i) - N_h u_i| \le M h^6$, *where* $u(x_i)$ *are the same as in Theorem* 1, *and* M *is independent of* h *and* ε.

Summarizing the results above, we have the main result of this paper.

Theorem 3. *Let condition* (2) *be satisfied, then* $\|u_h - V_h\|_\infty \le M h^4$, *where* $u_h = [u(x_0), u(x_1), \ldots, u(x_n)]^T \in R^{n+1}$ *is the restriction of solution* $u(x)$ *of problem* (1) *on mesh* (8), $V_h = [u_0, u_1, \ldots, u_n]^T$ *is the solution of problem* (7), *and* M *is independent of* h *and* ε.

Proof. By using Theorem 2 we have $|N_h u(x_i) - N_h u_i| \le M h^6$, where M is independent of h and ε. By using Theorem 1 we have $\|u_h - V_h\|_\infty \le \sigma^{-1} \|N_h u_h - N_h V_h\|_\infty$, where σ is a positive constant independent of ε and $\sigma^{-1} = O(h^{-2})$. So, $\|u_h - V_h\|_\infty \le M h^4$, where M is independent of h and ε. Thus, we have proven the theorem.

5. A Numerical Example

Consider the following example:

$$\begin{cases} -\varepsilon^2 u'' + u - x^2 + x + 1 + 2\varepsilon^2 = 0, \\ u(0) = u(1) = 0, \end{cases}$$

whose solution is $u(x) = \dfrac{\exp(-\frac{x}{\varepsilon}) + \exp(\frac{x-1}{\varepsilon})}{1 + \exp(-\frac{1}{\varepsilon})} + x^2 - x - 1$.

In the table we give the error $E = \|u_h - V_h\|_\infty$. Different values of ε and h are taken.

n \ ε	2^{-2}	2^{-4}	2^{-8}	2^{-16}
8	5.27690	5.46145(-2)	5.48783(-2)	5.48565(-2)
16	4.11203(-2)	9.29340(-3)	9.27858(-3)	9.29014(-3)
32	3.42830(-4)	2.71855(-3)	2.71941(-3)	2.71940(-3)
64	2.24534(-5)	1.38178(-4)	1.38218(-4)	1.38202(-4)
128	1.29380(-6)	9.00482(-6)	9.01035(-6)	9.01040(-6)
256	9.21414(-8)	5.48899(-7)	5.48889(-7)	5.48890(-7)

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