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ASYMPTOTIC ERROR EXPANSION FOR THE NYSTROM METHOD OF NONLINEAR VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND*1)

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Abstract

While the numerical solution of one-dimensional Volterra integral equations of the second kind with regular kernels is well understood, there exist no systematic studies of asymptotic error expansion for the approximate solution. In this paper, we analyse the Nystrom solution of one-dimensional nonlinear Volterra integral equation of the second kind and show that approximate solution admits an asymptotic error expansion in even powers of the step-size h, beginning with a term in h^2. So that the Richardson's extrapolation can be done. This will increase the accuracy of numerical solution greatly.

1. Introduction

Consider the nonlinear Volterra integral equation of the second kind

$$u(x) = \int_a^x K(x,t,u(t))dt + f(x), \quad x \in [a,b]. \tag{1}$$

Here, $u(x)$ is an unknown function, $f(x)$ and $K(x,t,u)$ are given continuous functions defined, respectively, on $[a,b]$ and $D = \{(x,t,u) \ : \ a \le x \le b, \ a \le t \le x, \ -\infty < u < \infty\}$, and assumes that $K(x,t,0) = 0$. Otherwise, since

$$u(x) = \int_a^x (K(x,t,u(t)) - K(x,t,0))dt + f(x) + \int_a^x K(x,t,0)dt.$$

we have

$$u(x) = \int_a^x \widetilde{K}(x,t,u(t))dt + \widetilde{f}(x) \tag{1'}$$

where $\widetilde{f}(x) = f(x) + \int_a^x K(x,t,0)dt$, $\widetilde{K}(x,t,u) = K(x,t,u) - K(x,t,0)$, so $\widetilde{K}(x,t,0) = 0$. We shall discuss (1′).

During the last ten years significant progress has been made in the numerical analysis of the one-dimensional Volterra integral equation (see for example [1], [2]). However,

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there exist no systematic studies of the asymptotic error expansion of the approximate solution. There is only a brief mention in [4: pp.300–309] for linear one-dimensional Volterra integral equation. In this paper, we consider the nonlinear one-dimensional Volterra integral equation of the second kind, and obtain asymptotic error expansion for the Nystrom method of the equation (1). For two-dimensional Volterra integral equations, similar results can be obtained. We shall discuss it in another paper.

2. The Nystrom Method and Its Asymptotic Expansion

Let Δ be an equidistant partition of $[a,b]$

$$\Delta:\ a=x_0<x_1<\ldots<x_N=b$$

and $h=(b-a)/N$.

The Nystrom method to (1) is: Find $u^h=(u_0^h,u_1^h,\ldots,u_N^h)^T$ such that

$$u_i^h=\frac{h}{2}\left[K(x_i,x_0,u_0^h)+K(x_i,x_i,u_i^h)+2\sum_{j=1}^{i-1}K(x_i,x_j,u_j^h)\right]+f_i, \tag{2}$$

where $f_i=f(x_i)$.

In order to guarantee the existence of a unique solution to equation (2) and obtain the asymptotic error expansion of Nystrom method. We assume throughout this paper that the following conditions (i)—(iii) be satisfied:

(i) $K(x,t,u)\in C^{r+1}(D)$, $f(x)\in C^{r+1}[a,b]$;

(ii) $K(x,t,u)$ satisfies the Lipschitz condition

$$|K(x,t,u)-K(x,t,v)|\le L|u-v|;$$

(iii) $q=Lh/2<1$.

Under these conditions there exists a unique solution in $C^{r+1}[a,b]$ for equation (1). This can be done by classical Volterra theory.

Lemma 1. *Suppose that $K(x,t,u)$ satisfies (ii)—(iii). Then the equation (2) has a unique solution and the estimate*

$$\|u^h\|_{\Delta,\infty}\le\frac{1}{1-q}\exp\left(\frac{L(b-a)}{1-q}\right)\|f\|_{\Delta,\infty}$$

holds for the solution of the equation (2), where $\|f\|_{\Delta,\infty}=\max_{0\le i\le N}|f_i|$.

Proof. The equation (2) is a system of nonlinear triangular algebraic equations and its diagonal elements

$$\left|u_i^h-\frac{h}{2}K(x_i,x_i,u_i^h)\right|\ge|u_i^h|-\frac{h}{2}|K(x_i,x_i,u_i^h)|\ge|u_i^h|-\frac{hL}{2}|u_i^h|=(1-q)|u_i^h|$$

differ from zero. So the equation (2) has a unique solution.

From (2) and condition (ii) we have

$$|u_i^h| \le \frac{h}{2}\left[|K(x_i,x_0,u_0^h)| + |K(x_i,x_i,u_i^h)| + 2\sum_{j=1}^{i-1}|K(x_i,x_j,u_j^h)|\right] + |f_i|$$
$$\le \frac{h}{2}\left[L|u_0^h| + L|u_i^h| + 2L\sum_{j=1}^{i-1}|u_j^h|\right] + |f_i|.$$

From this, using condition (iii), we have

$$|u_i^h| \le \frac{hL}{1-q}\sum_{j=0}^{i-1}|u_j^h| + \frac{1}{1-q}\|f\|_{\Delta,\infty}.$$

Using Gronwall's inequality (see, for example [2]) we obtain

$$|u_i^h| \le \frac{1}{1-q}\left(1+\frac{Lh}{1-q}\right)^i \|f\|_{\Delta,\infty}. \tag{3}$$

From (3) and $1+x \le \exp(x)$, for any $x \ge 0$, it follows

$$|u_i^h| \le \frac{1}{1-q}\cdot\exp\left(\frac{iLh}{1-q}\right)\|f\|_{\Delta,\infty} \le \frac{1}{1-q}\cdot\exp\left(\frac{L(b-a)}{1-q}\right)\|f\|_{\Delta,\infty},$$
$$i = 0,1,\dots,N.$$

The Lemma is proved.

Lemma 2. *If $f(x) \in C^r[a,b]$. Then, for any $i = 1,2,\dots,N$, we have*

$$\frac{h}{2}\left[f(x_0)+f(x_i)+2\sum_{j=1}^{i-1}f(x_j)\right] = \int_a^{x_i} f(x)dx + \sum_{j=1}^{[\frac{r}{2}]} h^{2j}\frac{B_{2j}}{(2j)!}[f^{(2j-1)}(x)]_{x=a}^{x_i} + O(h^{r+1})$$

where B_j are Bernoulli numbers. $[f(x)]_{x=a}^{x_i} = f(x_i) - f(a)$.

Using Taylor's expansion, we can easily get the following Lemma.

Lemma 3. *Suppose that $v(y) = \sum_{p=0}^{[\frac{r}{2}]} h^{2p}d_p(y)$, $d_p(y) \in C[a,b]$. Then*

$$K(x,y,v(y)) = K(x,y,d_0(y)) + \sum_{p=1}^{[\frac{r}{2}]} h^{2p}\left[\frac{\partial}{\partial u}K(x,y,d_0(y))d_p(y) + g_p(x,y)\right] + O(h^{r+1})$$

here

$$g_p(x,y) = \sum_{s=2}^{p}\frac{1}{s!}\left(\frac{\partial}{\partial u}\right)^s K(x,y,d_0(y))\cdot\sum_{\substack{k_1+\dots+k_s=p\\ k_i\ge 1}}\prod_{n=1}^{s} d_{k_n}(y).$$

Theorem 1. *Let $T_hv_i = \frac{h}{2}\left[K(x_i,x_0,v_0)+K(x_i,x_i,v_i)+2\sum_{j=1}^{i-1}K(x_i,x_j,v_j)\right]+f_i$,*

$v(x) = \sum_{p=0}^{[\frac{r}{2}]} h^{2p}d_p(x)$, $v_i = v(x_i)$. If $d_0(x) = u(x)$, $d_p(x)$ satisfy the following equations

$(p = 1, 2, \ldots, [\frac{r}{2}])$

$$d_p(x) - \int_a^x \frac{\partial}{\partial u} K(x,t,u(t)) d_p(t) dt = \int_a^x g_p(x,t) dt + \frac{B_{2p}}{(2p)!} \left[\frac{\partial^{2p-1}}{\partial t^{2p-1}} K(x,t,u(t)) \right]_{t=a}^x$$
$$+ \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!} \left[\left(\frac{\partial}{\partial t} \right)^{2k-1} \left(\frac{\partial}{\partial u} K(x,t,u(t)) d_{p-k}(t) + g_{p-k}(x,t) \right) \right]_{t=a}^x . \tag{4}$$

Then

$$v_i - T_h v_i = O(h^{r+1}), \quad i = 1, 2, \ldots, N. \tag{5}$$

Proof. Applying Lemma 3 and Lemma 2 we have

$$v_i - T_h v_i = \sum_{p=0}^{[\frac{r}{2}]} h^{2p} d_p(x_i) - \frac{h}{2} \sum_{j=0}^{i-1} \sum_{l=0}^{1} K(x_i, x_{j+1}, d_0(x_{j+1}))$$
$$- \sum_{p=1}^{[\frac{r}{2}]} h^{2p} \cdot \frac{h}{2} \sum_{j=0}^{i-1} \sum_{l=0}^{1} \left(\frac{\partial}{\partial u} K(x_i, x_{j+1}, d_0(x_{j+1})) d_p(x_{j+1}) + g_p(x_i, x_{j+1}) \right)$$
$$- f_i + O(h^{r+1})$$
$$= d_0(x_i) - \int_a^{x_i} K(x_i, t, d_0(t)) dt - f(x_i)$$
$$+ \sum_{p=1}^{[\frac{r}{2}]} h^{2p} \left\{ d_p(x_i) - \int_a^{x_i} \frac{\partial}{\partial u} K(x_i, t, d_0(t)) d_p(t) dt - \int_a^{x_i} g_p(x_i, t) dt \right.$$
$$- \frac{B_{2p}}{(2p)!} \left[\frac{\partial^{2p-1}}{\partial t^{2p-1}} K(x_i, t, d_0(t)) \right]_{t=a}^{x_i}$$
$$\left. - \sum_{k=1}^{[\frac{r}{2}]-p} h^{2k} \frac{B_{2k}}{(2k)!} \left[\left(\frac{\partial}{\partial t} \right)^{2k-1} \left(\frac{\partial}{\partial u} K(x_i, t, d_0(t)) d_p(t) + g_p(x_i, t) \right) \right]_{t=a}^{x_i} \right\}$$
$$+ O(h^{r+1})$$

Write the expression as polynomials in h, from (4) all term in h^{2p}, $p = 0, 1, \ldots, [\frac{r}{2}]$, cancel out, therefore

$$v_i - T_h v_i = O(h^{r+1}), \quad i = 1, 2, \ldots, N.$$

The Theorem is thus proved.

Theorem 2. *Assume that* (i)–(iii) *hold. Then the Nystrom solution u^h can be expanded as*

$$u_i^h = u(x_i) + \sum_{p=1}^{[\frac{r}{2}]} h^{2p} d_p(x_i) + O(h^{r+1}), \; i = 1, 2, \ldots, N. \tag{6}$$

where $d_p(x)$ are the solutions of the equations (4).

Proof. For $i = 0, 1, \ldots, N$, let $e_i = u_i^h - v_i$. By using the definition of T_h we have

$$u_i^h - T_h u_i^h = 0, \quad i = 1, 2, \ldots, N. \tag{7}$$

Subtracting (5) from (7) we have

$$|e_i| \le \frac{h}{2}\left[L|e_0| + L|e_i| + 2L\sum_{j=1}^{i-1}|e_j|\right] + O(h^{r+1}).$$

By using Lemma 1 we have

$$\|e\|_{\Delta,\infty} = O(h^{r+1}).$$

The proof of Theorem 2 is proved.

Remark 1. When $p=1$, $d_1(x)$ satisfies the equation

$$d_1(x) - \int_a^x \frac{\partial}{\partial u}K(x,t,u(t))d_1(t)dt = \frac{1}{12}\left[\frac{\partial}{\partial t}K(x,t,u(t))\right]_{t=a}^{x}.$$

When $p=2$, $d_2(x)$ satisfies the equation

$$d_2(x) - \int_a^x \frac{\partial}{\partial u}K(x,t,u(t))d_2(t)dt = -\frac{1}{720}\left[\left(\frac{\partial}{\partial t}\right)^3 K(x,t,u(t))\right]_{t=a}^{x}$$
$$+\frac{1}{12}\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}K(x,t,u(t))\right)d_1(t)\right]_{t=a}^{x} + \frac{1}{2}\int_a^x \left(\frac{\partial}{\partial u}\right)^2 K(x,t,u(t))d_1^2(t)dt.$$

Remark 2. We assume throughout this paper the sum $\sum_{t_1}^{t_2}$ equals to zero when $t_1 > t_2$.

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