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CONVERGENCE OF LEGENDRE METHODS FOR NAVIER-STOKES EQUATIONS*

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Abstract

This paper is concerned with spectral type of methods using Legendre polynomials. Both Galerkin and collocation approximations for the Navier-Stokes equations are considered and their rates of convergence are obtained. As a consequence, it is shown that these methods achieve spectral accuracy if the solutions to the Navier-Stokes equations are smooth.

1. Introduction

In this paper we study spectral type of methods based on Legendre polynomials. We prove stability and convergence results for these methods using energy estimates. The convergence results we obtained are nearly optimal in the sense that the error estimates for the numerical solution is of the same order as the error estimates in approximation theory [7, 14]. A trivial consequence is that these methods are indeed spectrally accurate.

Spectral methods have been used quite extensively in the past two decades. Because of their high resolution power, these methods receive particular attention in simulating incompressible flows in high Reynolds number. We refer to [6] for a review of applications of the spectral methods in the computation of fluid flows and for the computational issues involved in these applications.

There has also been quite extensive work on the analysis of these methods. The basic stability and convergence results are summarized in [10] for linear hyperbolic problems. The approximation theory in the setting of Sobolev spaces for projections and interpolations using Fourier, Legendre and Chebyshev polynomials are presented in [7]. For the steady Navier-Stokes equations on simple geometries, a complete theory has been established by Canuto, Maday, Quarteroni and their co-workers [3, 4, 5, 6]. For the time-dependent Navier-Stokes equations, previous work has been restricted to Fourier methods with periodic boundary conditions [8, 11, 12]. The present paper extends these results to Legendre methods.

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The key to convergence results and error estimates is stability. Roughly speaking, for the steady Navier-Stokes equations, the stability condition amounts to the inf-sup condition; whereas for the unsteady problem, the stability condition amounts to some unform (independent of the discretization parameter) a priori estimates for the numerical solutions. If we have uniform a priori estimates under sufficiently strong norms, e.g. the $W^{1,\infty}$ norm, convergence and error estimates follow as a consequence of the Gronwall inequality. If the numerical method is linearly stable, then we have a uniform L^2 estimate. If the method is sufficiently accurate, then we also get uniform control of higher norms and L^∞ norms using inverse inequality. If the method is not accurate enough, we may still obtain these estimates by applying Strang's trick. Spectral methods are high order methods and Strang's argument can be replaced by standard smoothness assumptions on the exact solutions.

There is a link between the stability estimates for the steady and unsteady problems. This is explored in [9] for general parabolic equations. For the Navier-Stokes equations, the idea can be summarized as follows. The inf-sup condition for the Stokes equations usually implies some estimates for the resolvent of the Stokes equations. These estimates can then be used to prove stability for the time-dependent Stokes equations using semi-group formulations. Finally stability for the nonlinear time-dependent Navier-Stokes equations can be obtained using the ideas outlined in the last paragraph.

In this paper we more or less follow the argument indicated above, although we will not use semi-group formulations explicitly. The important difference between this work and the work of Canuto, Maday and Quarteroni is that we are concerned only with the approximation of velocity, not pressure. For this purpose it is not necessary to identify all the spurious modes in pressure, whereas in their work spurious modes in pressure are directly linked to the inf-sup condition.

This paper is organized as follows. In the next section we study the Legendre-Galerkin method. After a brief introduction of the method, we summarize the results on the Stokes problem. We then use these results to prove the error estimates for the full time-dependent Navier-Stokes equations. Similar results are proved for the Legendre-collocation method in section 3.

2. The Legendre-Galerkin method

2.1. Preliminaries on the Legendre-Galerkin method

We will use standard notations for Sobolev spaces (see [1]). We use $\|\cdot\|_{m,p}$ to denote the $W^{m,p}$ norm and $\|\cdot\|_m$ to denote the H^m norm. A generic point in the plane R^2 will be denoted by $x = (x_1, x_2)$. $\Omega = (-1,1) \times (-1,1)$. For $u(x,t) \in C([0,T], H^s(\Omega))$, we let

$$|||u|||_s = \max_{0\leq t\leq T}\{||u(\cdot,t)||_s\}. \tag{2.1}$$

We will work frequently with incompressible vector fields. Thus we define V to be the closure of the set $\{v = (v_1, v_2),\ v \in C_0^\infty(\Omega),\ \nabla \cdot v = 0\}$ in $H_0^1(\Omega)$.

Throughout this paper, we will use C to denote generic constants which do not depend on the norms of the data, and use K to denote constants which do depend on the data (but is independent of the discretization parameter). K and C may have different values in different appearance.

Let us now recall some basic facts about the Legendre polynomials. Denote by $L_m(x)$, $m = 0, 1, \cdots$, the Legendre polynomials normalized so that $L_m(1) = 1$, then we have the following orthogonality property

$$(L_m(x), L_k(x)) = \frac{2}{2m+1}\delta_{m,k} \tag{2.2}$$

where $(f, g) = \int_{-1}^{1} f(x)g(x)dx$, for $f, g \in L^2(-1,1)$, and $\delta_{m,k}$ is the Kronecker symbol.

Let $P_N(-1,1)$ be the space of algebraic polynomials of degree $\le N$ in one variable, restricted to $(-1,1)$. We denote by $P_N(\Omega) = P_N(-1,1) \times P_N(-1,1)$ the space of algebraic polynomials on R^2 having degree $\le N$ in each variable. We set $P_N^0(\Omega) = P_N(\Omega) \cap H_0^1(\Omega)$, $X_N = P_N^0(\Omega) \times P_N^0(\Omega)$, $X = H_0^1(\Omega) \times H_0^1(\Omega)$.

The projection operator P_N: $L^2 \to P_N(\Omega)$ is defined by

$$(v - P_N v, \phi) = 0, \quad \text{for} \quad v \in L^2(\Omega), \quad \phi \in P_N(\Omega).$$

It is proved in [7] that

$$\|v - P_N v\|_\mu \le C N^{e(\mu,s)} \|v\|_s \tag{2.3}$$

where

$$e(\mu, s) = \begin{cases} 2\mu - \frac{1}{2} - s & \mu \ge 1 \\ \frac{3}{2}\mu - s & 0 \le \mu \le 1. \end{cases}$$

When $\mu = 0$, this result has been improved by Maday [14]:

$$\|v - P_N v\|_0 \le C N^{-s} \|v\|_s. \tag{2.3$'$}$$

We also denote by P_N the obvious extension of the above projection operator from $L^2(\Omega) \times L^2(\Omega)$ to $P_N(\Omega) \times P_N(\Omega)$.

We will be concerned with the full time-dependent Navier-Stokes equation written in the following form

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla u) + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

with initial condition

$$u(x, 0) = u_0(x).$$

Here u, p stand for velocity and pressure respectively. For simplicity, we have normalized the density and viscosity to be one. We refer to [19] for the basic results concerning (2.4).

The semi-discrete version of the Legendre-Galerkin approximation can be formulated as the following problem:

$$\begin{cases} \text{Find } u_N(\cdot): [0,\infty) \to X_N, p_N(\cdot): [0,\infty) \to P_N(\Omega), \quad \text{such that} \\ (\frac{\partial u_N}{\partial t}, \phi) + a(u_N, \phi) + b(u_N, u_N, \phi) + (\nabla p_N, \phi) = (f, \phi), \quad \text{for any } \phi \in X_N \\ (\nabla \cdot u_N, q) = 0, \quad \text{for any } q \in P_N(\Omega) \\ \text{at } t = 0, \ u_N = P_N u_0 \end{cases} \tag{2.5}$$

where the bilinear form $a(\cdot,\cdot)$ and the tri-linear form $b(\cdot,\cdot,\cdot)$ are defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \text{for } u, v \in X \tag{2.6a}$$

$$b(u,v,\phi) = \int_\Omega u \cdot \nabla v \cdot \phi dx, \text{for } u, v, \phi \in X \tag{2.6b}$$

with $\nabla u \cdot \nabla v = \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$, $u \cdot \nabla v \cdot \phi = \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i} \phi_j$.

Remark 1. (2.5) is an ill-posed problem in terms of the computation of pressure. Spurious modes are included in the space $P_N(\Omega)$ which make the solution $p_N(t)$ nonunique. These are the functions ϕ in $P_N(\Omega)$, which satisfy $(\phi, \nabla \cdot v) = 0$, for any $v \in X_N$. A list of the spurious modes has been given in [5]. If a good approximation for the pressure is desired, one has to filter out these spurious modes. However, these spurious modes will not affect the computation of the velocity field which is our primary concern in this paper. Once we have the estimates on the velocity fields, estimates of the pressure can be obtained from using standard techniques for saddle-point problems. We refer to [5] for some results in this direction.

2.2. The Stokes problem and the operator S_N

The steady state Stokes problem is described by a set of linear equations

$$\begin{cases} -\Delta v + \nabla q = f & \text{in}\,\Omega \\ \nabla v = 0 & \text{in}\,\Omega \end{cases} \tag{2.7}$$

with the boundary condition

$$v = 0, \qquad \text{on} \quad \partial\Omega.$$

The Legendre-Galerkin approximation v_N to v, is determined from the following problem: Find (v_N, q_N) in $X_N \times P_N(\Omega)$ such that

$$\begin{cases} \text{for any } \phi \in X_N, & a(v_N, \phi) + (\nabla q_N, \phi) = (f, \phi) \\ \text{for any } \phi \in P_N(\Omega), & (\nabla v_N, \phi) = 0. \end{cases} \tag{2.8}$$

We mention that Remark 1 on the issue of spurious modes also applies here. However, it is proved in [4] (see also [5]) that there exists a unique function v_N in X_N, which, together with some (nonunique) $q_N \in P_N(\Omega)$, satisfies (2.8). Moreover, the following estimate holds

$$||v - v_N||_1 \le C N^{1-s} ||v||_s. \tag{2.9}$$

A standard duality argument gives:

$$||v - v_N||_0 \leq CN^{-s}||v||_s. \tag{2.10}$$

A maximum norm estimate is needed later in our proof of Theorem 1. Applying Gagliardo-Nirenberg inequality, we get

$$||v - P_N v||_{0,\infty} \leq C||v - P_N v||_2^{\frac{1}{4}}||v - P_N v||_1^{\frac{3}{4}} \leq CN^{2-s}||v||_s.$$

An inverse inequality can be proved easily:

$$||\phi||_{0,\infty} \leq CN^2||\phi||_0 \quad \text{for all} \quad \phi \in P_N(\Omega). \tag{2.11}$$

Using this, we get,

$$\begin{aligned} ||v - v_N||_{0,\infty} &\leq ||v - P_N v||_{0,\infty} + ||P_N v - v_N||_{0,\infty} \\ &\leq ||v - P_N v||_{0,\infty} + CN^2||P_N v - v_N||_0 \\ &\leq CN^{2-s}||v||_s + CN^2(||v - P_N v||_0 + ||v - v_N||_0) \\ &\leq CN^{2-s}||v||_s. \end{aligned} \tag{2.12}$$

In the sequel, we will denote v_N by $S_N v$. S_N is a well-defined linear operator from V to $V \cap X_N$ which enjoys the properties discussed above.

2.3. Error estimates for the Legendre-Galerkin method

Our main result in this section is the following theorem.

Theorem 1. *Assume $u, u_t \in C([0,T], H^s(\Omega) \cap H_0^1(\Omega))$, for $T > 0$, $s > 2$. Then (2.5) has a unique solution $u_N(t)$ up to time T. Moreover, we have the following estimates for $0 \leq t \leq T$*

$$||u(t) - u_N(t)||_0 \leq KN^{-s}(|||u|||_s + |||u_t|||_s) \tag{2.13a}$$

$$\int_0^T ||S_N u(t) - u_N(t)||_1 dt \leq KN^{-s}(|||u|||_s + |||u_t|||_s) \tag{2.13b}$$

$$||u(t) - u_N(t)||_1 \leq KN^{-s+3/2}(|||u|||_s + |||u_t|||_s) \tag{2.13c}$$

$$||u_N(t)||_{0,\infty} \leq K. \tag{2.13d}$$

Some remarks are in order.

Remark 2. An immediate question is whether the regularity assumption in Theorem 1 is realistic. The answer is "no" because we are working on a square, the usual corner singularity may restrict the smoothness of the solutions no matter how smooth the initial data is . For the Navier-Stokes equations, regularity also relies on some non-local compatibility conditions for the data [13]. This apparent controversy seems to appear also in many other works on the Navier-Stokes equations where regularity of the solutions is simply assumed. It is of special importance for spectral methods because often times regularity alone determines the rate of convergence. Some work has appeared recently to clarify this issue for finite element methods [2, 13]. We hope to be able to address this issue in the context of spectral methods in our future publications.

Remark 3. Here only the simplest domain, namely a square, is treated. The implementation of spectral methods on complicated geometry is a subtle issue and has received considerable attention. Besides the domain decomposition techniques [6] and the spectral element methods [18], we also mention the work of Babuska et. al. on p-version finite element method.

Remark 4. In actual computations, (2.5) has to be coupled with an ODE solver. Implicit methods are usually preferred because of the severe time-step restriction for the explicit methods. Based on Theorem 1, the fully discretized methods can be analyzed by standard techniques.

Proof of the theorem. First observe that the following evolutionary equation is satisfied by $S_N u$

$$\begin{aligned}(\frac{\partial S_N u}{\partial t},\phi) &+ a(S_N u,\phi)+b(u,u,\phi)+(\nabla p,\phi)+(f,\phi)\\ &= (\frac{\partial S_N u}{\partial t}-\frac{\partial u}{\partial t},\phi)+a(S_N u-u,\phi),\text{for any }\phi\in X_N. \qquad (2.14)\end{aligned}$$

Let $\xi = S_N u - u_N$, $\eta = u - S_N u$, $e = u - u_N = \xi + \eta$. Then from (2.10), we have

$$\|\eta\|_0 \le CN^{-s}|||u|||_s. \qquad (2.15)$$

Subtracting (2.5) from (2.14), we get

$$\begin{aligned}(\frac{\partial \xi}{\partial t},\phi) &+ a(\xi,\phi)+b(u,u,\phi)-b(u_N,u_N,\phi)+(\nabla p-\nabla p_N,\phi)\\ &= (\frac{\partial S_N u}{\partial t}-\frac{\partial u}{\partial t},\phi)+a(S_N u-u,\phi),\text{for any }\phi\in X_N\ . \qquad (2.16)\end{aligned}$$

We set $\phi = \xi$ in the above equation. Notice that

$$\begin{aligned}&(\frac{\partial \xi}{\partial t},\xi) = \frac{1}{2}\frac{d}{dt}||\xi||_0^2,\\ &a(\xi,\xi)\le \alpha||\xi||_1^2,\\ &(\nabla p-\nabla p_N,\xi)=0,\\ &a(S_N u-u,\xi)=0.\end{aligned}$$

Notice also that $\frac{\partial S_N u}{\partial t} = S_N(\frac{\partial u}{\partial t})$. Therefore,

$$(\frac{\partial S_N u}{\partial t}-\frac{\partial u}{\partial t},\xi)\le \|\frac{\partial u}{\partial t}-S_N(\frac{\partial u}{\partial t})\|_0\|\xi\|_0 \le CN^{-s}||u_t||_s||\xi||_0.$$

To treat the nonlinear term, we write

$$b(u,u,\xi)-b(u_N,u_N,\xi)=b(e,u,\xi)+b(u_N,e,\xi).$$

Integrating by parts gives us $b(u_N,\xi,\xi)=0$, and $b(u_N,e,\xi)=b(u_N,\eta,\xi)=-b(u_N,\xi,\eta)$. Therefore,

$$\begin{aligned}&|b(u,u,\xi)-b(u_N,u_N,\xi)| = |b(e,u,\xi)+b(u_N,e,\xi)|\\ &\le \|\nabla u\|_{0,\infty}||e||_0||\xi||_0 + ||u_N||_{0,\infty}||\xi||_1||\eta||_0.\end{aligned}$$

Using Sobolev inequality, we have $||\nabla u||_{0,\infty} \le C||u||_3 \le K$. Let $M(t) = ||u_N||_{0,\infty}$ which may depend on N. Then from (2.16) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\xi||_0^2 + \alpha||\xi||_1^2 &\le CN^{-2s}||u_t||_s^2 + ||\xi||_0^2 + K(||\xi||_0^2 + ||\eta||_0||\xi||_0) + M(t)||\xi||_1||\eta||_0 \\ &\le KN^{-2s}(||u||_s^2 + ||u_t||_s^2) + K||\xi||_0^2 + CM^2(t)||\eta||_0^2 + \frac{\alpha}{2}||\xi||_1^2\end{aligned}$$

or

$$\frac{d}{dt}||\xi||_0^2 + \alpha||\xi||_1^2 \le CM^2(t)||\eta||_0^2 + KN^{-2s}R^2 + K||\xi||_0^2 \tag{2.17}$$

where $R = |||u|||_s + |||u_t|||_s$. This implies,

$$||\xi(t)||_0^2 \le e^{Kt}||\xi(0)||_0^2 + e^{Kt}\int_0^t e^{-k\tau}(CM^2(\tau)||\eta||_0^2 + KN^{-2s}R^2)d\tau.$$

Note that

$$||\xi(0)||_0 = ||S_N u_0 - P_N u_0||_0 \le ||u_0 - P_N u_0||_0 + ||u_0 - S_N u_0||_0 \le CN^{-s}||u_0||_s.$$

From this and (2.15), we obtain, for $0 \le t \le T$

$$||\xi(t)||_0^2 \le KN^{-2s}|||u|||_s^2 \int_0^t M^2(\tau)d\tau + KN^{-2s}R^2. \tag{2.18}$$

From the inverse inequality (2.11), we get

$$||\xi(t)||_{0,\infty} \le CN^4||\xi||_0^2 \le KN^{4-2s}|||u|||_s^2 \int_0^t M^2(\tau)d\tau + KN^{4-2s}R^2.$$

Using (2.10) we have for $s \ge 2$ and $0 \le t \le T$, $||S_N u||_{L^\infty} \le K$. Since $||u_N(t)||_{0,\infty} \le ||S_N u||_{0,\infty} + ||\xi||_{0,\infty}$, we have

$$||u_N(t)||_{0,\infty}^2 \le K + KN^{4-2s}|||u|||_s^2 \int_0^t M^2(\tau)d\tau + KN^{4-2s}R^2 \tag{2.19}$$

or

$$M^2(t) \le K + KN^{4-2s}|||u|||_s^2 \int_0^t M^2(\tau)d\tau + KN^{4-2s}R^2.$$

Let $y(t) = 1 + \int_0^t M^2(\tau)d\tau$, then for $t \ge 0$

$$y\prime(t) \le K + KN^{4-2s}R^2 y(t). \tag{2.20}$$

Solving this differential inequality, we get for $0 \le t \le T$

$$y(t) \le K.$$

Substituting back in (2.18), we have

$$||\xi(t)||_0 \le KN^{-s}R.$$

This, together with (2.15) , proves (2.13a).

Integrating (2.17), we get (2.13b).

Next, we prove (2.13c). Set $\phi = \frac{\partial\xi}{\partial t}$ in (2.16), we get

$$\begin{aligned}(\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial t}) + a(\xi, \frac{\partial\xi}{\partial t}) &\le ||\frac{\partial S_N u}{\partial t} - \frac{\partial u}{\partial t}||_0||\frac{\partial\xi}{\partial t}||_0 + |b(u, u, \frac{\partial\xi}{\partial t}) - b(u_N, u_N, \frac{\partial\xi}{\partial t})| \\ &\le 4||\frac{\partial S_N u}{\partial t} - \frac{\partial u}{\partial t}||_0^2 + \frac{1}{4}||\frac{\partial\xi}{\partial t}||_0^2 + K||u - u_N||_1^2 + \frac{1}{4}||\frac{\partial\xi}{\partial t}||_0^2.\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}a(\xi,\xi) = a(\xi,\frac{\partial\xi}{\partial t}) \le K(\|u-u_N\|_1^2 + \|\frac{\partial S_N u}{\partial t} - \frac{\partial u}{\partial t}\|_0^2)$$

By (2.3), we have $a(\xi(0),\xi(0)) \le C\|\xi(0)\|_1^2 \le \|u_0\|_s$. From these, and (2.13b), we obtain

$$\|\xi(t)\|_1 \le Ca(\xi(t),\xi(t))^{\frac{1}{2}} \le CN^{-s+\frac{3}{2}}R.$$

We get (2.13c) by combining this estimate and (2.9).

(2.13d) is a direct consequence of (2.19). The proof is now complete.

Remark 5. It can be seen from the proof that if, in our formulation of the Legendre-Galerkin method, we use the initial condition $u_N = S_N u_0$ instead of the one in (2.5), then $\xi(0) = 0$, and we obtain an estimate better than (2.13c).

$$\|u-u_N\|_1 \le CN^{-s+1}(|||u|||_s + |||u_t|||_s), \qquad \text{for } ,0 \le t \le T.$$

The other estimates are still valid.

§ 3. The Legendre-collocation method

3.1. Preliminaries on the Legendre-collocation method

A more commonly used version of the Legendre methods is the collocation method which will be discussed in this section. Especially when nonlinear terms or variable coefficients are involved, collocation type of methods are much more flexible than Galerkin methods.

The stability issue for the collocation scheme is more subtle since the aliasing error introduced by the interpolation operator may cause what is called the "aliasing instability". For example the Galerkin method usually inherits an energy estimate from the PDE. This is no longer the case for the collocation method. For this reason smoothing techniques are introduced [16] and aliasing-free formulations are suggested [17] to avoid instability. These techniques usually complicate the algorithms to certain extend. For practical purposes it is important to identify the cases when the simplest collocation scheme works. For the Fourier-collocation method stability and error estimates has been proved in [15] for the KdV equation and in [8, 12] for the Navier-Stokes equations. Here we will prove similar results for the Legendre-collocation method.

Let us denote by $\{\xi_j\}_{j=0}^N$ the nodes in the Gauss-Lobatto integration formula with weight function $\omega(x) = 1$, and $\{\rho_j\}_{j=0}$ the associated weights. We have

$$-1 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$$

and for $\phi \in P_{2N-1}(-1,1)$,

$$\int_{-1}^1 \phi(x)dx = \sum_{j=0}^N \phi(\xi_j)\rho_j. \tag{3.1}$$

Our grid is defined by

$$G_N = \{x_{j,k} = (\xi_j, \xi_k), \quad 0 \le j, k \le N\}$$

with the corresponding weights, $\omega_{j,k} = \rho_j \rho_k$. For ϕ, $\psi \in C(\bar{\Omega})$, we define a discrete inner product by

$$(\phi, \psi)_c = \sum_{j,k=0}^{N} \phi(x_{j,k})\psi(x_{j,k})\omega_{j,k}. \tag{3.2}$$

The subscript c stands for "collocation". We also define the corresponding discrete norm $||\phi||_c = (\phi, \phi)_c^{\frac{1}{2}}$. This norm is defined for any function in $C(\bar{\Omega})$. It is easy to show for $\phi \in P_N(\Omega)$ (see [7]),

$$||\phi||_0 \le ||\phi||_c \le C||\phi||_0. \tag{3.3}$$

Finally for $\phi \in C^0(\bar{\Omega})$, denote by $P_c\phi$ the unique polynomial in $P_N(\Omega)$ which interpolates ϕ at the grid G_N, i.e.

$$(P_c\phi)(x) = \phi(x), \quad \text{for } x \in G_N.$$

P_c can also be viewed as a projection operator under the inner product defined in (3.2)

$$(\phi - P_c\phi, \psi) = 0, \quad \text{for} \quad \phi \in C_0(\bar{\Omega}), \psi \in P_N(\bar{\Omega}). \tag{3.4}$$

The approximation properties of P_c is studied in [14]. Following result was proved: for $s > 1$, $0 \le \mu \le s$, there exists a constant C such that

$$||u - P_c u||_\mu \le CN^{\mu - s}||u||_s, \quad \mu = 0, 1. \tag{3.5}$$

Assuming $f(x) \in C([0,T] \times \bar{\Omega})$, we consider the following version of the semi-discrete Legendre-collocation approximation to (2.1): Find $u_c(\cdot) \in C([0,T], X_N)$, $p_c(\cdot) \in C([0,T], P_N(\Omega))$, such that

$$\begin{cases} \frac{\partial u_c}{\partial t}(x_{j,k}) - \Delta u_c(x_{j,k}) + (u_c \nabla u_c)(x_{j,k}) & +(\nabla p_c)(x_{j,k}) = f(x_{j,k}), \\ & \text{for} \quad x_{j,k} \in G_N \cap \Omega \\ \nabla \cdot u_c(x_{j,k}) = 0, & \text{for} \quad x_{j,k} \in G_N \\ \text{at } t = 0, & u_c(x,0) = (P_c u_0)(x). \end{cases} \tag{3.6}$$

Here the time variable t was suppressed for notational simplicity. The second equation guarantees that the approximate velocity field is divergence free. The remark after (2.5) on the issue of spurious modes also applies here.

We begin our analysis by putting (3.6) into a variational form. For this purpose, we define the bilinear form a_c: $X_N \times X_N \to R$ and tri-linear form $b_c : X_N \times X_N \times X_N \to R$ by

$$a_c(v, \phi) = -(\Delta v, \phi)_c \tag{3.7a}$$

$$b_c(u, v, \phi) = (u \nabla v, \phi)_c \tag{3.7b}$$

where u, v, $\phi \in X_N$. Arguing as in [3], using (3.1) and integrating by parts, we obtain

$$a_c(v, \phi) = (\nabla v, \nabla \phi)_c \tag{3.8a}$$

$$(\nabla q, \phi) = -(q, \nabla \phi)_c. \tag{3.8b}$$

Now it is easy to see that (3.6) is equivalent to the following problem

$$\begin{cases} (\frac{\partial u_c}{\partial t}, \phi)_c + a_c(u_c, \phi) + b_c(u_c, u_c, \phi) + (p_c, \nabla \cdot \phi)_c = (f, \phi)_c & \text{for all } \phi \in X_N \\ (\nabla \cdot u_c, q)_c = 0 & \text{for all } q \in P_N(\Omega). \end{cases} \tag{3.9}$$

3.2. Error estimates for the Legendre-collocation method

The convergence properties for the scheme (3.9) are summarized in the following theorem. These are nearly optimal estimates in the sense that the error in the numerical solution is of the same order as the error in the initial interpolation.

Theorem 2. *Assume u, $u_t \in C([0,T], H^s(\Omega) \cap H_0^1(\Omega))$, $f \in C([0,T], H^{s-1}(\Omega))$, for some $T > 0$, $s > 3$. Then there exist some constants N_0, and K, depending only on T and $R = |||u|||_s^2 + |||u|||_s + |||u_t|||_s + |||f|||_{s-1}$, such that for $N > N_0$, (3.9) or equivalently (3.6), has a unique solution up to time T. Furthermore, the following estimates hold on $[0,T]$*

$$||u(t) - u_c(t)||_0 \le KN^{-s+1}R \tag{3.10a}$$

$$\int_0^T ||S_N u(t) - u_c(t)||_1 dt \le KN^{-s+1}R \tag{3.10b}$$

$$||u(t) - u_c(t)||_1 \le KN^{-s+2}R. \tag{3.10c}$$

Proof. As in [13], we will break the total error into two parts, the part due to discretization and the part due to the presence of nonlinearity. Let (u_{N-1}, p_{N-1}) be the solution of (2.5) in the space $X_{N-1} \times P_{N-1}(\Omega)$. By (3.1) and (2.5), we have,

$$\begin{cases} (\frac{\partial u_{N-1}}{\partial t}, \phi)_c + a_c(u_{N-1}, \phi) + b(u_{N-1}, u_{N-1}, \phi) + & (p_{N-1}, \nabla \cdot \phi)_c = (f, \phi), \\ & \text{for all } \phi \in X_{N-1} \\ \nabla \cdot u_{N-1} = 0. \end{cases} \tag{3.11}$$

Let $(\bar{u}_c, \bar{p}_c)$ be the solution of the following auxiliary problem: $\bar{u}_c(t)$: $[0,\infty) \to X_{N-1}$, $\bar{p}_c(t)$: $[0,\infty) \to P_{N-1}(\Omega)$

$$\begin{cases} (\frac{\partial \bar{u}_c}{\partial t}, \phi)_c + a_c(\bar{u}_c, \phi) + b(u_{N-1}, u_{N-1}, \phi) + & (\bar{p}_c, \nabla \cdot \phi)_c = (f, \phi)_c, \\ & \text{for all } \phi \in X_{N-1} \\ (\nabla \cdot \bar{u}_c, q)_c = 0 & \text{for } q \in P_{N-1}(\Omega) \end{cases} \tag{3.12}$$

with initial condition $\bar{u}_c(x,0) = P_{N-1}u_0(x) = u_{N-1}(x,0)$. Notice here we are still using the discrete inner product defined in (3.2) with the grid G_N.

Let $\xi = u_{N-1} - \bar{u}_c$, $\eta = \bar{u}_c - u_c$, $\rho = u - u_{N-1}$, $e = u - u_c = \rho + \xi + \eta$. We have from (2.13a)

$$||\rho||_c \le CN^{-s}(|||u|||_s + |||u_t|||_s). \tag{3.13}$$

Taking the difference of (3.11) and (3.12), we get

$$(\frac{\partial \xi}{\partial t}, \phi)_c + a_c(\xi, \phi) + (p_{N-1} - \bar{p}_c, \nabla \cdot \phi)_c = (f, \phi) - (f, \phi)_c. \tag{3.14}$$

Recall an inequality proved in[7]:

$$|(f,\phi)-(f,\phi)_c| \le C\|\phi\|_0(\|f-P_cf\|_0+\|f-P_{N-1}f\|_0). \tag{3.15}$$

We have

$$|(f,\xi)-(f,\xi)_c| \le \|\xi\|_0^2 + CN^{-2s+2}\|f\|_{s-1}^2.$$

Let $\phi=\xi$ in (3.14). Since

$$a_c(\xi,\xi)=(\nabla\xi,\nabla\xi)_c \ge \|\nabla\xi\|_0^2 \ge \alpha\|\xi\|_1^2$$
$$\nabla\cdot\xi=0,$$

we have

$$\frac{d}{dt}\|\xi\|_0^2+\alpha\|\xi\|_1^2 \le \|\xi\|_0^2 + CN^{-2s+2}\|f\|_{s-1}^2.$$

This implies, for $0\le t\le T$

$$\|\xi\|_0 \le CN^{-s+1}|||f|||_{s-1} \tag{3.16}$$

$$\Big(\int_0^T \|\xi\|_1^2 dt\Big)^{1/2} \le CN^{-s+1}|||f|||_{s-1}. \tag{3.17}$$

If we set $\phi=\frac{\partial\xi}{\partial t}$ in (3.14), we get

$$\Big\|\frac{\partial\xi}{\partial t}\Big\|_0^2 + a_c\Big(\xi,\frac{\partial\xi}{\partial t}\Big) = \Big(f,\frac{\partial\xi}{\partial t}\Big)-\Big(f,\frac{\partial\xi}{\partial t}\Big)_c.$$

Since

$$\Big|\Big(f,\frac{\partial\xi}{\partial t}\Big)-\Big(f,\frac{\partial\xi}{\partial t}\Big)_c\Big| \le \Big\|\frac{\partial\xi}{\partial t}\Big\|_0^2 + CN^{-2s+2}\|f\|_{s-1}^2,$$

we obtain then

$$\frac{1}{2}\frac{d}{dt}a_c(\xi,\xi) \le \Big|\Big(f,\frac{\partial\xi}{\partial t}\Big)-\Big(f,\frac{\partial\xi}{\partial t}\Big)_c\Big|.$$

Notice that $\xi(0)=0$. We obtain for $0\le t\le T$

$$\|\xi(t)\|_1 \le CN^{-s+1}|||f|||_{s-1}. \tag{3.18}$$

Next we turn to the estimate of η. We first study the nonlinear term. Write

$$b(u_{N-1},u_{N-1},\phi)-b_c(u_c,u_c,\phi) = b(u_{N-1},u_{N-1},\phi)-b(u,u,\phi)+b(u,u,\phi)$$
$$-b_c(u,u,\phi)+b_c(u,u,\phi)-b_c(u_c,u_c,\phi).$$

Using Theorem 1 (more precisely the improved version stated in Remark 5) and (3.13), we get

$$|b(u_{N-1},u_{N-1},\phi)-b(u,u,\phi)| \le |b(u_{N-1},\rho,\phi)|+|b(\rho,u,\phi)|$$
$$\le \|u_{N-1}\|_{0,\infty}\|\nabla\rho\|_0\|\phi\|_0+\|\nabla u\|_{0,\infty}\|\rho\|_0\|\phi\|_0$$
$$\le KN^{-s+1}(|||u|||_s+|||u_t|||_s)\|\phi\|_0,$$

$$|b(u,u,\phi)-b_c(u,u,\phi)| \le |(u\nabla u,\phi)-(u\nabla u,\phi)_c|+|b(\rho,u,\phi)|$$
$$\le CN^{-s+1}\|u\nabla u\|_{s-1}\|\phi\|_0 \le CN^{-s+1}\|u\|_s^2\|\phi\|_0.$$

Here we have used the fact that H^s is a Banach algebra for $s > 1$,

$$||uv||_s \leq C||u||_s||v||_s.$$

Let $M(t) = ||u_c(t)||_{0,\infty}$ which may depend on N, then

$$\begin{aligned}
&b_c(u,u,\phi) - b_c(u_c,u_c,\phi) = (u\nabla u,\phi)_c - (u_c\nabla u_c,\phi)_c \\
&= ((u-u_c)\nabla u,\phi)_c - (u_c\nabla(u-u_c),\phi)_c \\
&\leq ||\nabla u||_{0,\infty}||u-u_c||_c||\phi||_0 + ||u_c||_{0,\infty}||\nabla u - \nabla u_c||_c||\phi||_0 \\
&\leq K||\phi||_0(||P_c u - u_{N-1}||_0 + ||\eta||_0 + ||\xi||_0) \\
&\qquad +M(t)||\phi||_0(||P_c(\nabla u) - \nabla u_{N-1}||_0 + ||\eta||_1 + ||\xi||_1) \\
&\leq K(||\phi||_0^2 + ||\rho||_0^2) + CN^{-2s+2}(|||u|||_s^2 + |||f|||_{s-1}^2 + |||u_t|||_s^2) \\
&+M^2(t)||\phi||_0^2 + \frac{\alpha}{2}||\eta||_1^2
\end{aligned}$$

where in the last step, we have used (3.5), (3.14) , (3.16) and the fact that

$$\begin{aligned}
||P_c(\nabla u) - \nabla u_{N-1}||_0 &\leq ||\nabla u - P_c(\nabla u)||_0 + ||u - u_{N-1}||_1 \\
&\leq CN^{-s+1}||u||_s + CN^{-s+1}(|||u|||_s + |||u_t|||_s).
\end{aligned}$$

Taking the difference of (3.9) and (3.12), and setting $\phi = \eta$, we get

$$(\frac{\partial\eta}{\partial t},\eta)_c + a_c(\eta,\eta) + b(u_{N-1},u_{N-1},\eta) - b_c(u_c,u_c,\eta) = 0.$$

Using the estimates we have just derived, we get,

$$\frac{d}{dt}||\eta||_c^2 + a_c||\eta||_1^2 \leq (CM^2(t) + K)||\eta||_c^2 + CN^{-2s+2}R^2. \tag{3.19}$$

For notational simplicity, let $B(t) = CM^2(t) + K$, $z(t) = e^{\int_0^t B(\tau)d\tau}$. Solving the above inequality, we obtain

$$||\eta(t)||_c^2 \leq z(t)(CR^2N^{-2s+2} + ||\eta(0)||_c^2) \leq z(t)CR^2N^{-2s+2}. \tag{3.20}$$

From (2.11) and (2.13a), we have

$$\begin{aligned}
M^2(t) &= ||u_c||_{0,\infty}^2 \leq 2(||u_{N-1} - u_c||_{0,\infty}^2 + ||u_{N-1}||_{0,\infty}^2) \leq CN^4||u_{N-1} - u_c||_0^2 + K \\
&\leq CN^{-2s+6}||f||_{s-1}^2 + CR^2N^{-2s+6}z(t) + K = CR^2N^{-2s+6}z(t) + K
\end{aligned}$$

or

$$\begin{aligned}
\frac{z\prime(t)}{z(t)} &= B(t) \leq K + CRN^{-2s+6}z(t) \\
z\prime(t) &\leq Kz(t) + CR^2N^{-2s+6}z^2(t). \qquad (3.21)
\end{aligned}$$

We get here a nonlinear differential inequality instead of the linear one in (2.20). Solving this inequality, we get

$$z(t) \leq \frac{e^{Kt}}{K + CR^2N^{-2s+6}}(K + CR^2N^{-2s+6}z(t)) \leq e^{Kt} + \frac{e^{Kt}}{K}CR^2N^{-2s+6}z(t). \tag{3.22}$$

If we choose N_0, such that

$$\frac{e^{Kt}}{K}CR^2N_0^{-2s+9} < \frac{1}{2}, \tag{3.23}$$

then for $N > N_0$, we get from (3.22) that for $0 \leq t \leq T$

$$z(t) \leq 2e^{KT}.$$

Substituting back in (3.20), we get

$$||\eta||_c \leq KN^{-s+1}R.$$

This, together with (3.13), (3.16), proves (3.10a).

The rest of the proof is the same as the proof of (2.13b) and (2.13c).

Remark 6. Again if we replace the initial condition in (3.6) by $u_c(x,0) = S_N u_0(x)$, then we can get an improved version of (3.10c)

$$||u(t) - u_c(t)||_1 \leq CR^2N^{-s+1}$$

while the other estimates remain unchanged.

A different version of the Legendre-collocation is often suggested and analyzed in the literature. Instead of defining $b_c(u,v,\phi)$ by (3.7b), one defines it by [3]

$$b_c(u,v,\phi) = \sum_{i,j=1}^{2}(\frac{\partial}{\partial x_i}P_c(u_iv_j),\phi_j)_c.$$

For this problem, one can prove the same estimates as in Theorem 2 by the argument presented in this section.

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