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Grothendieck Property for the Symmetric Projective Tensor Product

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Abstract. For a Banach space E, we give sufficient conditions for the Grothendieck property of $\hat{\otimes}_{n,s,\pi}E$, the symmetric projective tensor product of E. Moreover, if E^* has the bounded compact approximation property, then these sufficient conditions are also necessary.

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1 Results

Recall that a Banach space is said to have the *Grothendieck property* (GP in short) if every weak* convergent sequence in its dual is weakly convergent (see, e.g., [6, 10]). González and Gutiérrez in [8] showed that if $n \geqslant 2$ then $\hat{\otimes}_{n,s,\pi}E$, the symmetric projective tensor product of a Banach space E, has GP if and only if $\hat{\otimes}_{n,s,\pi}E$ is reflexive. In this short paper, we show that for any $n \geqslant 1$, if E has GP and every scalar-valued continuous n-homogeneous polynomial on E is weakly continuous on bounded sets, then $\hat{\otimes}_{n,s,\pi}E$ has GP. Moreover, if E^* has the bounded compact approximation property, then these sufficient conditions for $\hat{\otimes}_{n,s,\pi}E$ having GP are also necessary.

Let E and F be Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ and let n be a positive integer. A map $P: E \to F$ is said to be an n-homogeneous polynomial if there is a symmetric n-linear operator T from $E \times \cdots \times E$ (a product of n copies of E) into F such that $P(x) = T(x,\dots,x)$. Indeed, the symmetric n-linear operator $T_P: E \times \cdots \times E \to F$ associated to P can be given by the *Polarization Formula*:

$$T_P(x_1,\dots,x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P\Big(\sum_{i=1}^{n} \epsilon_i x_i\Big), \quad \forall x_1,\dots,x_n \in E.$$

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Let $\mathcal{P}(^nE;F)$ denote the space of all continuous n-homogeneous polynomials from E into F with its norm

$$\|P\|=\sup\{\|P(x)\|: x\in E, \|x\|\leqslant 1\},$$

and let $\mathcal{P}_w(^nE;F)$ denote the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets. In particular, if $F=\mathbb{R}$ or $\mathbb{C}$, then $\mathcal{P}(^nE;F)$ and $\mathcal{P}_w(^nE;F)$ are simply denoted by $\mathcal{P}(^nE)$ and $\mathcal{P}_w(^nE)$ respectively.

Let $\otimes_n E$ denote the *n-fold algebraic tensor product* of E. For $x_1\otimes\cdots\otimes x_n\in\otimes_n E$, let $x_1\otimes_s\cdots\otimes_s x_n$ denote its symmetrization, that is,

$$x_1\otimes_s\cdots\otimes_s x_n=\frac{1}{n!}\sum_{\sigma\in\pi(n)}x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)},$$

where $\pi(n)$ is the group of permutations of $\{1,\ldots,n\}$. Let $\otimes_{n,s}E$ denote the *n-fold symmetric algebraic tensor product* of E, that is, the linear span of $\{x_1\otimes_s\cdots\otimes_s x_n : x_1,\ldots,x_n\in E\}$ in $\otimes_n E$. It is known that each $u\in\otimes_{n,s}E$ has a representation $u=\sum_{k=1}^m\lambda_k x_k\otimes\cdots\otimes x_k$ where $\lambda_1,\ldots,\lambda_m$ are scalars and $x_1,\ldots,x_m$ are vectors in E. Let $\hat{\otimes}_{n,s,\pi}E$ denote the *n-fold symmetric projective tensor product* of E, that is, the completion of $\otimes_{n,s}E$ under the *symmetric projective tensor norm* on $\otimes_{n,s}E$ defined by

$$\|u\|=\inf\left\{\sum_{k=1}^m|\lambda_k|\cdot\|x_k\|^n : x_k\in E, u=\sum_{k=1}^m\lambda_k x_k\otimes\cdots\otimes x_k\right\},\quad u\in\otimes_{n,s}E.$$

For each n-homogeneous polynomial $P:E\to F$, let $A_P:\otimes_{n,s}E\to F$ denote its linearization, that is,

$$A_P(x\otimes\cdots\otimes x)=P(x),\quad \forall x\in E.$$

Then under the isometry: $P\to A_P$,

$$\mathcal{P}(^nE;F)=\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F),$$

where $\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$ is the space of all continuous linear operators from $\hat{\otimes}_{n,s,\pi}E$ to F. In particular,

$$\mathcal{P}(^nE)=(\hat{\otimes}_{n,s,\pi}E)^*,$$

where $(\hat{\otimes}_{n,s,\pi}E)^*$ is the topological dual of $\hat{\otimes}_{n,s,\pi}E$.

For the basic knowledge about homogeneous polynomials and symmetric projective tensor products, we refer to [7,12,13].

For a Banach space E, let E^* denote its dual and E^{**} denote its second dual. For every $P\in\mathcal{P}(^nE)$, let $\widetilde{P}\in\mathcal{P}(^nE^{**})$ denote the *Aron-Berner extension* of P (see, e.g., [1,5]). To obtain $\hat{\otimes}_{n,s,\pi}E$ having GP, we first need the following lemma, which is a special case of [9, Corollary 5].

Lemma 1.1. ([9]) *Let $P_k,P\in\mathcal{P}_w(^nE)$ for each $k\in\mathbb{N}$. Then $\lim_k P_k=P$ weakly in $\mathcal{P}_w(^nE)$ if and only if $\lim_k\widetilde{P}_k(z)=\widetilde{P}(z)$ for every $z\in E^{**}$.*

Now we give sufficient conditions to ensure that $\hat{\otimes}_{n,s,\pi}E$ has GP.

Theorem 1.1. *If E has GP and $\mathcal{P}(^nE)=\mathcal{P}_w(^nE)$, then $\hat{\otimes}_{n,s,\pi}E$ has GP.*

Proof. Take $P_k,P\in\mathcal{P}(^nE)=(\hat{\otimes}_{n,s,\pi}E)^*$ for each $k\in\mathbb{N}$ such that $\lim_k P_k=P$ weak* in $\mathcal{P}(^nE)$. Then $\lim_k P_k(x)=P(x)$ for every $x\in E$. Let T_{P_k} denote the symmetric n-linear operator associated to P_k. By the Polarization Formula, for every $x_1,\ldots,x_n\in E$,

$$\lim_k T_{P_k}(x_1,\ldots,x_n)=T_P(x_1,\ldots,x_n). \tag{1.1}$$

For every fixed $x_2,\ldots,x_n\in E$, define $\phi_k(x)=T_{\widetilde{P}_k}(x,x_2,\ldots,x_n)$ and $\phi(x)=T_{\widetilde{P}}(x,x_2,\ldots,x_n)$ for every $x\in E$, respectively. Then $\phi_k,\phi\in E^*$, and $\langle\phi_k,z_1\rangle=T_{\widetilde{P}_k}(z_1,x_2,\ldots,x_n)$ and $\langle\phi,z_1\rangle=T_{\widetilde{P}}(z_1,x_2,\ldots,x_n)$ for every $z_1\in E^{**}$. By (1), $\lim_k\phi_k=\phi$ weak* in E^* and hence, $\lim_k\phi_k=\phi$ weakly in E^*. Thus, for every $z_1\in E^{**}$ and every $x_2,\ldots,x_n\in E$,

$$\lim_k T_{\widetilde{P}_k}(z_1,x_2,\ldots,x_n)=T_{\widetilde{P}}(z_1,x_2,\ldots,x_n).$$

Using the induction, we can show that for every $z_1,z_2,\ldots,z_n\in E^{**}$,

$$\lim_k T_{\widetilde{P}_k}(z_1,z_2,\ldots,z_n)=T_{\widetilde{P}}(z_1,z_2,\ldots,z_n).$$

In particular, $\lim_k\widetilde{P}_k(z)=\widetilde{P}(z)$ for every $z\in E^{**}$. It follows from Lemma 1 that $\lim_k P_k=P$ weakly in $\mathcal{P}_w(^nE)=\mathcal{P}(^nE)$, and hence $\hat{\otimes}_{n,s,\pi}E$ has GP. □

To ensure that the sufficient conditions for GP of $\hat{\otimes}_{n,s,\pi}E$ in Theorem 1.1 are also necessary, we need the bounded compact approximation property. Recall that a Banach space E is said to have the *bounded compact approximation property* (BCAP in short) (see, e.g., [4, p. 308]), if there exists $\lambda\geqslant 1$ so that for every compact subset C of E and for every $\varepsilon>0$, there is a compact operator $T:E\to E$ such that $\|T\|\leqslant\lambda$ and $\|T(x)-x\|\leqslant\varepsilon$ for all $x\in C$. It is well known that the bounded approximation property implies the bounded compact approximation property, but the converse is not true (see, e.g., [14] or [4, p. 309]).

Theorem 1.2. *If E^* has the BCAP, then $\hat{\otimes}_{n,s,\pi}E$ has GP if and only if E has GP and $\mathcal{P}_w(^nE)=\mathcal{P}(^nE)$.*

Proof. Suppose that $\hat{\otimes}_{n,s,\pi}E$ has GP. By [2, Theorem 3], E is a complemented subspace of $\hat{\otimes}_{n,s,\pi}E$ and hence, E has GP. It is known that every dual Banach space is weak* sequentially complete (see, e.g., [11, p. 230, Corollary 2.6.21]). This fact yields that $\mathcal{P}(^nE)=(\hat{\otimes}_{n,s,\pi}E)^*$ is weakly sequentially complete and hence, $\mathcal{P}_w(^nE)$ also is weakly sequentially complete. It follows from [3, Theorem 3.5] that $\mathcal{P}_w(^nE)=\mathcal{P}(^nE)$. □

It is worth while to mention here that González and Gutiérrez in [8] showed that if $n\geqslant 2$ then $\hat{\otimes}_{n,s,\pi}E$ has GP if and only if $\hat{\otimes}_{n,s,\pi}E$ is reflexive. Thus Theorem 1.1 yields the following corollary.

Corollary 1.1. *If E has GP and $\mathcal{P}(^nE)=\mathcal{P}_w(^nE)$ for some $n\geqslant 2$, then $\hat{\otimes}_{n,s,\pi}E$ is reflexive. In particular, E is reflexive.*

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