Metric Subregularity for a Multifunction

Xi Yin Zheng*

Department of Mathematics, Yunnan University, Kunming 650091, P. R. China.

Abstract. Metric subregularity is an important and active area in modern variational analysis and nonsmooth optimization. Many existing results on the metric suregularity were established in terms of coderivatives of the multifunctions concerned. This note tries to give a survey of the metric subregularity theory related to the coderivatives and normal cones.

AMS subject classifications: 90C31, 90C25, 49J52

Key words: Metric subregularity, coderivative, normal cone.

1 Introduction

Let X and Y be Banach spaces and $\Phi: X \rightrightarrows Y$ be a multifunction such that its graph $\mathrm{gph}(\Phi) := \{(x,y) \in X \times Y : y \in \Phi(x)\}$ is closed. Recall that Φ is metrically subregular at $(a,b) \in \mathrm{gph}(\Phi)$ if there exist $\tau,\delta \in (0,+\infty)$ such that

$$d(x,\Phi^{-1}(b)) \leq \tau d(b,\Phi(x)) \quad \forall x \in B(a,\delta), \tag{1.1}$$

where $d(x,\Phi^{-1}(b)) := \inf\{\|x-u\| : u \in \Phi^{-1}(b)\}$ and $B(a,\delta) := \{u \in X : \|x-a\| < \delta\}$. This property provides an estimate of how far a candidate x can be from the solution set of generalized equation (GE)

$$b \in \Phi(x). \tag{GE}$$

Also recall that a multifunction $M: Y \rightrightarrows X$ is said to be calm at $(b,a) \in \mathrm{gph}(M)$ if there exists $L \in (0,+\infty)$ such that

$$d(x,M(b)) \leq L\|y-b\| \quad \text{for all } (y,x) \in \mathrm{gph}(M) \text{ close to } (b,a).$$

It is known that Φ is metrically subregular at (a,b) if and only if $M = \Phi^{-1}$ is calm at (b,a) (cf. [9]). The metric subregularity and calmness have been already studied by many authors under various names (see [2, 3, 7–10, 14–19, 25, 26, 28–31] and therein references).

*Corresponding author. *Email addresses:* `xyzheng@ynu.edu.cn` (X. Y. Zheng)

Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and consider the special case that $Y = \mathbb{R}$, $b = 0$ and

$$\Phi(x) := [f(x), +\infty) \quad \forall x \in X. \tag{1.2}$$

In this case, generalized equation (GE) reduces to the following inequality

(IE) $$f(x) \leq 0,$$

while metric subregularity (1.1) reduces to

$$d(x, S) \leq \tau[f(x)]_+ \quad \forall x \in B(a, \delta), \tag{1.3}$$

where $S = \{x \in X | f(x) \leq 0\}$ and $[f(x)]_+ = \max\{f(x), 0\}$. Usually inequality (IE) is said to have a local error bound at a if there exist $\tau, \delta \in (0, +\infty)$ such that (1.3) holds. Error bound properties have important applications in sensitivity analysis and convergence analysis of mathematical programming. The research on error bounds has attracted the interest of many researchers and there are a vast number of publications reporting the progress in this area (cf. [4, 11, 12, 20–22, 24, 26, 32,35] and references therein). In particular, studies on error bounds have been well carried out in terms of subdifferentials; these studies are mainly carried out in two directions of approach. The first direction is described by the subdifferentials of f at points *inside* the solution set S and the normal cones of S. In this direction, it is known that if f is convex then inequality (IE) has a local error bound at a if and only if there exist $\tau, \delta \in (0, +\infty)$ such that

$$N(S, x) \cap B_{X^*} \subset [0, \tau] \partial f(x) \quad \forall x \in S \cap B(a, \delta)$$

(cf. [5, 11, 12, 20, 28]). The second direction is described only by the subdifferentials of f at points *outside* the solution set S. In this direction, Ioffe [13] first studied error bound (under a different name) and proved that the following implication holds:

$$d(0, \partial_c f(x)) \geq \kappa \quad \forall x \in B(a, \delta) \setminus S \Longrightarrow (\text{IE}) \text{ has a local error bound at } a. \tag{1.4}$$

Note that the coderivative for a multifunction is the counterpart of the subdifferential for a real-valued function and that the subdifferential $\partial f(x)$ of f at x is equal to the coderivative $D^*\Phi(x, f(x))(1)$ (where Φ is defined by (1.2)). So it is natural to study the more general metric subregularity for a closed multifunction between two Banach spaces in terms of coderivatives also along two directions of approach. In this note, we will give a survey of the research of the metric subregularity along these two directions.

2 Preliminaries

Let X be a Banach space with topological dual X^*. Let B_X and S_X denote the closed unit ball and unit sphere of X, respectively. For a closed subset A of X and a point x in A,

the Clarke tangent cone and normal cone of A at x are denoted by $T_c(A,x)$ and $N_c(A,x)$, respectively, that is, $h \in T_c(A,x)$ if and only if for each sequence $\{x_n\}$ in A converging to x and each sequence $\{t_n\}$ in $(0,\infty)$ decreasing to 0 there exists a sequence $\{h_n\}$ in X converging to h such that $x_n + t_n h_n \in A$ for all n, and

$$N_c(A,x) := \left\{ x^* \in X^* \mid \langle x^*, h \rangle \leq 0 \ \forall h \in T_c(A,x) \right\}.$$

For $\varepsilon \geq 0$ and $a \in A$, the nonempty set

$$\hat{N}_\varepsilon(A,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{A} x} \frac{\langle x^*, u-x \rangle}{\|u-x\|} \leq \varepsilon \right\}$$

is called the set of Fréchet ε-normals of A at x. When $\varepsilon = 0$, $\hat{N}_\varepsilon(A,x)$ is a convex cone and is called the Fréchet normal cone of A at x; it will also be denoted by $\hat{N}(A,x)$. Let $N(A,x)$ denote the limiting (Mordukhovich) normal cone of A at x, that is,

$$N(A,x) = \limsup_{u \xrightarrow{A} x, \varepsilon \to 0^+} \hat{N}_\varepsilon(A,x).$$

Thus, $x^* \in N(A,x)$ if and only if there exists a sequence $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (x,0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \hat{N}_{\varepsilon_n}(A, x_n)$ for each n. It is known that

$$\hat{N}(A,x) \subset N(A,x) \subset N_c(A,x)$$

(cf. [6, 22, 23]). If A is convex, then

$$N_c(A,a) = \hat{N}(A,a) = \left\{ x^* \in X^* \mid \langle x^*, x \rangle \leq \langle x^*, a \rangle \text{ for all } x \in A \right\}.$$

Given a proper lower semicontinuous function $\phi : X \to \mathbb{R} \cup \{+\infty\}$, let

$$\mathrm{dom}(\phi) := \{x \in X \mid \phi(x) < +\infty\}, \quad \mathrm{epi}(\phi) := \{(x,t) \in X \times \mathbb{R} \mid \phi(x) \leq t\}.$$

For $x \in \mathrm{dom}(\phi)$ and $h \in X$, let $\phi^\uparrow(x,h)$ denote the generalized directional derivative

$$\phi^\uparrow(x,h) := \lim_{\varepsilon \downarrow 0} \limsup_{z \xrightarrow{\phi} x, t \downarrow 0} \inf_{w \in h + \varepsilon B_X} \frac{\phi(z+tw) - \phi(z)}{t},$$

where the expression $z \xrightarrow{\phi} x$ means that $z \to x$ and $\phi(z) \to \phi(x)$. Let $\partial_c \phi(x)$ denote the Clarke subdifferential of ϕ at x, that is,

$$\partial_c \phi(x) := \left\{ x^* \in X^* \mid \langle x^*, h \rangle \leq \phi^\uparrow(x,h) \ \forall h \in X \right\}.$$

Recall that the Fréchet subdifferential of ϕ at $x \in \text{dom}(\phi)$ is defined as

$$\hat{\partial}\phi(x) := \left\{ x^* \in X^* \,\Big|\, \liminf_{z \to x} \frac{\phi(z) - \phi(x) - \langle x^*, z - x \rangle}{\|z - x\|} \geq 0 \right\}.$$

It is well known (cf. [6,22]) that

$$\hat{\partial}\phi(x) \subset \partial_c\phi(x).$$

If ϕ is convex, then

$$\partial_c\phi(x) = \hat{\partial}\phi(x) = \left\{ x^* \in X^* \,|\, \langle x^*, y - x \rangle \leq \phi(y) - \phi(x) \;\forall y \in X \right\} \quad \forall x \in \text{dom}(\phi).$$

For a closed set A in X, let δ_A denote the indicator function of A. It is known (see [6, 22, 23]) that

$$N_c(A,a) = \partial_c\delta_A(a), \quad \hat{N}(A,a) = \hat{\partial}\delta_A(a) \quad \forall a \in A$$

$$\partial_c\phi(x) = \left\{ x^* \in X^* \,|\, (x^*, -1) \in N_c(\text{epi}(\phi), (x, \phi(x))) \right\} \quad \forall x \in \text{dom}(\phi). \tag{2.1}$$

Recall that a Banach space X is called an Asplund space if every continuous convex function on X is Fréchet differentiable at each point of a dense subset of X. It is well known (cf. [27]) that X is an Asplund space if and only if every separable subspace of X has a separable dual space. In particular, every reflexive Banach space is an Asplund space. In the case when X is an Asplund space, Mordukhovich and Shao [23] proved that

$$N_c(A,a) = \text{cl}^*(\text{co}(N(A,a))), \quad N(A,a) = \limsup_{x \xrightarrow{A} a} \hat{N}(A,x).$$

The following sum rule and fuzzy sum rule (cf. [6, 23]) play important roles in variational analysis and are useful for our analysis.

Lemma 2.1. *Let X be a Banach space and $\phi_1, \phi_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. Let $x \in \text{dom}(\phi_1) \cap \text{dom}(\phi_2)$ be a local minimizer of $\phi_1 + \phi_2$. Suppose that one of ϕ_1 and ϕ_2 is locally Lipschitz around x. Then*

$$0 \in \partial_c\phi_1(x) + \partial_c\phi_2(x).$$

If, in addition, X is an Asplund space, then for any $\varepsilon > 0$ there exist $x_1, x_2 \in B(x,\varepsilon)$ such that $|\phi_i(x_i) - \phi_i(x)| < \varepsilon$ (i=1,2) and

$$0 \in \hat{\partial}\phi_1(x_1) + \hat{\partial}\phi_2(x_2) + \varepsilon B_{X^*}.$$

For a multifunction Φ from X to Y, as usual, Φ is said to be closed (resp. convex) if its graph $\text{gph}(\Phi)$ is a closed (resp. convex) subset of $X \times Y$. By virtue of different kinds of

normal cones of gph(Φ), one defines corresponding different kinds of coderivatives of Φ at $(x,y) \in$ gph(Φ) as follows: For any $y^* \in Y^*$,

$$\hat{D}^*\Phi(x,y)(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in \hat{N}(\text{gph}(\Phi),(x,y)) \right\},$$

$$D^*\Phi(x,y)(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in N(\text{gph}(\Phi),(x,y)) \right\},$$

$$D_c^*\Phi(x,y)(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in N_c(\text{gph}(\Phi),(x,y)) \right\}.$$

In the case when $Y = \mathbb{R}$ and $\Phi(x) = [\varphi(x), +\infty)$ for all $x \in X$, it is easy from (2.1) to verify that

$$D_c^*\Phi(x,\varphi(x))(1) = \partial\varphi(x).$$

The coderivative $D^*\Phi(x,y)$, as a generalization of the subdifferential, plays a key role in variational analysis. The history of the coderivatives can be found in Mordukhovich's book [22].

3 BCQ, strong BCQ, and metric subregularity

Let φ be a proper lower semicontinuous extended real convex function on a Banach space X and consider the following convex inequality

$$\varphi(x) \leq 0. \tag{CIE}$$

Let S denote the solution set of (CIE), that is, $S := \{x \in X : \varphi(x) \leq 0\}$. Let $a \in \text{bd}(S)$ and recall that (CIE) satisfies basic constraint qualification (BCQ) at a if

$$N(S,a) = \mathbb{R}_+ \partial\varphi(a). \tag{3.1}$$

BCQ is a basic notion in convex optimization (cf. [11,12,21]). If φ is not continuous at a, it is possible that $\partial\varphi(a)$ is empty. To study error bound for (CIE), in terms of the singular subdifferential $\partial^\infty\varphi(x)$, BCQ is extended to the lower semicontinuity case in [28]:

$$N(S,a) = \partial^\infty\varphi(a) + \mathbb{R}_+ \partial\varphi(a), \tag{BCQ$'$}$$

where $\mathbb{R}_+ \partial\varphi(a)$ is understood as $\{0\}$ if $\partial\varphi(a) = \varnothing$. Moreover, strong BCQ is also introduced in [28]; convex inequality (CIE) is said to satisfy strong BCQ at a if there exists $\tau \in (0, +\infty)$ such that

$$N(S,a) \cap B_{X^*} \subset \partial^\infty\varphi(a) + [0, \tau]\partial\varphi(a). \tag{SBCQ}$$

In terms of the coderivative replacing the subdifferential and the singular subdifferential, the concept of the BCQ$'$ and strong BCQ are further extended to the multifunction case in [29]. Let Φ be a closed convex multifunction between Banach spaces X and Y. Recall that Φ has BCQ at $a \in \Phi^{-1}(b)$ if

$$N(\Phi^{-1}(b),a) = D^*\Phi(a,b)(Y^*), \tag{3.2}$$

and that Φ has strong BCQ at $a \in \Phi^{-1}(b)$ if there exists $\tau \in (0, +\infty)$ such that

$$N(\Phi^{-1}, a) \cap B_{X^*} \subset \tau D^* \Phi(a, b)(B_{Y^*}). \tag{3.3}$$

In the special case when $\Phi(x) = [\varphi(x), +\infty)$, (3.2) and (3.3) reduce to (BCQ$'$) and (SBCQ), respectively. From the definitions concerned, it is easy to verify that $D^* \Phi(a, b)(Y^*)$ is always a subset of $N(\Phi^{-1}(b), a)$ and

$$\text{strong BCQ} \Rightarrow \text{BCQ}.$$

The following proposition (cf. [29, Proposition 3.4]) shows that strong BCQ is equivalent to BCQ in some case.

Proposition 3.1. *Let Φ be a closed convex multifunction between Banach spaces X and Y and $(a, b) \in \text{gph}(\Phi)$. Suppose that $\Phi^{-1}(b)$ is a polyhedron. Then Φ has BCQ if and only if Φ has strong BCQ.*

As the main result in [29], in terms of strong BCQ, the following theorem provides a characterization of the metric subregularity for a closed convex multifunction.

Theorem 3.1. *Let Φ be a closed convex multifunction between Banach spaces X and Y and $(a, b) \in \text{gph}(\Phi)$. Then Φ is metrically subregular at (a, b) if and only if there exists $\delta > 0$ such that Φ has strong BCQ at each $x \in \text{bd}(\Phi^{-1}(b)) \cap B(a, \delta)$ with the same constant.*

Theorem 3.1 recaptures some earlier results dealing only with numerical valued functions. When $\Phi(x) = [\varphi(x), +\infty)$ and $b = 0$, then Theorem 3.1 is obtained in [28]. A slightly earlier result is due to Burke and Deng who showed in [4, Theorem 5.2] that if X is a Hilbert space, $\Phi(x) = [\varphi(x), +\infty)$ and $b = \inf_{x \in X} \varphi(x)$, then Φ is metrically subregular at a if and only if there exist $\tau, \delta \in (0, +\infty)$ such that

$$N(\Phi^{-1}(b), x) \cap B_{X^*} \subset \tau \text{cl}^*(\partial \varphi(x)) \quad \forall x \in F^{-1}(b) \cap B(a, \delta),$$

where cl^* denotes the weak* closure.

In general, the strong BCQ of Φ only at $a \in \Phi^{-1}(b)$ does not imply the metric subregularity of Φ at (a, b). This makes the following result meaningful (see [29, Theorem 3.2])

Theorem 3.2. *Let Φ be a closed convex multifunction between Banach spaces X and Y and $(a, b) \in \text{gph}(\Phi)$. Suppose that there exists a cone C and a neighborhood V of a such that $\Phi^{-1}(b) \cap V = C \cap V$. Then Φ is metrically subregular at (a, b) if and only if Φ has strong BCQ at a.*

In many cases, global metric subregularity, a stronger notion than the local metric subregularity in the sense of (1.1), is more useful. Recall that Φ is globally metrically subregular at b if there exists $\tau \in (0, +\infty)$ such that

$$d(x, \Phi^{-1}(b)) \leq \tau d(b, \Phi(x)) \quad \forall x \in X.$$

To consider the global metric subregularity, we adopt the recession core notion introduced in [29]. Let K be a closed convex subset of X and K^∞ denote the recession cone of K, that is,

$$K^\infty := \{h \in X : K + \mathbb{R}_+ h \subset K\}.$$

Clearly, $K = K + K^\infty$, and it is well-known that if K is a closed convex subset of $\mathbb{R}^n$ containing no lines, then $K = \mathrm{co}(\mathrm{ext}(K)) + K^\infty$, where $\mathrm{ext}(K)$ denotes the set of all extreme points of K. As a generalization of $\mathrm{ext}(K)$, we adopt the so-called recession core of K: a subset C of K is said to be a recession core of K if

$$K = \mathrm{co}(C) + K^\infty.$$

Next, in terms of recession cores and the BCQs, we provide some characterizations for the global metric subregularity which are established in [29].

Theorem 3.3. *Let Φ be a closed convex multifunction between Banach spaces X and Y. Let $b \in \Phi(X)$, $\tau \in (0, +\infty)$, and let C be a recession core of $\Phi^{-1}(b)$. Consider the following statements:*

 (i) Φ has the strong BCQ at each $x \in C$ with the constant τ.

 (ii) Φ has the strong BCQ at each $x \in \Phi^{-1}(b)$ with the constant τ.

 (iii) Φ is metrically subregular at each point in C with the constant τ.

 (iv) Φ is metrically subregular at each point in $\Phi^{-1}(b)$ with the constant τ.

 (v) Φ is globally metrically subregular with the constant τ.

 (vi) Φ has the BCQ at each $x \in C$.

Then, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)$\Rightarrow$(vi). If, in addition, $\Phi^{-1}(b)$ is a polyhedron, then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi).

All the above results require that Φ is convex. For the points of view of theoretical interest as well as for applications, it is useful to relax the convexity assumption in the above results. As an interesting extension of the convexity, smoothness and prox-regularity, Aussel et al. [1] introduced and studied the following subsmoothness: *a subset A of a Banach space X is said to be subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle x^* - u^*, x - u \rangle \geq -\varepsilon \|x - u\|$$

whenever $x, u \in B(a, \delta) \cap A$, $x^* \in N_c(A, x) \cap B_{X^*}$ and $u^* \in N_c(A, u) \cap B_{X^*}$. It is easy to verify that A is subsmooth at $a \in A$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - u \rangle \leq \varepsilon \|x - u\|$$

whenever $x, u \in B(a, \delta) \cap A$ and $u^* \in N_c(A, u) \cap B_{X^*}$. Clearly, if A is subsmooth at a then

$$N_c(A, a) = N(A, a) = \hat{N}(A, a).$$

It is known (and easily verified) that

$$\text{convexity} \;\Rightarrow\; \text{prox-regularity} \;\Rightarrow\; \text{subsmoothness}.$$

We say that a closed multifunction $\Phi\colon X\rightrightarrows Y$ is subsmooth at $(a,b)\in\mathrm{gph}(\Phi)$ if $\mathrm{gph}(\Phi)$ is subsmooth at (a,b).

The following proposition on the subsmoothness is established in [31].

Proposition 3.2. *Suppose that $\Phi\colon X\rightrightarrows Y$ is defined by $\Phi(x)=\Psi(g(x))$ for all $x\in X$, where $g\colon X\to Z$ is a smooth function and $\Psi\colon Z\rightrightarrows Y$ is a convex multifunction. Let $(a,b)\in\mathrm{gph}(\Phi)$ and suppose that $g'(a)$ is surjective. Then, Φ is subsmooth at (a,b).*

The following theorem is proved in [31] and provides necessary and sufficient conditions for the metric subregularity of a closed (not necessarily convex) multifunction.

Theorem 3.4. *Let Φ be a closed multifunction between Banach spaces X and Y and let $(a,b)\in \mathrm{gph}(\Phi)$. Then the following statements hold:*

(i) *If Φ is metrically subregular at (a,b), then there exist $\eta,\delta\in(0,+\infty)$ such that*

$$\hat{N}(\Phi^{-1}(b),x)\cap B_{X^*}\subset\eta D^*\Phi(x,b)(B_{Y^*})\quad \forall x\in\Phi^{-1}(b)\cap B(\bar{x},\delta).$$

(ii) *If Φ is subsmooth at (a,b) and there exist $\eta,\delta\in(0,+\infty)$ such that*

$$N(\Phi^{-1}(b),x)\cap B_{X^*}\subset\eta D^*\Phi(x,b)(B_{Y^*})\quad \forall x\in\Phi^{-1}(b)\cap B(\bar{x},\delta),$$

then Φ is metrically subregular at (a,b).

An counterexample given in [31] shows that Theorem 3.4(ii) does not hold if the subsmoothness assumption of Φ at (a,b) is dorpped.

4   Nonconvex case

In contrast to Section 3, this section considers another direction to study the metric subregularity for a general closed multifunction, which is described only by the coderivative of the multifunction concerned at points *outside* the solution set.

Let J denote the normal dual mapping of a Banach space Y, that is,

$$J(y):=\{y^*\in S_{Y^*}\,|\,\langle y^*,y\rangle=\|y\|\}\quad \forall y\in Y\setminus\{0\}.$$

Thus,

$$J(y)=\partial\|\cdot\|(y)\quad \forall y\in Y\setminus\{0\}.$$

For any $\varepsilon>0$, let

$$J_\varepsilon(y):=\{y^*\in S_{Y^*}\,|\,d(y^*,J(y))<\varepsilon\}\quad \forall y\in Y\setminus\{0\}.$$

For a subset A of Y and $b \in Y$, let $P_A(b)$ and $P_A^\varepsilon(b)$ denote respectively the projection and ε-projection of b to A, that is,

$$P_A(b) := \left\{ y \in A \mid \|b-y\| = d(b,A) \right\}$$

$$P_A^\varepsilon(b) := \left\{ y \in A \mid \|y-b\| < d(b,A) + \varepsilon \right\}.$$

The following theorem is proved in [32] and extends Ioffe's result (cf. [13]) to the general closed multifunction case.

Theorem 4.1. *Let Φ be a closed multifunction between Banach spaces X and Y and let $(a,b) \in \mathrm{gph}(\Phi)$. Let $\varepsilon, \eta, \delta \in (0, +\infty)$ be such that*

$$d(0, D_c^* \Phi(x,y)(J_\varepsilon(y-b))) \geq \eta$$

for all $x \in B(a,\delta) \setminus \Phi^{-1}(b)$ and all $y \in P_{\Phi(x)}^\varepsilon(b) \cap B(b,\delta)$. Then

$$d(x, \Phi^{-1}(b)) \leq \frac{1}{\eta} d(b, \Phi(x)) \quad \forall x \in B\left(a, \frac{\delta}{2+\eta}\right).$$

Consequently, Φ is metrically subregular at (a,b).

Letting $\delta \to +\infty$ in Theorem 4.1, we have the following global metric subregularity result.

Corollary 4.1. *Let Φ and $(a.b)$ be as in Theorem 4.1. Suppose that there exist $\varepsilon, \eta \in (0, +\infty)$ such that*

$$d(0, D_c^* \Phi(x,y)(J_\varepsilon(y-b))) \geq \eta \quad \forall x \in X \setminus \Phi^{-1}(b) \text{ and } \forall y \in P_{\Phi(x)}^\varepsilon(b).$$

Then

$$d(x, \Phi^{-1}(b)) \leq \frac{1}{\eta} d(b, \Phi(x)) \quad \forall x \in X.$$

An example given in [32] shows that the converses of Theorem 4.1 and Corollary 4.1 do not hold. As a partial converse of Theorem 4.1, we have the following necessity result for Φ to be metrically subregular.

Theorem 4.2. *Suppose that the closed multifunction $\Phi : X \rightrightarrows Y$ is convex and that Φ is metrically subregular at $(a,b) \in \mathrm{gph}(\Phi)$. Then, there exist $\delta > 0$ and a decreasing function $\eta : [0,1) \to (0, +\infty)$ such that*

$$d(0, D^* \Phi(x,y)(J_\varepsilon(y-b))) \geq \eta(\varepsilon)$$

for all $\varepsilon \in [0,1)$, $x \in B(a,\delta) \setminus \Phi^{-1}(b)$ and $y \in \Phi(x)$.

In the finite dimension case, Theorem 4.1 can be simplified. More precisely, J_ε and $P_{\Phi(x)}^\varepsilon(b)$ can be replaced by J and $P_{\Phi(x)}(b)$, respectively.

Proposition 4.1. *Let X,Y be finite dimensional Banach spaces and $\Phi:X\rightrightarrows Y$ be a closed multi-function with $(a,b)\in\mathrm{gph}(\Phi)$. Suppose there exist $\eta,\delta\in(0,+\infty)$ such that*

$$d(0,D^*\Phi(x,y)(J(y-b)))\geq\eta\|y-b\|\quad\forall x\in B(a,\delta)\setminus\Phi^{-1}(b)\text{ and }\forall y\in P_{\Phi(x)}(b).$$

Then

$$d(x,\Phi^{-1}(b))\leq\frac{1}{\eta}d(b,F(x))\quad\forall x\in B(a,\frac{\delta}{2+\eta}).$$

Under some mild restrictions, the coderivative $D_c^*\Phi(x,y)$ in Theorem 4.1 can be replaced with $\hat{D}^*\Phi(x,y)$, as the following result shows.

Theorem 4.3. *Let X be an Asplund space and Y be a Hilbert space. Let $\varepsilon,\eta,\delta\in(0,+\infty)$ be such that*

$$d(0,\hat{D}^*\Phi(x,y)(J_\varepsilon(y-b)))\geq\eta$$

for all $x\in B(a,\delta)\setminus\Phi^{-1}(b)$ and all $y\in P^\varepsilon_{\Phi(x)}(b)\cap B(b,\delta)$. Then

$$d(x,\Phi^{-1}(b))\leq\frac{1}{\eta}d(b,\Phi(x))\quad\forall x\in B\left(a,\frac{\delta}{2+\eta}\right).$$

The following lemma plays key roles in the proofs of Theorems 4.1 and 4.2 and Proposition 4.1 and is of independent interest.

Lemma 4.1. *Let $u\in X$ and $\tau,r\in(0,+\infty)$ be such that*

$$\tau d(b,\Phi(u))<r<d(u,\Phi^{-1}(b)),$$

and let $\eta,\varepsilon\in(0,+\infty)$. Then there exist $\bar{x}\in X$ and $\bar{y}\in\Phi(\bar{x})$ satisfying the following properties:

$$\|\bar{x}-u\|<r,\ 0<\|\bar{y}-b\|<\min\left\{\frac{r}{\tau},d(b,\Phi(\bar{x}))+\varepsilon\right\},$$

$$\|\bar{y}-b\|\leq\|y-b\|+\frac{1}{\tau}(\|x-\bar{x}\|+\eta\|y-\bar{y}\|)\ \forall(x,y)\in\mathrm{gph}(\Phi),$$

$$(0,0)\in\{0\}\times J(\bar{y}-b)+\frac{1}{\tau}(B_{X^*}\times\eta B_{Y^*})+N_c(\mathrm{gph}(\Phi),(\bar{x},\bar{y})).$$

If, in addition, X and Y are Asplund spaces then for any $\sigma>0$ there also exist $\tilde{x}_\sigma\in X$ and $y_\sigma,\tilde{y}_\sigma\in Y\setminus\{b\}$ such that

$$\max\left\{\|\tilde{x}_\sigma-\bar{x}\|,\|\tilde{y}_\sigma-\bar{y}\|,\|y_\sigma-\bar{y}\|\right\}<\sigma,\quad\tilde{y}_\sigma\in\Phi(\tilde{x}_\sigma)$$

$$(0,0)\in\{0\}\times J(y_\sigma-b)+\frac{1}{\tau}(B_{X^*}\times\eta B_{Y^*})+\hat{N}(\mathrm{gph}(\Phi),(\tilde{x}_\sigma,\tilde{y}_\sigma)).$$

Based on (1.4), one can provide some sufficient conditions of the metric subregularity for Φ by finding some conditions which imply that the function $x \mapsto d(b,\Phi(x))$ is lower semicontinuous, and that $\partial_c d(b,\Phi(\cdot))$ can be described by $D_c^*\Phi(\cdot,\cdot)$. In this direction, Ledyaev and Zhu [19, Lemmas 3.3 and 3.5] provided the following relationship between $\partial d(b,\Phi(\cdot))$ and $D^*\Phi(\cdot,\cdot)$.

Lemma LZ. *Let X and Y be Banach spaces with Fréchet-smooth Lipschitz bump functions and Φ be a closed upper semicontinuous multifunction between X and Y. Let $U \subset X$ be an open set and suppose that for any $x \in U \setminus \Phi^{-1}(b)$*

$$\sigma \leq \lim_{\varepsilon \to 0^+} \inf \left\{ d(0, D^*\Phi(x',y')(S_{Y^*})) \mid x' \in B(x,\varepsilon), y' \in P_{\Phi(x')}^\varepsilon(b) \right\}. \tag{4.1}$$

Then $\sigma \leq d(0, \partial d(b,\Phi(\cdot))(\bar{x}))$ for any $x \in U \setminus \Phi^{-1}(b)$. If, in addition, Φ is compact-valued, then (4.1) can be replaced with

$$\sigma \leq \inf\{d(0, D^*\Phi(x,y)(S_{Y^*})) \mid y \in P_{\Phi(x)}(b)\}.$$

Under the convexity and reflexivity assumption, we have the following exact formula for $\partial d(b,\Phi(\cdot))$ (see [32]).

Lemma 4.2. *Let X be a Banach space and Y be a reflexive Banach space. Suppose that $\Phi : X \rightrightarrows Y$ is a closed convex multifunction. Then the following statements hold.*

(i) *For any $x \in \mathrm{dom}(\Phi)$, $P_{\Phi(x)}(b) \neq \varnothing$.*

(ii) *The function $x \mapsto d(b,\Phi(x))$ is lower semicontinuous.*

(iii) *For any $x \in \mathrm{dom}(\Phi) \setminus \Phi^{-1}(b)$ and $y \in P_{\Phi(x)}(b)$,*

$$\partial d(b,\Phi(\cdot))(x) = D^*\Phi(x,y)(J(y-b)).$$

Based on (1.4) and Lemma 4.2, we have the following characterizations of the metric subregularity for a convex closed multifunction (see [32]).

Proposition 4.2. *Let Y be a reflexive Banach space and suppose that Φ is convex and closed. Then the following statements are equivalent.*

(i) *Φ is metrically subregular at (a,b).*

(ii) *There exist $\kappa, \delta \in (0, +\infty)$ such that*

$$d(0, D^*\Phi(x,y)(J(y-b))) \geq \kappa \quad \forall x \in B(a,\delta) \setminus \Phi^{-1}(b) \text{ and } \forall y \in P_{\Phi(x)}(b) \cap B(b,\delta).$$

(iii) *There exist $\kappa, \delta \in (0, +\infty)$ such that*

$$d(0, D^*\Phi(x,y)(J(y-b))) \geq \kappa \quad \forall x \in B(a,\delta) \setminus \Phi^{-1}(b) \text{ and } \forall y \in \Phi(x).$$

Adopting an admissible function φ (namely an increasing $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\varphi(0)=0$ and $[\varphi(t)\to0\Rightarrow t\to0]$), one can consider the following more general metric subtrgularity: Φ *is said to be metrically φ-subregular at* $(a,b)\in\mathrm{gph}(\Phi)$ *if there exist* $\tau,\delta\in(0,+\infty)$ *such that*

$$\varphi(d(x,\Phi^{-1}(b)))\le\tau d(b,\Phi(x))\quad\forall x\in B(\bar{x},\delta).$$

In the special case when $\varphi(t)=t$, the φ-metric subregularirty reduces the usual metric subregularity. In the very recent paper [34], Theorem 4.1 is improved and extended to the φ-subregularirty case.

For $(a,b)\in\mathrm{gph}(\Phi)$ and $\varepsilon,\delta\in(0,+\infty)$, let

$$\mathcal{B}(\Phi,a,b,\varepsilon,\delta):=\left\{(x,y)\in\mathrm{gph}(\Phi): x\in B(a,\delta)\setminus\Phi^{-1}(b), y\in P_{\Phi(x)}^{\varepsilon}(b)\cap B(b,\delta)\right\}.$$

For $\beta\in(0,+\infty]$ and $(a,b)\in X\times Y$, let

$$K_\beta(a,b):=\{(x,y)\in X\times Y:\|y-b\|\le\beta\|x-b\|\}.$$

$K_\beta(a,b)$ is a cone with the vertex (a,b), and it is of an arbitrarily small "angle" if β is sufficiently small.

Theorem 4.4. *Let φ be a convex admissible function and Φ be a closed multifunction between two Banach spaces X and Y. Let $\alpha\in(0,1)$, $\varepsilon,\delta\in(0,+\infty)$, $\beta\in(0,+\infty]$ and $(a,b)\in\mathrm{gph}(\Phi)$ be such that*

$$\frac{1}{\alpha}\varphi_+'\left(\frac{d(x,\Phi^{-1}(b))}{1-\alpha}\right)\le d(0,D^*\Phi(x,y)(J_\varepsilon(y-b)))$$

for all $(x,y)\in\mathcal{B}(\Phi,a,b,\varepsilon,\delta)\cap K_\beta(a,b)$. Let

$$\delta':=\min\left\{\frac{\delta}{1+\alpha},\varphi^{-1}(\delta)\right\}\text{ and }\kappa:=\max\left\{1,\frac{\varphi_+'(\delta')}{\alpha\beta}\right\}.$$

Then

$$\varphi(d(x,\Phi^{-1}(b)))\le\kappa d(b,\Phi(x))\quad\forall x\in B_X(\bar{x},\delta').$$

Acknowledgments

References

[1] D. Aussel, A. Daniilidis, L. Thibault, Subsmooth sets: functional characterizations and related concepts, *Trans. Amer. Math. Soc.*, **357**(2005), 1275-1301.

[2] D. Aze, A unified theory for metric regularity of multifunctions, *J. Convex Anal.*, **13** (2006), 225-252.

[3] J.F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.

[4] J.V. Burke, S. Deng, Weak sharp minima revisited, Part I: Basic theory, *Control Cybernetics*, **31**(2002), 399-469.

[5] J.V. Burke, M.C.Ferris, Weak sharp minima in mathematical programming. *SIAM J. Optim. Control*, **36**(1993), 1340-1359.

[6] F.H. Clarke, Optimization and Nonsmooth Analysis, *Wiley*, NewYork, 1983.

[7] A.L. Dontchev, R.T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis, *Set-Valued Anal.*, **12**(2004), 79-109.

[8] R. Henrion, A. Jourani, Subdifferential conditions for calmness of convex constraints, *SIAM J. Optim.*, **13**(2002), 520-534.

[9] R. Henrion, A. Jourani, J. Outrata, On the calmness of a class of multifunctions, *SIAM J. Optim.*, **13**(2002), 603-618.

[10] R. Henrion, J. Outrata, Calmness of constraint systems with applications, *Math. Program.*, **104**(2005), 437-464.

[11] H. Hu, Characterizations of the strong basic constraint qualification, *Math. Oper. Res.*, **30**(2005), 956-965.

[12] H. Hu, Characterizations of local and global error bounds for convex inequalities in Banach spaces, *SIAM J. Optim.*, **18**(2007), 309-321.

[13] A.D. Ioffe, Regular Points of Lipschitz functions, *Trans. Amer. Math. Soc.*, **251**(1979), 61-69.

[14] A.D. Ioffe, Metric regularity and subdifferential calculus, *Russian Math. Surveys*, **55**(2000), 501-558.

[15] A.D. Ioffe, J.V. Outrata, On metric and calmness qualification conditions in subdifferential calculus, *Set-Valued Anal.*, **16**(2008), 199-277.

[16] A. Jourani, L. Thibault, Metric regularity and subdifferential calculus in Banach spaces, *Set-Valued Anal.*, **3** (1995), 87-100.

[17] A. Jourani, L. Thibault, Verifiable conditions for openness and metric regularity of multivalued mappings, *Trans. Amer. Math. Soc.*, **347** (1995), 1225-1268.

[18] D. Klatte, B. Kummer, Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications, *Nonconvex Optimization and its Application*, Kluwer Academic Publishers, Dordrecht, **(60)** 2002.

[19] Y. Ledyaev, Q.J. Zhu, Implicit multifunction theorem, *Set-Valued Anal.*, **7** (1999), 209-238.

[20] A. Lewis, J.S. Pang, Error bounds for convex inequality systems, in *Generalized Convexity, Generalized Monotonicity: Recent Results, Proceedings of the Fifth Symposium on Generalized Convexity*, Luminy, June 1996; J.-P. Crouzeix, J.-E. Martinez-Legaz, and M. Volle, eds., Kluwer Academic Publishers, Dordrecht, 1997, 75–100.

[21] W. Li, Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities, *SIAM J. Optim.*, **7** (1997), 966-978.

[22] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation I/II, Springer-verlag, Berlin Heidelberg, 2006.

[23] B. S. Mordukhovich and Y. Shao, Nonsmooth sequential analysis in Asplund spaces, *Trans. Amer. Math. Soc.*, **348** (1996), 1235-1280.

[24] K.F. Ng, X.Y. Zheng, Error bound for lower semicontinuous functions in normed spaces, *SIAM J. Optim.*, **12** (2001), 1-17.

[25] H.V. Ngai, M. Thera, Metric inequality, subdifferential calculus and applications, *Set-Valued Anal.*, **9** (2001), 187-216.

[26] H.V. Ngai, M. Thera, Error bounds and implicit multifunctions in smooth Banach spaces and

applications to optimization, *Set-Valued Anal.*, **12** (2004), 195-223.

[27] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability, *Lecture Notes in Math.*, **1364**, Springer, New York, 1989.

[28] X.Y. Zheng, K.F. Ng, Metric regularity and constraint qualifications for convex inequalities on Banach spaces, *SIAM J. Optim.*, **14** (2003), 757-772.

[29] X.Y. Zheng, K.F. Ng, Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces, *SIAM. J. Optim.*, **18** (2007), 437-460.

[30] X.Y. Zheng, K.F. Ng, Linear regularity for a collection of subsmooth sets in Banach spaces, *SIAM J. Optim.*, **19** (2008), 62-76.

[31] X.Y. Zheng, K.F. Ng, Calmness for L-subsmooth multifunctions in Banach spaces, *SIAM J. Optim.*, **19** (2009), 1648-1673.

[32] X.Y. Zheng, K.F. Ng, Metric subregularity and calmness for nonconvex generalized equations in Banach spaces, *SIAM J. Optim.*, **20** (2010), 2119-2136.

[33] X.Y. Zheng, X.Q. Yang, Weak sharp minima for semi-infinite optimization problems with applications, *SIAM J. Optim.*, **18** (2007), 573-588.

[34] X.Y. Zheng, J. Zhu, Generalized metric subregularity and regularity with respect to an admissible function, *SIAM J. Optim.*, **26** (2016), 535-563.

[35] C. Zălinescu, Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces, *in Proceedings of the 12th Baikal International Conference on Optimization Methods and Their Applications*, Irkutsk, Russia, 2001, 272-284.

[36] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.