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A Unified Instability Region for the Extended Taylor–Goldstein Problem of Hydrodynamic Stability

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Abstract. We consider inviscid, incompressible shear flows with variable density and variable cross section. For this problem, we derived a new estimate for the growth rate of an unstable mode and a parabolic instability region which intersects semiellipse instability region under some condition.

AMS subject classifications: 76E05

Key words: Hydrodynamic stability, shear flows, variable bottom, sea straits.

1 Introduction

The study of inviscid, incompressible shear flows in sea straits with variable density and variable cross section was initiated by [1] and [2] developed the mathematical approach. In the above work, if density remains constant then it leads to extended Rayleigh problem of hydrodynamic stability, while the variable density leads to the extended Taylor-Goldstein problem of hydrodynamic stability. In our present work, we consider extended Taylor-Goldstein problem. For this extended Taylor-Goldstein problem, many general analytical results have been obtained in [2–7].

In [3], they defined the function

$$\phi(z)=\frac{U_0''-TU_0'+2\pi^2[U_{0\max}-U_{0\min}]b_{\min}}{D^2 b_{\max}}>0,$$
$$\psi(z)=\frac{U_0''-TU_0'-2\pi^2[U_{0\max}-U_{0\min}]b_{\min}}{D^2 b_{\max}}<0.$$

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For flows satisfying $\phi(z)>0$ or $\psi(z)<0$, the semiellipse instability region of [3] is reduced. [4] extended their work and derived a parabolic instability region which depends $U_{0\min}>0$. In the present paper, we have obtained a new estimate for the growth rate of an unstable mode and a parabolic instability region which intersects semiellipse instability region given in [3] without any conditions as given in [3,4]. We illustrate our results with examples.

2 Extended Taylor–Goldstein problem

The extended Taylor–Goldstein Problem [2] is an eigen value problem given by

$$\left[\frac{(bW)'}{b}\right]'+\left[\frac{N^2}{(U_0-c)^2}-\frac{b(\frac{U_0'}{b})'}{U_0-c}-k^2\right]W=0, \tag{2.1}$$

with boundary conditions

$$W(0)=0=W(D). \tag{2.2}$$

Here W is the complex eigen function, $c=c_r+ic_i$ is the complex phase velocity, $N^2>0$ is the Brunt Vaisala frequency, $k>0$ is the wave number, U_0 is the basic velocity profile, $b(z)$ is the breadth function.

3 Estimate for growth rate

Theorem 3.1. *An estimate for the growth rate of an unstable mode is given by*

$$k^2c_i^2\le\left[\frac{\left[\frac{|N^2_{\max}|}{2}-|N^2_{\min}|\right]+|b(\frac{U_0'}{b})'|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{4}\right]}{\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}k^2}+1\right]}\right].$$

Proof. Multiplying (2.1) by (bW^*), where $*$ stands for complex conjugation; integrating over $[0,D]$ and using (2.2), we get

$$\int\frac{|(bW)'|^2}{b}dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c)}b|W|^2dz+k^2\int b|W|^2dz-\int\frac{N^2}{(U_0-c)^2}b|W|^2dz=0. \tag{3.1}$$

Equating real part of (3.1), we get

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz\\&-\int\frac{N^2}{|U_0-c|^4}[(U_0-c_r)^2-c_i^2]b|W|^2dz=0.\end{aligned} \tag{3.2}$$

Eq. (3.2) can be written as

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\\&=\left|\int\frac{N^2}{|U_0-c|^4}[(U_0-c_r)^2-c_i^2]b|W|^2dz-\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz\right|.\end{aligned}$$

Using triangular inequality, we get

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\\&\le\left|\int\frac{N^2}{|U_0-c|^4}[(U_0-c_r)^2-c_i^2]b|W|^2dz\right|+\left|-\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz\right|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\\&\le\int\frac{N^2}{|U_0-c|^4}[(U_0-c_r)^2-c_i^2]b|W|^2dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz.\end{aligned}$$

Since $(U_0-c_r)^2-c_i^2=2(U_0-c_r)^2-|U_0-c|^2$, we get

$$\begin{aligned}&\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\\&\le2\int\frac{N^2(U_0-c_r)^2}{|U_0-c|^4}b|W|^2dz-\int\frac{N^2}{|U_0-c|^2}b|W|^2dz+\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)b|W|^2dz,\end{aligned}$$

since

$$\frac{(U_0-c_r)}{|U_0-c|^2}\le\frac{1}{2c_i},\quad\frac{(U_0-c_r)^2}{|U_0-c|^4}\le\frac{1}{4c_i^2}\quad\text{and}\quad\frac{1}{|U_0-c|^2}\le\frac{1}{c_i^2},$$

and using Rayleigh–Ritz inequality, we have

$$\left[\frac{\pi^2b_{\min}}{D^2b_{\max}}+k^2\right]\int b|W|^2dz\le\left[\frac{2|N^2_{\max}|}{4c_i^2}-\frac{N^2_{\min}}{c_i^2}+\frac{|b(\frac{U_0'}{b})'|_{\max}}{2c_i}\right]\int b|W|^2dz,$$

i.e.,

$$\begin{aligned}&\left[\frac{\pi^2b_{\min}}{D^2b_{\max}}+k^2\right]\int b|W|^2dz\\&\le\frac{\frac{|N^2_{\max}|}{2}-|N^2_{\min}|+|b(\frac{U_0'}{b})'|_{\max}\frac{c_i}{2}}{c_i^2}\int b|W|^2dz.\end{aligned}$$

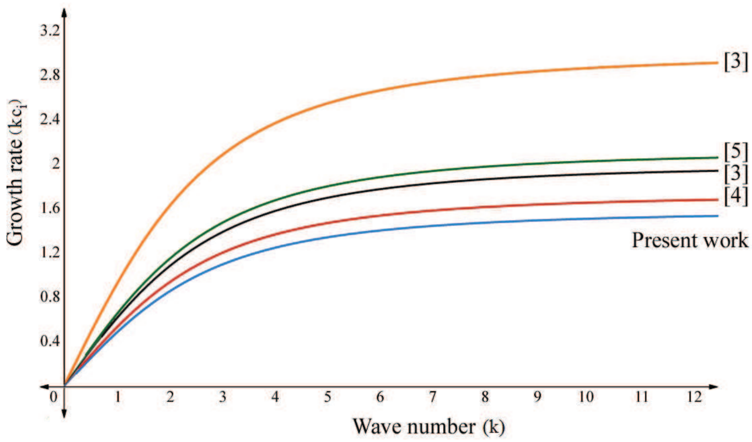

Figure 1: Wave number vs Growth rate.

Table 1: Comparison of growth rates.

Sl.No	Growth	Numerical value	References
1	$k^2c_i^2 \leq \frac{4.5}{\frac{\pi^2}{k^2}+1}$	0.643	[5]
2	$k^2c_i^2 \leq \frac{9}{\frac{\pi^2}{k^2}+1}$	0.910	[3]
3	$k^2c_i^2 \leq \frac{4}{\frac{\pi^2}{k^2}+1}$	0.6066	[3]
4	$k^2c_i^2 \leq \frac{3}{\frac{\pi^2}{k^2}+1}$	0.5257	[4]
5	$k^2c_i^2 \leq \frac{2.5}{\frac{\pi^2}{k^2}+1}$	0.480	Present work

On using the fact that

$$c_i \leq \left[\frac{U_{0\max}-U_{0\min}}{2}\right],$$

we have

$$k^2c_i^2 \leq \frac{\frac{|N^2_{\max}|}{2} - |N^2_{\min}| + |b(\frac{U_0'}{b})'|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{4}\right]}{\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max} k^2}+1\right]}. \tag{3.3}$$

Let us consider a basic flow with velocity profile $U_0(z)=4z(1-z)$ in the region $0\leq z\leq 1$ given in [5], $b(z)=1$, $N^2(z)=z$.

The numerical computation of already existing growth rates are presented in Fig. 1 and Table 1.

From the above Fig. 1 and Table 1, it is evident that our estimate is sharper than the estimates given in [3–5]. For any basic flow when b =constant (or) $T=\frac{b'}{b}=[\ln b]'=0$, the estimate for the growth rate obtained using (3.3) is sharper than the growth rates obtained for standard Taylor-Goldstein problem. $\square$

4 Instability region

Introducing the transformation $W=(U_0-c)^{\frac{1}{2}}G$, [2] derived the equation satisfied by G to be

$$\left[(U_0-c)\frac{(bG)'}{b}\right]'-\frac{b\left(\frac{U_0'}{b}\right)'}{2}G-k^2(U_0-c)G-\frac{\left[\frac{(U_0')^2}{4}-N^2\right]G}{(U_0-c)}=0, \tag{4.1}$$

with boundary conditions

$$G(0)=0=G(D). \tag{4.2}$$

Theorem 4.1. *If $c_i>0$ and U_{os} is an arbitrary real number then the following integral relations are true*

$$\int(U_0-c_r)Q\,dz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}(U_0-U_{0s})b|G|^2dz$$
$$-(c_r-U_{0s})\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}b|G|^2dz=0, \tag{4.3a}$$
$$-c_i\int Qdz+c_i\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]G}{|U_0-c|^2}b|G|^2dz=0. \tag{4.3b}$$

Proof. Multiplying (4.1) by (bG^*), integrating over $[0,D]$ and using (4.2), we get

$$\int(U_0-c)\left[\frac{|(bG)'|^2}{b}+k^2b|G|^2\right]dz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz$$
$$+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{(U_0-c)}b|G|^2dz=0.$$

Let

$$Q=\frac{|(bG)'|^2}{b}+k^2b|G|^2,$$

then the above equation becomes

$$\int(U_0-c)Qdz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{(U_0-c)}b|G|^2dz=0. \tag{4.4}$$

Equating real part of (4.4), we get

$$\int(U_0-c_r)Qdz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right](U_0-c_r)}{|U_0-c|^2}b|G|^2dz=0.$$

Writing (U_0-c_r) as $(U_0-U_{0s})-(c_r-U_{0s})$, where U_{0s} is an arbitrary real number, we get

$$\begin{aligned}&\int(U_0-c_r)Qdz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right](U_0-U_{0s})}{|U_0-c|^2}b|G|^2dz\\&-(c_r-U_{0s})\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}b|G|^2dz=0.\end{aligned}\tag{4.5}$$

Equating imaginary parts of (4.4), we get

$$-c_i\int Qdz+c_i\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]G}{|U_0-c|^2}b|G|^2dz=0.\tag{4.6}$$

So, we complete the proof. □

Theorem 4.2. *For the existence of unstable mode, the following integral relation is true*

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz\geq\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}(c_r-U_0+U_{0\min}-U_{0\max})b|G|^2dz.$$

Proof. Multiplying (4.6) by $(\frac{c_r+U_{0s}}{c_i})$ and subtracting from (4.5), we get

$$\int(U_0+U_{0s})Qdz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G^2|dz+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}(U_0-U_{0s}-2c_r)b|G^2|dz=0.\tag{4.7}$$

Multiplying (4.6) by $(\frac{U_{0\max}-U_{0\min}}{c_i})$ and adding the resultant with (4.5), we get

$$\begin{aligned}&\int(U_0-c_r-U_{0\max}+U_{0\min})Qdz+\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz\\&+\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}(U_0-c_r+U_{0\max}-U_{0\min})b|G|^2dz=0.\end{aligned}$$

Since $(U_0-c_r-U_{0\max}+U_{0\min})\leq 0$, dropping the term, we get

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)'b|G|^2dz\geq\int\frac{\left[\frac{(U_0')^2}{4}-N^2\right]}{|U_0-c|^2}(c_r-U_0+U_{0\min}-U_{0\max})b|G|^2dz.\tag{4.8}$$

Thus, we complete the proof. □

Theorem 4.3. *A necessary condition for the existence of unstable mode is*

$$c_i^2 \leq \lambda \left[c_r + \frac{3U_{0\max} - U_{0\min}}{2} \right],$$

where

$$\lambda = \frac{\left| \frac{(U_0')^2}{4} - N^2 \right|_{\max}}{\left| \frac{3U_{0\min} + U_{0\max}}{2} \right| \left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2 \right]}.$$

Proof. Substituting (4.8) in (4.7), we get

$$\begin{aligned} &\int (U_0 + U_{0s}) \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2 \right] dz \\ &\quad + \int \frac{\left[\frac{(U_0')^2}{4} - N^2 \right]}{|U_0 - c|^2} (U_{0\min} - U_{0\max} - c_r - U_{0s}) b|G|^2 dz \leq 0. \end{aligned}$$

Since

$$\frac{1}{|U_0 - c|^2} \leq \frac{1}{c_i^2},$$

and using Rayleigh-Ritz inequality, we have

$$\begin{aligned} &|U_{0\min} + U_{0s}| \left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2 \right] \int b|G|^2 dz \\ \leq & \frac{\left| \frac{(U_0')^2}{4} - N^2 \right|_{\max}}{c_i^2} (c_r + U_{0s} - U_{0\min} + U_{0\max}) \int b|G|^2 dz. \end{aligned}$$

Let U_{0s} be an arbitrary real number given by $U_{0s} = \frac{U_{0\min} + U_{0\max}}{2}$ then we have

$$c_i^2 \leq \frac{\left| \frac{(U_0')^2}{4} - N^2 \right|_{\max} \left[c_r + \frac{3U_{0\max} - U_{0\min}}{2} \right]}{\left| \frac{3U_{0\min} + U_{0\max}}{2} \right| \left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2 \right]}.$$

Then we have

$$c_i^2 \leq \lambda \left[c_r + \frac{3U_{0\max} - U_{0\min}}{2} \right], \tag{4.9}$$

where

$$\lambda = \frac{\left| \frac{(U_0')^2}{4} - N^2 \right|_{\max}}{\left| \frac{3U_{0\min} + U_{0\max}}{2} \right| \left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2 \right]}.$$

Thus, we complete the proof. □

Theorem 4.4. *If $\lambda<\lambda_c$, where*

$$\lambda_c=\left[1+\sqrt{1-4J_0}\right]\left[2U_{0\max}-\frac{\sqrt{15U_{0\max}^2-U_{0\min}^2+2U_{0\max}U_{0\min}}}{2}\right],$$

then the parabola

$$c_i^2\le\lambda\left[c_r+\frac{3U_{0\max}-U_{0\min}}{2}\right]$$

intersects the semi-ellipse

$$\left[c_r-\left(\frac{U_{0\max}+U_{0\min}}{2}\right)\right]^2+\frac{2c_i^2}{1+\sqrt{1-4J_0}}\le\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2.$$

Proof. The semiellipse given in [3] is

$$\left[c_r-\left(\frac{U_{0\max}+U_{0\min}}{2}\right)\right]^2+\frac{2c_i^2}{1+\sqrt{1-4J_0}}\le\left(\frac{U_{0\max}-U_{0\min}}{2}\right)^2, \tag{4.10}$$

where

$$J_0=\left[\frac{N^2}{(U_0')^2}\right]_{\min}.$$

It is easily seen that the parabola given in (4.9) touches the semiellipse given in (4.10), if

$$\lambda=(1+\sqrt{1-4J_0})\left[2U_{o\max}\pm\frac{\sqrt{15U_{0\max}^2-U_{0\min}^2+2U_{0\max}U_{0\min}}}{2}\right]. \tag{4.11}$$

The value given in (4.11) with positive sign is invalid as it leads to $c_r<U_{0\min}$, which violates $U_{0\min}<c_r<U_{0\max}$.

Therefore, if $\lambda<\lambda_c$, where

$$\lambda_c=\left[1+\sqrt{1-4J_0}\right]\left[2U_{0\max}-\frac{\sqrt{15U_{0\max}^2-U_{0\min}^2+2U_{0\max}U_{0\min}}}{2}\right],$$

then the parabola given in (4.9) intercepts the semiellipse (4.10). □

Remark 4.1. (1) If U_0 is a constant or $U_{0\min}=0$ then our result is valid.

(2) If U_0 changes its sign then

$$\left|\frac{3U_{0\min}+U_{0\max}}{2}\right|>0$$

and hence the instability region is valid.

5 Examples

Now the below examples will illustrate the applicability of the reduction of the semielliptical instability region by the parabolic instability region.

(i) Let $U_0(z)=(z-\frac{1}{2})^2$, $0\le z\le 1$, $b(z)=e^{T_0 z}$, $T=T_0$ (a constant), $N=N_0$ is a positive constant.

In this case,

$$U_{0\min}=0, \quad U_{0\max}=\frac{1}{4}, \quad b_{\min}=1, \quad b_{\max}=e^{T_0}, \quad U'_{0\max}=1, \quad \lambda=\frac{2(\frac{1}{4}-J_0)}{e^{-T_0}\pi^2+k^2}$$

and $\lambda_c=(1+\sqrt{1-4J_0})$ (0.015877081) is the critical value of λ for which the parabola touches the semiellipse and consequently the parabola intersects the semiellipse if $\lambda<\lambda_c$.

We can see after some computation that $\lambda<\lambda_c$ if

$$\begin{aligned}
&0.224<J_0<0.25, &&\text{when } T_0=0,\ k=0,\\
&0.241<J_0<0.25, &&\text{when } T_0=1,\ k=0,\\
&0.247<J_0<0.25, &&\text{when } T_0=2,\ k=0,\\
&0.221<J_0<0.25, &&\text{when } T_0=0,\ k=1,\\
&0.239<J_0<0.25, &&\text{when } T_0=1,\ k=1,\\
&0.245<J_0<0.25, &&\text{when } T_0=2,\ k=1,\\
&0.212<J_0<0.25, &&\text{when } T_0=0,\ k=2,\\
&0.231<J_0<0.25, &&\text{when } T_0=1,\ k=2,\\
&0.237<J_0<0.25, &&\text{when } T_0=2,\ k=2.
\end{aligned}$$

It is seen that the range of J_0 for which the intersection of the parabola and semiellipse takes place is reduced when T_0 is increased.

(ii) Let $U_0(z)=1+(z-\frac{1}{2})^2$, $0\le z\le 1$, $b(z)=e^{T_0 z}$, $T=T_0$ (a constant), $N=N_0$ is a positive constant.

In this case,

$$U_{0\min}=1, \quad U_{0\max}=\frac{5}{4}, \quad b_{\min}=1, \quad b_{\max}=e^{T_0}, \quad U'_{0\max}=1, \quad \lambda=\frac{2(1-4J_0)}{17(e^{-T_0}\pi^2+k^2)}.$$

Here $\lambda_c=(1+\sqrt{1-4J_0})$ (0.003126955) is the critical value of λ for which the parabola touches the semiellipse and consequently the parabola intersects the semiellipse if $\lambda<\lambda_c$. We can see after computation that

$$\begin{aligned}
&0.141<J_0<0.25, &&\text{when } T_0=0,\ k=0,\\
&0.217<J_0<0.25, &&\text{when } T_0=1,\ k=0,\\
&0.239<J_0<0.25, &&\text{when } T_0=2,\ k=0,
\end{aligned}$$

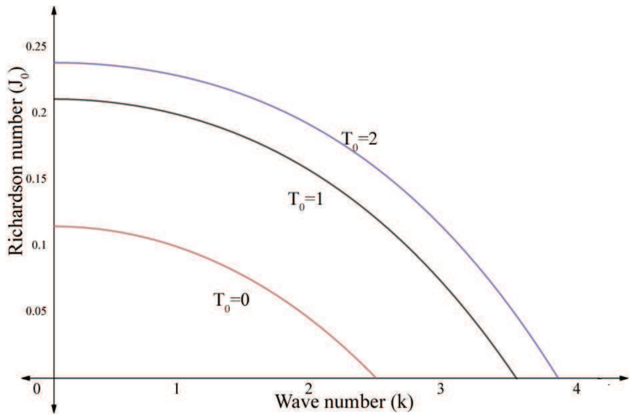

Figure 2: Wavenumber vs Richardson number.

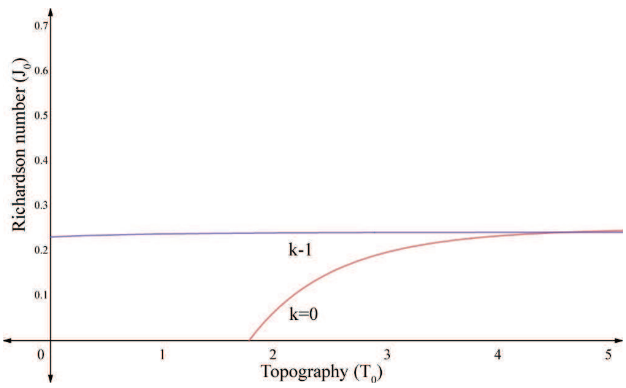

Figure 3: Topography vs Richardson number.

$$
\begin{aligned}
&0.127 < J_0 < 0.25, && \text{when } T_0 = 0,\ k = 1,\\
&0.206 < J_0 < 0.25, && \text{when } T_0 = 1,\ k = 1,\\
&0.230 < J_0 < 0.25, && \text{when } T_0 = 2,\ k = 1,\\
&0.082 < J_0 < 0.25, && \text{when } T_0 = 0,\ k = 2,\\
&0.171 < J_0 < 0.25, && \text{when } T_0 = 1,\ k = 2,\\
&0.198 < J_0 < 0.25, && \text{when } T_0 = 2,\ k = 2.
\end{aligned}
$$

It is seen that the range of J_0 for which the intersection of the parabola and semiellipse takes place is reduced when T_0 is increased.

(iii) Let $U_0 = (z - \frac{1}{2})$, $0 \le z \le 1$, $b(z) = e^{T_0 z}$, $T = T_0$ (a constant), $N = N_0$ is a positive constant.

In this case,

$$
U_{0\min} = -\frac{1}{2}, \quad U_{0\max} = \frac{1}{2}, \quad b_{\min} = 1, \quad b_{\max} = e^{T_0}, \quad U'_{0\max} = 1, \quad \lambda = \frac{(1-4J_0)}{2(e^{-T_0}\pi^2 + k^2)}.
$$

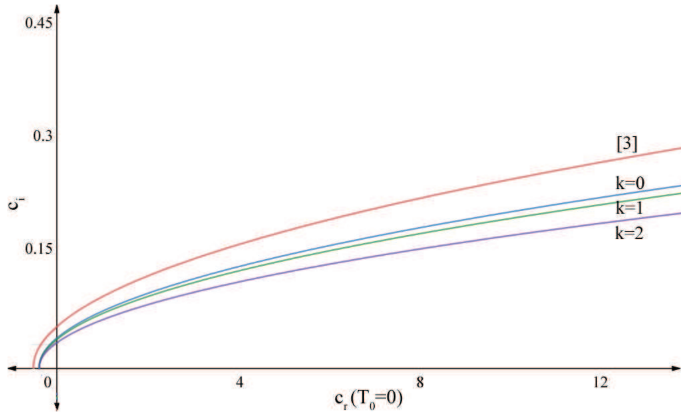

Figure 4: c_r vs c_i.

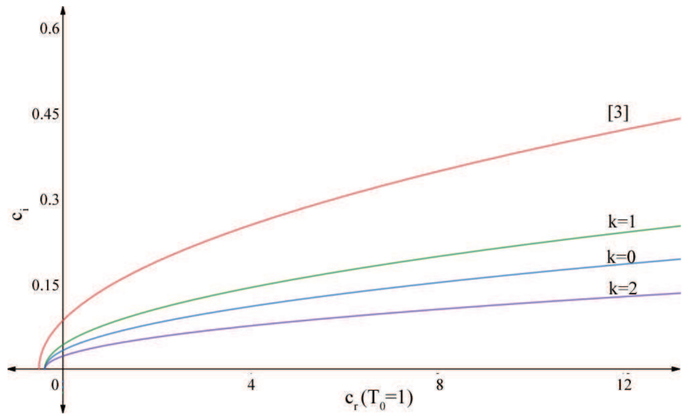

Figure 5: c_r vs c_i.

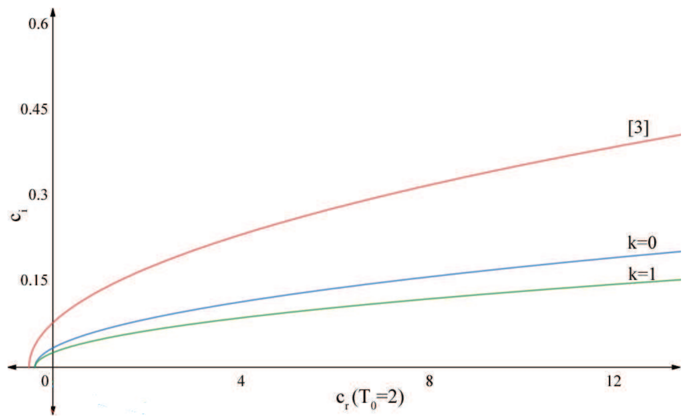

Figure 6: c_r vs c_i.

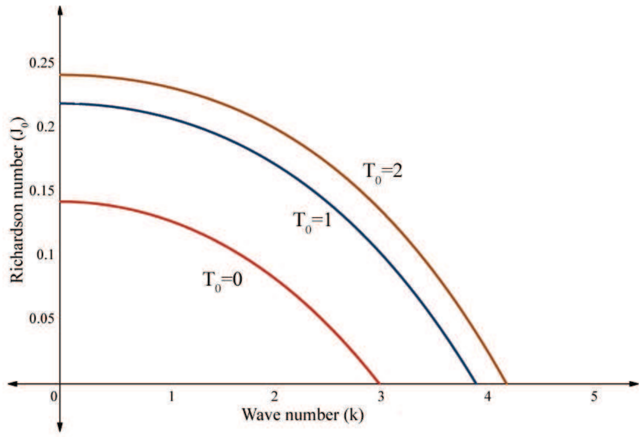

Figure 7: Wavenumber vs Richardson number.

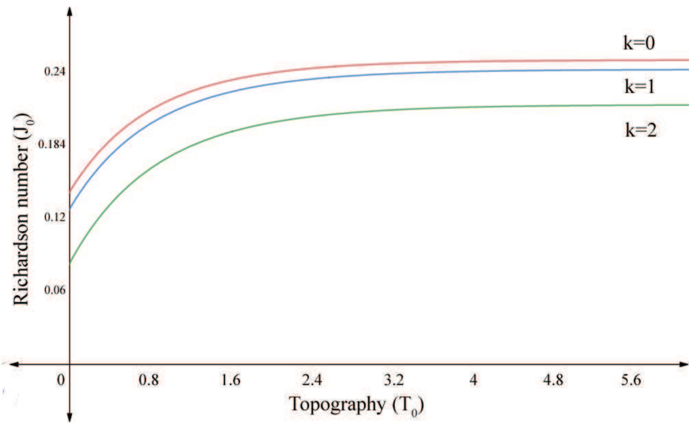

Figure 8: Topography vs Richardson number.

Here $\lambda_c=(1+\sqrt{1-4J_0})$ (0.133974596) is the critical value of λ for which the parabola touches the semiellipse and consequently the parabola intersects the semiellipse if $\lambda<\lambda_c$. We can see after computation that $0<J_0<0.25$.

(iv) Let $U_0=1+(z-\frac{1}{2})$, $0\leq z\leq 1$, $b(z)=e^{T_0 z}$, $T=T_0$ (a constant), $N=N_0$ is a positive constant.

In this case,

$$U_{0\min}=1,\quad U_{0\max}=\frac{3}{2},\quad b_{\min}=1,\quad b_{\max}=e^{T_0},\quad U'_{0\max}=1,\quad \lambda=\frac{(1-4J_0)}{9(e^{-T_0}\pi^2+k^2)}.$$

Here $\lambda_c=(1+\sqrt{1-4J_0})$ (0.010434814) is the critical value of λ for which the parabola touches the semiellipse and consequently the parabola intersects the semiellipse if $\lambda<\lambda_c$.

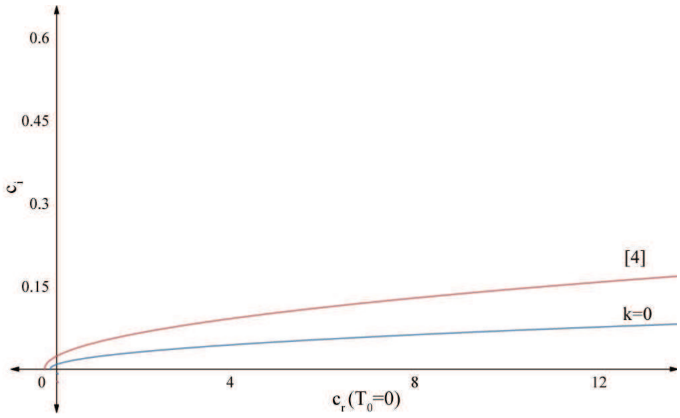

Figure 9: c_r vs c_i.

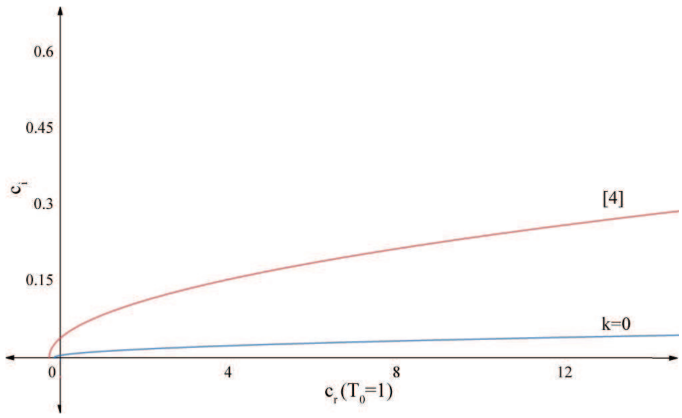

Figure 10: c_r vs c_i.

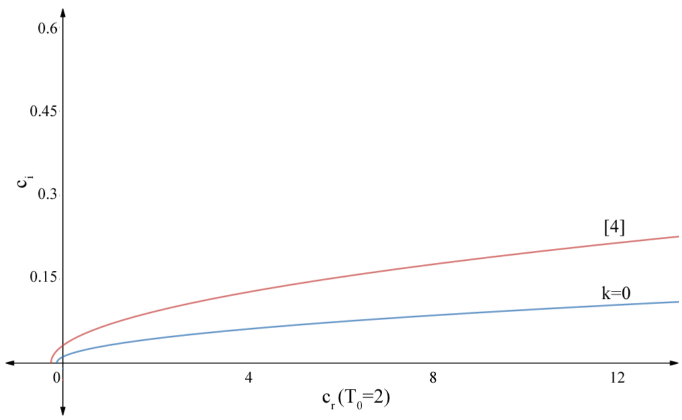

Figure 11: c_r vs c_i.

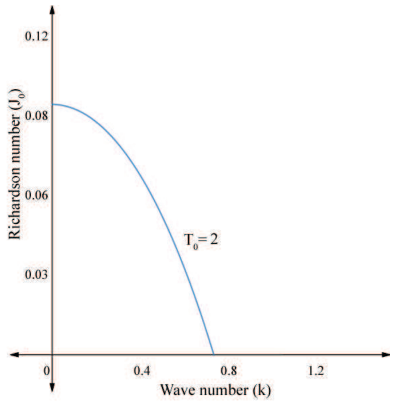

Figure 12: Wavenumber vs Richardson number.

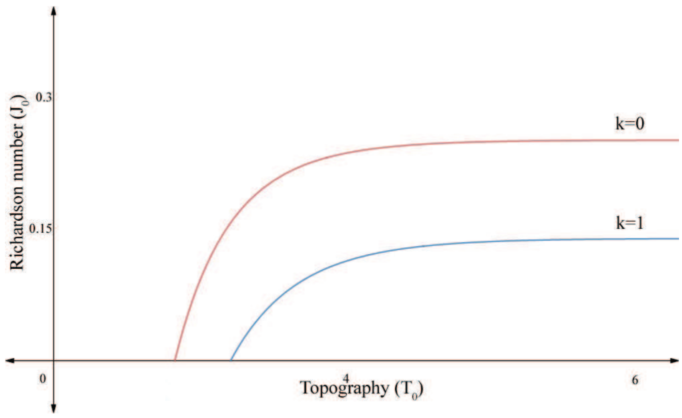

Figure 13: Topography vs Richardson number.

We can see after computation that

$$
\begin{aligned}
&0 < J_0 < 0.25, && \text{when } T_0 = 0, \ k = 0,\\
&0.098 < J_0 < 0.25, && \text{when } T_0 = 1, \ k = 0,\\
&0.205 < J_0 < 0.25, && \text{when } T_0 = 2, \ k = 0,\\
&0 < J_0 < 0.25, && \text{when } T_0 = 0, \ k = 1,\\
&0.042 < J_0 < 0.25, && \text{when } T_0 = 1, \ k = 1,\\
&0.163 < J_0 < 0.25, && \text{when } T_0 = 2, \ k = 1,\\
&0 < J_0 < 0.25, && \text{when } T_0 = 0, \ k = 2,\\
&0 < J_0 < 0.25, && \text{when } T_0 = 1, \ k = 2,\\
&0 < J_0 < 0.25, && \text{when } T_0 = 2, \ k = 2.
\end{aligned}
$$

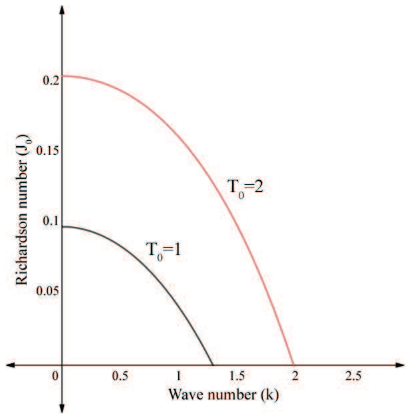

Figure 14: Wavenumber vs Richardson number.

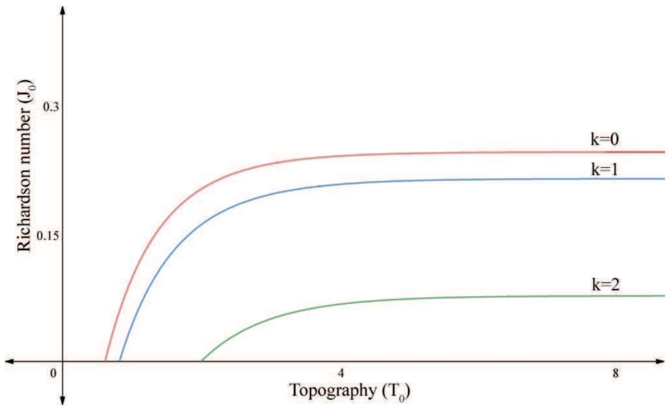

Figure 15: Topography vs Richardson number.

6 Concluding remarks

For the extended Taylor-Goldstein problem of hydrodynamic stability dealing with inviscid, incompressible shear flows with variable density and variable cross section, we derive a new estimate for the growth rate of unstable mode. This estimate is sharper than the already existing bound for the same problem. Furthermore, we have obtained parabolic instability region which intersects and reduce the semielliptical instability region under some conditions and this is illustrated in examples of basic flows satisfying these intersection conditions.

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