The Distributional Henstock-Kurzweil Integral and Applications: a Survey

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Abstract. This study presents a summary of the current state of research on the distributional Henstock-Kurzweil integral. Basic properties such as integration by parts, Hake theorem, inner product, Hölder inequality, second mean value theorem, orderings, Banach lattice, convergence theorems, fixed point theorems, are shown. This study also summarizes its applications in integral and differential equations.

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Key words: Distributional Henstock-Kurzweil integral, Banach lattice, convergence theorem, cone, fixed point theorem, integral and differential equations.

1 Introduction

In the integration theory, there is a simple way to define integrals, defined by their primitives. Take the Lebesgue integral as an example, if a function f is the derivative of an absolute continuous (AC) function F, then f is Lebesgue integrable. In symbol, if $F \in AC$ and $F' = f$ a.e., then $f \in L$. The Henstock-Kurzweil (HK) integral has a similar definition, that is, if $F \in ACG^*$ and $F' = f$ a.e., then $f \in HK$ ([1–7]). However, if the primitive F is a continuous function or a regulated function, i.e., $F \in C$ or $F \in G$ (here C and G denote the spaces of the continuous functions and regulated functions, respectively), then the Schwartz distribution (or generalized function) and distributional derivative are needed here ([9, 10, 13, 15, 20]), because there are plenty of continuous functions that are differentiable nowhere ([32]). For simplicity, relationships between primitives and integrands for some major integrals are shown in Figure 1.

This survey is an outline of some results on the distributional Henstock-Kurzweil integral (for short, D_{HK}). Namely, a distribution f is distributionally Henstock-Kurzweil integrable on an interval $[a,b]$ if there exists a continuous function F such that the distributional derivative of F is f and denote by $\int_a^b f = F(b) - F(a)$, and F is called the

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$$\textit{Primitive } F: \quad C^1 \quad \subsetneq \quad AC \quad \subsetneq \quad ACG^* \quad \subsetneq \quad C \quad \subsetneq \quad G$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\textit{Integrand } f: \quad C \quad \subsetneq \quad L \quad \subsetneq \quad HK \quad \subsetneq \quad D_{HK} \quad \subsetneq \quad G_{HK}$$
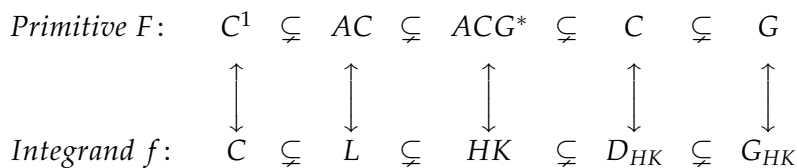
Figure 1: Relationships between primitives and integrands for some major integrals.

primitive of f. This integral seems to have been first introduced by Mikusiński and Ostaszewski [11]. Then, it was developed in detail in the plane by Ang, Schmitt and Vy [12] and on $[-\infty, +\infty]$ by Talvila [13].

According to its definition, this integral comprises Riemann, Lebesgue, Henstock-Kurzweil, Perron, Denjoy, and improper integrals as special cases ([1–7, 12, 13, 20]). Denote the spaces of the Henstock-Kurzweil integrable functions and the Henstock-Kurzweil integrable distributions by HK and D_{HK}, respectively. The space D_{HK} is a separable Banach space with the Alexiewicz norm and it is isometrically isomorphic to the space of continuous functions on a closed interval with uniform norm. The spaces L and HK are dense in the space D_{HK}. Some other basic properties such as integration by parts, Hake theorem, inner product, Hölder inequality, second mean value theorem are also introduced in Section 2. Moreover, orderings and Banach lattice are shown in Section 3. Section 4 is devoted to the weak and strong convergence theorems and quasi-convergence. In the last section, we show that D_{HK} is an ordered Banach space with a regular cone under certain ordering and then give a fixed point theorem in D_{HK}. Applications in functional Urysohn integral equation, Darboux problem and Measure differential equation are presented.

2 Basic definitions and preliminaries

For convenience, we use the same notations as in [20] and list some basic facts as follows.

Let (a,b) be an open interval in $\mathbb{R}$, we define

$$\mathcal{D}((a,b)) = \left\{ \phi : (a,b) \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has a compact support in } (a,b) \right\}.$$

The distributions on (a,b) are defined to be the continuous linear functionals on $\mathcal{D}((a,b))$. The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

For all $f \in \mathcal{D}'((a,b))$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f', \phi \rangle = -\langle f, \phi' \rangle$, where $\phi \in \mathcal{D}((a,b))$ is a test function. Further, we write distributional derivative as f' and its pointwise derivative as $f'(t)$ where $t \in \mathbb{R}$. From now on, all derivative in this paper will be distributional derivatives unless stated otherwise.

Denote the space of continuous functions on $[a,b]$ by $C([a,b])$. Let

$$C_0 = \left\{ F \in C([a,b]) : F(a) = 0 \right\}. \tag{2.1}$$

Then C_0 is an Banach space under the norm

$$\|F\|_\infty = \sup_{t\in[a,b]} |F(t)| = \max_{t\in[a,b]} |F(t)|.$$

Definition 2.1. ([20, Definition 1]). *A distribution $f \in \mathcal{D}'((a,b))$ is said to be Henstock–Kurzweil integrable (shortly D_{HK}) on an interval $[a,b]$ if there exists a continuous function $F \in C_0$ such that $F' = f$, i.e., the distributional derivative of F is f. The distributional Henstock–Kurzweil integral of f on $[a,b]$ is denoted by $\int_a^b f(t)dt = F(b) - F(a)$. The function F is called the primitive of f. For short, $\int_a^b f = F(b) - F(a)$.*
For every $f \in D_{HK}$, $\phi \in \mathcal{D}((a,b))$, we write

$$\langle f, \phi \rangle = \int_a^b f(t)\phi(t)dt = -\int_a^b F(t)\phi'(t)dt.$$

Remark 2.1. If take $a = -\infty$ and $b = +\infty$, we obtain distributional Henstock-Kurzweil integral on $\overline{\mathbb{R}} = [-\infty, +\infty]$, see [13]. For the multidimensional case, we refer the interested reader to [12].

Notice that if $f \in D_{HK}$ then f has many primitives in $C([a,b])$, all differing by a constant, but f has exactly one primitive in C_0. For simplicity of notation, in what follows we use the letters $F, G, \dots$ for the primitives of $f, g, \dots$ in D_{HK}. Unless otherwise stated, "$\int$" denotes the D_{HK}-integral throughout this paper.

For $F \in C_0$, let the positive part $F^+ = \max_{t\in[a,b]}\{F(t), 0\}$ and the negative part $F^- = \max_{t\in[a,b]}\{-F(t), 0\}$, and hence $F = F^+ - F^-$ and the absolute value $|F| = F^+ + F^-$. Moreover, F^+, F^-, $|F|$ all belong to C_0. Let $f \in D_{HK}$ with the primitive $F \in C_0$, as in [13], define

$$f^+ = (F^+)', \quad f^- = (F^-)', \quad |f| = |F|'. \tag{2.2}$$

Then,

$$f = f^+ - f^-, \quad |f| = f^+ + f^-. \tag{2.3}$$

Let $f, g \in D_{HK}$ with the primitive $F, G \in C_0$. We say that $f = g$ if $F(t) = G(t)$ everywhere.

Lemma 2.1. ([13, Theorem 4]). **(Fundamental theorem of calculus)**

(a) *Let $f \in D_{HK}$ and $F(x) = \int_a^x f$. Then $F \in C_0$ and $F' = f$.*

(b) *Let $F \in C([a,b])$. Then $\int_a^x F' = F(x) - F(a)$ for all $x \in [a,b]$.*

Remark 2.2. The distributional Henstock-Kurzweil integral is very wide and it includes the Riemann integral, the Lebesgue integral, the Henstock-Kurzweil integral, the restricted and wide Denjoy integral (see [1,13,20]). For example, let

$$F(x) = \sum_{n=1}^\infty \frac{\sin n^2 x}{n^2}, \quad x \in [0,1].$$

Then F is continuous on $[0,1]$ and $F(0)=0$. Moreover, the function F, apart from certain exceptional points, indeed is not pointwise differentiable on $[0,1]$.

Let $f=F'$. Then $f\in D_{HK}$ and

$$\int_0^1 f = F(1) = \sum_{n=1}^{\infty} \frac{\sin n^2}{n^2}.$$

However, f is neither Henstock-Kurzweil integrable nor Lebesgue integrable on $[0,1]$, because the primitive of Henstock-Kurzweil integrable function is ACG^* and differentiable almost everywhere ([1,7]).

Lemma 2.2. ([13, Theorem 25]). **(Hake theorem)** *Suppose $f\in \mathcal{D}'((a,b))$ and $f=F'$ for some $F\in C((a,b))$. If $\lim_{x\to a}F(x)$ and $\lim_{x\to b}F(x)$ exist in $\mathbb{R}$, then $f\in D_{HK}$ and*

$$\int_a^b f = \lim_{x\to a}\int_x^0 f + \lim_{x\to b}\int_0^x f.$$

For $f\in D_{HK}$, define the Alexiewicz norm in D_{HK} as

$$\|f\| = \|F\|_{\infty} = \sup_{t\in[a,b]}|F(t)| = \max_{t\in[a,b]}|F(t)|. \tag{2.4}$$

Under the Alexiewicz norm, D_{HK} is a Banach space, see [13, Theorem 2].

Indeed, D_{HK} is a separable space ([13, Theorem 3]). The space of all Lebesgue integrable functions and the spaces of restricted Denjoy and wide Denjoy integrable functions are dense in D_{HK}. Since the HK-integral and the restricted Denjoy integral are equivalent, the space HK is also dense in D_{HK}.

The norm $\|\cdot\|$ in (2.4) does not induce an inner product in D_{HK}, because $\|\cdot\|$ do not satisfy the parallelogram law.

Let

$$\langle f,g\rangle = \langle F,G\rangle = \int_a^b F(t)G(t)dt. \tag{2.5}$$

(2.5) defines an inner product in D_{HK}, and so the space D_{HK} becomes an inner product space. Moreover, the inner product (2.5) induces a norm

$$\|f\|_{\langle,\rangle} = \left(\int_a^b F^2(t)dt\right)^{\frac{1}{2}}. \tag{2.6}$$

It is easy to obtain

$$\|f\|_{\langle,\rangle} \le (b-a)^{\frac{1}{2}}\|f\|.$$

This means that the norm $\|\cdot\|$ is stronger than $\|\cdot\|_{\langle,\rangle}$. However, the two norms $\|\cdot\|_{\langle,\rangle}$ and $\|\cdot\|$ in D_{HK} are not equivalent, because D_{HK} is complete under the norm $\|\cdot\|$ but not under the norm $\|\cdot\|_{\langle,\rangle}$. Therefore, D_{HK} is not a Hilbert space under the norm $\|\cdot\|_{\langle,\rangle}$.

Let $g : [a,b] \to \mathbb{R}$, its variation is $V(g) = \sup \sum_n |g(s_n) - g(t_n)|$ where the supremum is taken over every sequence $\{(t_n, s_n)\}$ of disjoint intervals in $[a,b]$. A function g is of bounded variation on $[a,b]$ if $V(g)$ is finite. Denote the space of functions of bounded variation by $\mathcal{BV}$ (see [16]). The space $\mathcal{BV}$ is a Banach space with norm $\|g\|_{\mathcal{BV}} = |g(a)| + V(g)$.

The dual of D_{HK} is $\mathcal{BV}$ (see cf. [13]) and we have

Lemma 2.3. ([10, Lemma 1.5]). *Let $F \in C[a,b]$ and $g \in \mathcal{BV}$. Then*

$$F'g = \left(\int_a^t g \, dF \right)', \tag{2.7}$$

and

$$Fg' = \left(\int_a^t F \, dg \right)'. \tag{2.8}$$

Lemma 2.4. ([20, Lemma 2]). **(Integration by parts)** *Let $f \in D_{HK}$, and $g \in \mathcal{BV}$. Then $fg \in D_{HK}$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F \, dg.$$

By Lemmas 2.3 and 2.4, it is easy to see that

Lemma 2.5. *Let $f \in D_{HK}$ be the distributional derivative of $F \in C[a,b]$, and $g \in \mathcal{BV}$. Then*

$$(Fg)' = fg + Fg'.$$

Moreover, we have the following results.

Lemma 2.6. ([13, Theorem 7]). **(Hölder inequality)** *Let $f \in D_{HK}$. If $g \in \mathcal{BV}$, then*

$$\left| \int_a^b fg \right| \le 2\|f\|\|g\|_{\mathcal{BV}}. \tag{2.9}$$

Lemma 2.7. ([13, Theorem 26]). **(Second mean value theorem)** *Let $f \in D_{HK}$ and let $g : [a,b] \to \mathbb{R}$ be monotonic. Then for some $\xi \in [a,b]$,*

$$\int_a^b fg = g(a) \int_a^\xi f + g(b) \int_\xi^b f.$$

3 The orderings in D_{HK} and Banach lattice

In C_0 there exists a pointwise order: for $F,G \in C_0$, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in [a,b]$. For $f,g \in D_{HK}$ with primitives $F,G \in C_0$, respectively, we say

$$f \overset{(p)}{\preceq} g \ (\text{or } g \overset{(p)}{\succeq} f) \quad \text{if and only if} \quad F(t) \le G(t), \ \forall t \in [a,b], \tag{3.1}$$

and

$$f \overset{(m)}{\preceq} g \ (\text{or } g \overset{(m)}{\succeq} f) \quad \text{if and only if} \quad \int_I f \le \int_I g, \tag{3.2}$$

where I is arbitrary subinterval of $[a,b]$. Obviously,

$$f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g, \tag{3.3}$$

but the converse is not true. Particularly, if f,g are integrable functions, then

$$f(t) \le g(t) \ (\forall t \in [a,b]) \quad \Leftrightarrow \quad f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g. \tag{3.4}$$

It is easy to see that the orderings defined in (3.1) and (3.2) are quite different. By (3.2), $f \overset{(m)}{\preceq} g$ in D_{HK} implies $F \le G$ in C_0. But the converse is not true. For example, let $F(x) = \sin x$ on $[0,\pi]$, so $f(x) = \cos x$, $x \in [0,\pi]$. Obviously, $F(x) \ge 0$ on $[0,\pi]$, which implies $f \overset{(p)}{\succeq} 0$ under the ordering (3.1). However, for two intervals $J_1 = [\pi/6, \pi/2]$ and $J_2 = [\pi/2, 5\pi/6]$, one has

$$\int_{J_1} f = 1 - 1/2 = 1/2 > 0, \quad \int_{J_2} f = 1/2 - 1 = -1/2 < 0.$$

Therefore, f can not be compared with the zero function under the ordering (3.2).

We now present some basic properties of order Banach spaces.

A closed subset X_+ of a Banach space X is called an *order cone* if $X_+ + X_+ \subseteq X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subseteq X_+$ for each $c \ge 0$. It is easy to see that the order relation $\preceq$, defined by

$$x \preceq y \ \text{ if and only if } \ y - x \in X_+,$$

is a partial ordering in X, and that $X_+ = \{y \in X \mid 0 \preceq y\}$ is an order cone in X. The space X, equipped with this partial ordering, is called an *ordered Banach space*. For any $r > 0$, $B_r = \{x \in X : \|x\| \le r\}$ is called a *closed ball* in X. The order interval $[y,z] = \{x \in X \mid y \preceq x \preceq z\}$ is a closed subset of X for all $y,z \in X$. A sequence (subset) of X is called *order bounded* if it is contained in an order interval $[y,z]$ of X. We say that an order cone X_+ of a Banach space is *normal* if there exists such a constant $\gamma \ge 1$ that

$$0 \preceq x \preceq y \ \text{ in } X \text{ implies } \ \|x\| \le \gamma \|y\|. \tag{3.5}$$

X_+ is called *regular* if all increasing and order bounded sequences of X_+ converge. If all norm-bounded and increasing sequences of X_+ converge, we say that X_+ is *fully regular*.

Denote by $D_{HK}^{(p)}$ and $D_{HK}^{(m)}$ the Banach space D_{HK} with orderings (3.1) and (3.2), respectively. We define

$$\begin{aligned} D_{HK+}^{(p)} &= \{f \in D_{HK} : f \overset{(p)}{\succeq} 0\}, \\ D_{HK+}^{(m)} &= \{f \in D_{HK} : f \overset{(m)}{\succeq} 0\}. \end{aligned} \tag{3.6}$$

Then $D_{HK+}^{(p)}$ and $D_{HK+}^{(m)}$ are order cones.

For $f, g \in D_{HK}$, denote $F(x) = \int_a^x f$ and $G(x) = \int_a^x g$. Since

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} g \Rightarrow 0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} g \Rightarrow 0 \leq F(x) \leq G(x), \quad \forall x \in [a,b]$$
$$\Rightarrow \|F\|_\infty \leq \|G\|_\infty \Rightarrow \|f\| \leq \|g\|.$$

It shows that

Theorem 3.1. $D_{HK+}^{(p)}$ and $D_{HK+}^{(m)}$ are normal cones in D_{HK}.

Remark 3.1. In fact, the cone $D_{HK+}^{(m)}$ rather than $D_{HK+}^{(p)}$ is regular, see details in Theorem 5.1 and Remark 5.1.

Assume that X is an order linear space. If for every $x, y \in X$, there exists $z \in X$ such that $x \preceq z$, $y \preceq z$, and if $x \preceq u$, $y \preceq u$ then $z \preceq u$, then X is called a *Riesz space* (or *lattice*) and denote $z = x \vee y$.

A vector subspace M of a Riesz space X is said to be a *Riesz subspace* (or a *vector sublattice*) whenever M is closed under the lattice operations of X, i.e., whenever for each pair $x, y \in M$ the vector $x \vee y$ (taken in X) belongs to M.

For a vector x in a lattice X, define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$, then we call them *the positive part, the negative part and the absolute value* (or *modulus*) respectively. Moreover, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Note that $|x| = 0$ if and only if $x = 0$.

Assume that X is a Banach space, if X is a lattice and

$$|x| \preceq |y| \text{ in } X \text{ implies } \|x\| \leq \|y\|, \tag{3.7}$$

then X is called a *Banach lattice* and the norm $\|\cdot\|$ satisfying (3.7) is called a *lattice norm*.

Theorem 3.2. ([13, Theorem 23]). $D_{HK}^{(p)}$ is a Banach lattice that is isomorphic to $C([a,b])$.

Remark 3.2. For $f, g \in D_{HK}^{(m)}$, define $f \vee g = (F \vee G)'$, where F, $G \in C_0$ are the primitives of f, g, respectively, and $F \vee G = \sup(F(x), G(x)) = \max(F(x), G(x))$. It is easy to see that $f \vee g \in D_{HK}^{(m)}$. But we do not have $f \overset{(m)}{\preceq} f \vee g$ and $g \overset{(m)}{\preceq} f \vee g$. Hence, $D_{HK}^{(m)}$ is not a lattice and so is not a Banach lattice.

In [21], we show that $D_{HK}^{(p)}$ is an AM-space, which possesses not only the *Archimedean property* but also the *Dunford-Pettis property*. Moreover, the norm on $D_{HK}^{(p)}$ is σ-order continuous but $D_{HK}^{(p)}$ is not Dedekind complete.

4 Convergence theorems

We firstly show some definitions of the convergences in D_{HK}, see also in [13, Section 7].

Definition 4.1. *(a) A sequence $\{f_n\} \subset D_{HK}$ is said to converge in norm to $f \in D_{HK}$ if $\|f_n - f\| \to 0$ as $n \to \infty$.*

(b) $\{f_n\}$ converges weakly in $\mathcal{D}((a,b))$ if $\langle f_n - f, \phi \rangle = \int_a^b (f_n - f)\phi \to 0$ $(n \to \infty)$ for each $\phi \in \mathcal{D}((a,b))$.

(c) $\{f_n\}$ converges weakly in $\mathcal{BV}$ if $\langle f_n - f, g \rangle = \int_a^b (f_n - f)g \to 0$ $(n \to \infty)$ for each $g \in \mathcal{BV}$.

The following lemma is easy to obtain from their definitions.

Lemma 4.1. ([13, Theorem 12]). *(a) Convergence in norm implies weak convergence in $\mathcal{D}((a,b))$ and $\mathcal{BV}$.*

(b) Weak convergence in $\mathcal{BV}$ implies weak convergence in $\mathcal{D}((a,b))$.

(c) Weak convergence in $\mathcal{D}((a,b))$ does not imply weak convergence in $\mathcal{BV}$.

(d) Weak convergence in $\mathcal{BV}$ does not imply convergence in norm.

Here is a sufficient and necessary condition for the convergence of D_{HK} integrals.

Theorem 4.1. ([20, Theorem 1]). *Let $\{f_n\} \subset D_{HK}$ and $f \in D_{HK}$ with the primitives F_n and F, respectively. Then $\{F_n\}$ is uniformly bounded and $F_n(x) \to F(x)$ pointwise on $[a,b]$ as $n \to \infty$ if and only if $\int_a^b fg = \lim_{n \to \infty} \int_a^b f_n g$ for every $g \in \mathcal{BV}$.*

Let $O(\mathcal{BV})$ be the unit ball in $\mathcal{BV}$, i.e.,

$$O(\mathcal{BV}) = \{g \in \mathcal{BV} : \|g\|_{\mathcal{BV}} \leq 1\}. \tag{4.1}$$

One has

Theorem 4.2. ([20, Theorem 2]). *Let $\{f_n\} \subset D_{HK}$. If $f_n \to f \in \mathcal{D}'((a,b))$ weakly in $\mathcal{BV}$ uniformly on $O(\mathcal{BV})$ as $n \to \infty$, then $f \in D_{HK}$ and $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$.*

The convergence of the product sequence $\{fg_n\}$ for $f \in D_{HK}$ and $g_n \in \mathcal{BV}$, $n = 1, 2, \ldots$, is as follows.

Theorem 4.3. ([20, Theorem 3]). *Let $f \in D_{HK}$. If $g_n, g \in \mathcal{BV}$ and $g_n \to g$ in $\mathcal{BV}$ as $n \to \infty$. Then $fg \in D_{HK}$ and*

$$\int_a^b fg_n \to \int_a^b fg \quad (n \to \infty).$$

Remark 4.1. This theorem is a reduction of [13, Theorem 19]. The uniform boundedness condition for $\{g_n\}$ in [13, Theorem 19] is omitted here.

For more general convergence results, one has

Proposition 4.1. ([20, Proposition 1]). *Let $f \in D_{HK}$, $\{g_n\} \subset \mathcal{BV}$. If $\{g_n\}$ is bounded in $\mathcal{BV}$ and $\{\int_a^b f g_n\}$ converges then there exists $g \in \mathcal{BV}$ such that $fg \in D_{HK}$ and*

$$\int_a^b f g_n \to \int_a^b f g \quad (n \to \infty).$$

Proposition 4.2. ([20, Proposition 2]). *Let $\{g_n\} \subset \mathcal{BV}$. If $\int_a^b f g_n$ converges for every $f \in D_{HK}$ then there exists a function $g \in \mathcal{BV}$ such that for every $f \in D_{HK}$, $fg \in D_{HK}$ and*

$$\int_a^b f g_n \to \int_a^b f g \quad (n \to \infty).$$

On the other hand, the monotone convergence theorem and the dominated convergence theorem in D_{HK} can be modified from [12, Corollaries 4, 5], which are important tools to deal with differential and integral equations.

Lemma 4.2. ([12, Corollary 4]). **(Monotone convergence theorem)** *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in D_{HK} such that $f_0 \overset{(m)}{\preceq} f_1 \overset{(m)}{\preceq} \cdots \overset{(m)}{\preceq} f_n \overset{(m)}{\preceq} \cdots$, and that $A = \lim_{n \to \infty} \int_a^b f_n$. Then $f_n \to f$ in D_{HK} and $\int_a^b f = A$.*

Lemma 4.3. ([12, Corollary 5]). **(Dominated convergence theorem)** *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in D_{HK} such that $f_n \to f$ in $\mathcal{D}'$. Suppose there exist f_-, $f_+ \in D_{HK}$ satisfying $f_- \overset{(m)}{\preceq} f_n \overset{(m)}{\preceq} f_+$, $\forall n \in \mathbb{N}$. Then $f \in D_{HK}$ and $\lim_{n \to \infty} \int_a^b f_n = \int_a^b f$.*

Further, we can obtain a variant monotone convergence theorem.

Theorem 4.4. ([20, Theorem 4]). *Let $\{f_n\}$ be a sequence in D_{HK}, $h \in D_{HK}$ such that*

$$f_1 \overset{(m)}{\preceq} f_2 \overset{(m)}{\preceq} \ldots \overset{(m)}{\preceq} f_n \overset{(m)}{\preceq} \ldots \overset{(m)}{\preceq} h.$$

Then there exists $f \in D_{HK}$ satisfying $\lim_{n \to \infty} \int_a^b f_n = \int_a^b f$.

Now, we introduce the quasi-convergence in D_{HK}.

Firstly, we recall the concept of quasi-uniform convergence in $C([a,b])$, which states as follows.

Definition 4.2. ([17]). **(Quasi-uniform convergence)** *Let $F_n \subset C([a,b])$ and $F : [a,b] \to \mathbb{R}$. If $F_n(x) \to F(x)$ $(n \to \infty)$ at each point $x \in [a,b]$, then $F_n \to F$ $(n \to \infty)$ quasi-uniformly at $x \in [a,b]$ if for each $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $\delta > 0$ and a positive integer $n \geq N$ such that $|F_n(y) - F(y)| < \varepsilon$ for all $y \in [a,b]$ that satisfy $|y - x| < \delta$.*

Remark 4.2. In [18] another definition of quasi-uniform convergence reads:

P: A sequence $\{F_n\} \subset C([a,b])$ is quasi-uniformly convergent to F if for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists a finite number of indices $n_1, \ldots, n_k$ with $n_j \geq N, j = 1, \ldots, k$, such that for each $x \in [a,b]$

$$|F_{n_j}(x) - F(x)| < \varepsilon$$

for at least one n_j.

In fact, using the following Lemma and [18, Corollary 2.6], it is easy to prove that the two definitions (Definition 4.2 and **P**) of quasi-uniformly convergence above are equivalent.

Lemma 4.4. ([17]). *Let* $\{F_n\}$ *be a sequence in* $C([a,b])$ *and* $\lim_{n \to \infty} F_n(x) = F(x)$ *for each* $x \in [a,b]$. *Then* $F \in C([a,b])$ *if and only if* $F_n \to F$ ($n \to \infty$) *quasi-uniformly at each* $x \in [a,b]$.

For convenience, here and after we denote $I = (a,b)$.

Definition 4.3. (Quasi-convergence) *Let* $\{f_n\} \subset D_{HK}$, F_n *be the primitives of* f_n, $n \in \mathbb{N}$. *Assume that* $F : [a,b] \to \mathbb{R}$ *is such that* $\lim_{n \to \infty} F_n(x) = F(x)$ *for each* $x \in [a,b]$. *The sequence* $\{f_n\}$ *is said to be quasi-convergent to a distribution* $f \in \mathcal{D}'(I)$ *on* $[a,b]$ *if* $F_n \to F$ *quasi-uniformly at each point* $x \in [a,b]$ *and* $F' = f$.

Remark 4.3. Recall that in the space D_{HK} a sequence $\{f_n\} \subset D_{HK}$ converges to $f \in D_{HK}$ in norm is $\|f_n - f\| = \max_{x \in [a,b]} |F_n(x) - F(x)| \to 0$ as $n \to \infty$. This convergence means that the primitives F_n of f_n uniformly converges to F in $C([a,b])$. Hence, the quasi-convergence is weaker than the convergence in norm in D_{HK}.

Theorem 4.5. ([28, Theorem 2]). *Let* $f \in \mathcal{D}'(I)$, $\{f_n\} \subset D_{HK}$ *and* F_n *be the primitives of* f_n, $n \in \mathbb{N}$. *Assume that* $F : [a,b] \to \mathbb{R}$ *is such that* $\lim_{n \to \infty} F_n(x) = F(x)$ *for each* $x \in [a,b]$. *The following assertions are equivalent:*

(i) $f \in D_{HK}$ *and* $F' = f$.
(ii) $\{f_n\}$ *is quasi-convergent to* f *on* $[a,b]$.

5   Fixed point theorems and applications

In this section, we apply the conclusions in Section 4 to establish fixed point theorems in D_{HK}. For simplicity of notation, in what follows, unless otherwise stated, "$\preceq$" (or "$\succeq$") stands for "$\overset{(m)}{\preceq}$" (or "$\overset{(m)}{\succeq}$").

By Theorem 4.4 and the definition of regular cone, the following statement holds.

Theorem 5.1. ([20, Theorem 5]). *The space* $D_{HK}^{(m)}$ *is an ordered Banach space with the regular cone* $D_{HK+}^{(m)}$.

Remark 5.1. In fact, the monotone convergence theorem and dominated convergence theorem is not valid for $D_{HK}^{(p)}$, and therefore $D_{HK+}^{(p)}$ is not a regular cone in $D_{HK}^{(p)}$.

Since $D_{HK+}^{(m)}$ is a regular cone in $D_{HK}^{(m)}$, the next result follows from [29, Lemma 2.3.1].

Lemma 5.1. ([20, Lemma 5]). *(i) Each totally ordered subset with an upper bound in $D_{HK}^{(m)}$ has the supremum.*

(ii) Each totally ordered subset with an lower bound in $D_{HK}^{(m)}$ has the infimum.

Let X be an ordered normed space. The operator $T:X \to X$ is *increasing* if $Tu \preceq Tv$, whenever $u,v \in X$ and $u \preceq v$.

Using Lemma 5.1, we give a general fixed point theorem.

Theorem 5.2. ([20, Theorem 6]). *Assume $u_0,v_0 \in D_{HK}^{(m)}$, $u_0 \preceq v_0$, $T:[u_0,v_0] \to D_{HK}^{(m)}$ is an increasing operator and $u_0 \preceq Tu_0$, $Tv_0 \preceq v_0$, then T has least and greatest fixed points u_* and u^* on $[u_0,v_0]$ and*

$$u_* = \lim_{n\to\infty} u_n,\, u^* = \lim_{n\to\infty} v_n, \tag{5.1}$$

where $u_n = Tu_{n-1}$, $v_n = Tv_{n-1}$ $(n=1,2,...)$ satisfying

$$u_0 \preceq u_1 \preceq ... \preceq u_n ... \preceq u_* \preceq u^* \preceq ... \preceq v_n ... \preceq v_1 \preceq v_0. \tag{5.2}$$

Corollary 5.1. ([20, Corollary 1]). *Assume that the assumptions in Theorem 5.2 hold. If T has a unique fixed point $\overline{x}$ on $[u_0,v_0]$, then for every initial value $x_0 \in [u_0,v_0]$, the iterative sequence $x_n = Tx_{n-1}$ $(n=1,2,...)$ converges to $\overline{x} \in D_{HK}$, i.e., $\|x_n - \overline{x}\| \to 0$ $(n \to \infty)$.*

Moreover, the well-known Schauder's fixed point theorem will be needed later.

Lemma 5.2. ([30, Theorem 6.15]). *Let M be a nonempty closed convex subset of a normed vector space X. Let T be a continuous map of M into a compact subset K of M. Then T has a fixed point.*

Now, we show the applications of the D_{HK} integral in integral and differential equations [20, 22–27].

1. Functional Urysohn integral equation

In [20], the following functional Urysohn integral equation was investigated:

$$u(t) = \int_a^b f(t,s,u(s))ds,\, t \in [a,b]. \tag{5.3}$$

where $f:[a,b] \times [a,b] \times D_{HK} \to D_{HK}$, and $[a,b]$ is a compact real interval, $-\infty < a < b < \infty$.

Definition 5.1. *We say that $u \in D_{HK}$ is a* lower solution *of (5.3) if*

$$u(.) \preceq \int_a^b f(.,s,u(s)). \tag{5.4}$$

If the reversed inequality holds in (5.4), we say that u is an upper solution *of (5.3). If equality holds in (5.4), we say that u is a* solution *of (5.3).*

We assume that f satisfies the following hypotheses:

(f_1) $f(t,.,u(.))$ and $\int_a^b f(.,s,u(s))ds$ belong to D_{HK} for all $t \in [a,b]$ and $u \in D_{HK}$;

(f_2) $f(t,s,z)$ is increasing with respect to z for all $(t,s) \in [a,b] \times [a,b]$;

(f_3) there exist $u_0, v_0 \in D_{HK}$, $u_0 \preceq v_0$, such that $u_0 \preceq \int_a^b f(.,s,u(s))ds \preceq v_0$ for all $u \in D_{HK}$.

The main result is given as follows.

Theorem 5.3. ([20, Theorem 7]). *Assume that the hypotheses $(f_1) - (f_3)$ are satisfied. Then the integral equation* (5.3) *has least and greatest solutions, and they are increasing with respect to f.*

Next, by successive approximations, we obtain the least and greatest solutions of the integral equation (5.3).

Proposition 5.1. ([20, Proposition 4]). *Assume that the hypotheses $(f_1) - (f_3)$ are valid, the operator T satisfies*

$$Tu(t) = \int_a^b f(t,s,u(s))ds, \ t \in [a,b]. \tag{5.5}$$

Then the successive approximations:

(i) $u_{n+1} = Tu_n$ converges to the least solution u_* of (5.3);

(ii) $v_{n+1} = Tv_n$, converges to the greatest solution u^* of (5.3).

Now, we give two examples to illustrate our results.

Example 5.1. Given $p \in D_{HK}$ with $p \succeq 0$ on $[0,1]$, and define a function $f : [0,1] \times [0,1] \times D_{HK} \to D_{HK}$ by

$$f(t,s,y) = \begin{cases} Mp(t), & 0 \preceq y \preceq Mp(t), \\ (M+1)p(t), & y \succeq (M+1)p(t), \\ (M+2^{-m-1})p(t), & (M+2^{-m-1})p(t) \prec \\ & \prec y \prec (M+2^{-m})p(t), \\ -f(t,s,-y), & y \prec 0, \end{cases} \tag{5.6}$$

where $M \geq 0$, $m \in \mathbb{N}$. Thus, the nonlinear integral equation

$$u(t) = \int_0^1 f(t,s,u(s))ds, \quad t \in [0,1] \tag{5.7}$$

has extremal solutions. Moreover, the greatest solution $u^* = (M+1)p(t)$ and the least solution $u_* = Mp(t)$.

In Example 5.1, if

$$f(t,s,y) = \begin{cases} Mp(t), & 0 \preceq y \preceq Mp(t), \\ (M+1)p(t), & y \succ (M+1)p(t), \\ (M+2^{-m-1})p(t), & (M+2^{-m-1})p(t) \prec \\ & \prec y \preceq (M+2^{-m})p(t), \\ -f(t,s,-y), & y \prec 0. \end{cases}$$

Then, Equation (5.7) has extremal solutions $u^* = u_* = Mp(t)$. Moreover, by Corollary 5.1, for every initial value $u_0 \in [Mp(t),(M+1)p(t)]$, the iterative sequence $u_n = Tu_{n-1}$ ($n=1,2,\dots$) converges to $u^* \in D_{HK}$, i.e., $\|u_n - u^*\| \to 0$ ($n \to \infty$).

2. Darboux problem

In [26], the following Darboux equation was studied.

$$\begin{cases} \frac{\partial^2 z}{\partial x \partial y} = f(x,y,z), & \in \Omega, \\ z(x,0) = 0, & 0 \leq x \leq d_1 \\ z(0,y) = 0, & 0 \leq y \leq d_2, \end{cases} \tag{5.8}$$

where $K = \{(x,y): 0 \leq x \leq d_1, 0 \leq y \leq d_2\}$, $B = \{z \in C(K): \|z\|_\infty \leq b\}$, d_1, d_2, $b > 0$, $C(K)$ denotes the space of all continuous functions on K with the uniform norm $\|.\|_\infty$, $\frac{\partial^2 z}{\partial x \partial y}$ denotes the second-order mixed distributional derivative of u and f is a distribution.

Applying the Schauder's fixed point theorem, we obtain an existence result as follows.

Theorem 5.4. ([26, Theorem 3.3]) *Assume that the distribution f satisfies*

(D_1) *f is D_{HK}-integrable on K for every $z \in B$;*

(D_2) *$z \to f(x,y,z)$ is continuous for almost every $(x,y) \in K$;*

(D_3) *There exist g, $h \in D_{HK}$ such that $g(.,.) \preceq f(.,.,z) \preceq h(.,.)$ for every $z \in B$.*

Then there exists at least one solution of Darboux equation (5.8) on J for $J \subset K$.

Moreover, utilizing the well-known Vidossich theorem ([31, Corollary 1.2]), we prove that the solution set S of (5.8) defined on J is an R_δ.

Recall that if a set is an R_δ, it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Furthermore, G. Vidossich [31] pointed out that R_δ is a nonempty, compact and connected set.

Example 5.2. Consider the following Darboux problem

$$\begin{cases} \frac{\partial^2 z}{\partial x \partial y} = r(x) + \frac{2}{y^2+1} \sin z, & (x,y) \in (0,1)^2, \\ z(x,0) = 0, & 0 \leq x \leq 1, \\ z(0,y) = 0, & 0 \leq y \leq 1, \end{cases} \tag{5.9}$$

where $r(x)$ be the distributional derivative of Riemann function $R(x) = \sum_{k=1}^{\infty} \frac{\sin k^2 \pi x}{k^2}$. Then equation (5.9) has at least one solution. Moreover, the solution set is an R_δ.

3. Measure differential equation

In [22], the following measure differential equation (MDE) was considered:

$$\begin{cases} Dx = f(x,t) + g(x,t)Du, \\ x(a) = x_0, \end{cases} \tag{5.10}$$

where $t \in [a,b] \subset \mathbb{R}$, $x \in C([a,b])$, $u:[a,b] \to \mathbb{R}$ is a right continuous function of bounded variation, $f:\mathbb{R} \times [a,b] \to \mathbb{R}$ is distributionally Henstock-Kurzweil integrable, $g:\mathbb{R} \times [a,b] \to \mathbb{R}$ is Henstock-Stieltjes (HS) integrable.

Define

$$B = \{x \in C([a,b]): \|x - x_0\|_\infty \le r\}, r > 0, \quad E = \{(x,t): t \in [a,b], x \in B\}.$$

If f and g satisfy the following assumptions:

(C_1) $f(x(.),.)$ is D_{HK}-integrable for every $x \in B$;

(C_2) $f(.,t)$ is continuous for all $t \in [a,b]$;

(C_3) There exist $f_-, f_+ \in D_{HK}$ such that $f_-(.) \preceq f(x,.) \preceq f_+(.)$ for every $x \in B$;

(C_4) $g(x(.),.)$ is HS-integrable for every $x \in B$;

(C_5) $g(.,t)$ is continuous for all $t \in [a,b]$;

(C_6) There exist $g_-, g_+ \in HS$ such that $g_-(t) \le g(x,t) \le g_+(t)$ for every $(x,t) \in E$.

Then, by the Schauder's fixed point theorem, the following existence result holds.

Theorem 5.5. ([22, Theorem 3.3]) *Assume that* $(C_1) - (C_6)$ *hold. Then there exists at least one solution of MDE* (5.10) *on J for* $J \subset [a,b]$.

Example 5.3. Consider the following MDE

$$\begin{cases} Dx = w(t) + x(t) + A(t)x(t)DH, \quad t \in [a,b], \\ x(a) = 0, \end{cases} \tag{5.11}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A(t) = \text{diag}(e^t, e^{2t}, ..., e^{nt}),$$

and $x_i(t)$ is continuous, $w_i(t)$ is the distributional derivative of the Weierstrass function $\mathcal{W}(t) = \sum_{k=1}^{\infty} \frac{\sin 7^k \pi t}{2^k}$, $H(t)$ is the Heaviside function, i.e., $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \ge 0$, and $DH = \delta$ is the Dirac measure. Then (4.2) has at least one solution.

Remark 5.2. It is well known that the Riemann function $R(x) = \sum_{k=1}^{\infty} \frac{\sin k^2 \pi x}{k^2}$, and the Weierstrass function $\mathcal{W}(t) = \sum_{k=1}^{\infty} \frac{\sin 7^k \pi t}{2^k}$ are continuous but differentiable nowhere on $\mathbb{R}$, see details in [32]. Therefore, the distributional derivatives r and w are neither Henstock-Kurzweil integrable nor Lebesgue integrable. Hence, in Examples 5.2 and 5.3, approaches in the literatures [8–10] are no longer workable. This fact implies that our results are more general.

Remark 5.3. The Schauder's fixed point theorem and the Vidossich theorem can also be applied to study the existence of solutions and the structure of the set of solutions of the wave equation [27], the periodic boundary value problem [24], the nonlinear multi-point boundary value problem [23].

Questions

In the theory of integrals, there are many problems unknown, such as the topological structure of the spaces G_{HK} and D_{HK} and how to make a further study about applying the new integrals to integral and differential equations, especially to PDEs. Although the Henstock-Kurzweil integral has good applications in finance and mechanics [19], it still needs in-depth studies and more extensive applications in other fields. The above problems remain open.

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