J. Math. Study
doi: 10.4208/jms.v49n4.16.02

Vol. **49**, No. 4, pp. 325-378
December 2016

On Some Applications of Geometry of Banach Spaces and Some New Results Related to the Fixed Point Theory in Orlicz Sequence Spaces

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Received July 21, 2016; Accepted August 25, 2016

Abstract. We present some applications of the geometry of Banach spaces in the approximation theory and in the theory of generalized inverses. We also give some new results, on Orlicz sequence spaces, related to the fixed point theory. After a short introduction, in Section 2 we consider the best approximation projection from a Banach space X onto its non-empty subset and proximinality of the subspaces of order continuous elements in various classes of Köthe spaces. We present formulas for the distance to these subspaces of the elements exterior to them. In Section 3 we recall some results and definitions concerning generalized inverses of operators (metric generalized inverses and Moore-Penrose generalized inverses). We also recall some results on the perturbation analysis of generalized inverses in Banach spaces. The last part of this section concerns generalized inverses of multivalued linear operators (their definitions and representations). The last section starts with a formula for modulus of nearly uniform smoothness of Orlicz sequence spaces ℓ^{Φ} equipped with the Amemiya-Orlicz norm. From this result a criterion for nearly uniform smoothness of these spaces is deduced. A formula for the Domínguez-Benavides coefficient $R(a,l_{\Phi})$ is also presented, whence a sufficient condition for the weak fixed point property of the space ℓ^{Φ} is obtained.

MSC 2010: 46B20, 46E30, 47A05, 47A55, 47H14

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Key words: Approximative compactness, proximinality, Kadec-Klee property, uniform rotundity, Orlicz spaces, Banach lattices, quasi-linear projection, generalized inverses.

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1 Introduction

The paper is divided into four sections. The first is an Introduction. In Section 2 some applications of the geometry of Banach spaces to the approximation problems are presented. This section contains 9 subsections. The first two deal with the best approximation problems such as non-emptiness of $P_A(x)$, its uniqueness, proximinal subspaces as well as formulas for the distance of any element of a Köthe space from its subspace of order continuous elements. Subsection 2.3 recalls the necessary and sufficient conditions for approximative compactness in arbitrary Banach spaces and some definitions of the notions that are used further. Subsection 2.4 contains criteria for approximative compactness and full rotundity of Musielak-Orlicz spaces. In Subsections 2.5 and 2.6 criteria for approximative compactness of Orlicz and Orlicz-Lorentz spaces are presented, respectively. Short Subsection 2.7 focuses on monotonicity properties of Banach lattices and their relationships to the dominated best approximation problems. Subsection 2.8 deals with the problem of proximinality in Calderón-Lozanovskiĭ spaces E_φ of some their subspaces while in the last Subsection some interpretations of theorems from Subsection 2.8 in the class of Orlicz spaces are given.

Section 3 is devoted to the applications of geometry of Banach spaces to some problems in the theory of generalized inverses. At the beginning generalized inverses of linear operators were constructed only in Hilbert spaces which have the best possible geometric properties (both rotundity and smoothness types). In order to generalize those results to Banach spaces it was necessary to select these geometric properties of Banach spaces which give also a possibility of constructing various generalized inverses in much more general class of Banach spaces than their subclass of Hilbert spaces. Algorithms and perturbation analysis for some general inverses in Banach spaces with suitable geometric properties are also presented.

In the last section some new results on the modulus of nearly uniform smoothness in Orlicz sequence spaces equipped with the Amemiya-Orlicz norm and some its application to the fixed point theory are presented. Some useful formulas for this modulus are presented and their usefulness on some examples is illustrated.

Notions and definitions are established in any section separately; some of them are even repeated for the convenience of the readers.

2 Applications of geometry of Banach spaces to best approximation problems

2.1 Projections from a Banach space onto its subspace, their existence, uniqueness and properties

Let $X=(X,\|.\|)$ be a Banach space and let A be a non-empty subset of X. A function $d:X\to[0,\infty)$ defined by

$$d(x,A):=\inf\{\|x-y\|:y\in A\}$$

is called the distance of an element $x\in X$ from the set A. If A is a closed subspace of X, then $d(x,A)$ is the norm in the quotient space X/A and the quotient space X/A equipped with this norm is a Banach space. For arbitrary $x\in X$, the set

$$P_A(x)=\{y\in A:d(x,A)=\|x-y\|\}$$

is called the metric projection of an element $x\in X$ onto the set A, and its elements, if the set is non-empty, are called the best approximation elements in the set A for the element x (see [77], [87] and [88]). A set A is said to be a proximinal set in X if $P_A(x)$ is nonempty for any $x\in X$ and $P_A(x)$ is called a Chebyshev set in X if $P_A(x)$ is a singleton for each $x\in X$.

It seems to be natural to raise the question, for what kind of sets $A\subset X$, $A\neq\emptyset$, the condition $P_A(x)\neq 0$ holds for any $x\in X$ (equivalently for any $x\in X\backslash A$)? Natural is also the question, in which Banach spaces X, taking any non-empty convex and closed subset A of the space X and arbitrary $x\in X$, we have that $P_A(x)$ contains at most one element, that is, $P_A(x)=\emptyset$ or $P_A(x)$ is a singleton.

Let us now present two useful theorems which are presented below as lemmas.

Lemma 2.1. ([33], Godini theorem) *Let $(X,\|.\|)$ be a normed linear space and let M be its closed linear subspace. Then M is a proximinal subspace of X if and only if $B(X/M)=q(B(X))$, where X/M is a quotient space of X modulo M and q is the canonical mapping of X on X/M given for any $x\in X$ by the formula:*

$$q(x)=[x]:=\{y\in X:\ x-y\in M\}.$$

Lemma 2.2 (Mazur theorem). *If $(X,\|.\|)$ is a normed space, $x_n\in X$ for any $n\in\mathbb{N}$, $x_0\in X$ and $x_n\to x_0$ weakly in X, then there exists a sequence $(y_n)_{n=1}^{\infty}$ of convex combinations of elements of the sequence $(x_n)_{n=1}^{\infty}$, namely, $y_n=\sum_{k=1}^{n}\alpha_{n_k}x_k$, where $\alpha_{n_k}\geq 0$ for $n\in\mathbb{N}$, $k\in\{1,2,3,...,n\}$ and $\sum_{k=1}^{n}\alpha_{n_k}=1$, that is strongly convergent to some x_0 in X.*

In order to get sufficient conditions for non-emptiness of the set $P_A(x)$ for any $x\in X$ and for the sets A from the quite wide class of weakly compact subsets of a Banach space X, the following theorem would be helpful.

Theorem 2.1. *In any Banach space $(X,\|.\|)$ the norm $\|.\|$ is a lower semi-continuous function under the weak topology, that is, if $(x_n)_{n=1}^{\infty}$ is a sequence of elements of X which is weakly convergent to $x\in X$ (shortly $x_n \stackrel{w}{\to} x$), then*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Proof. From the Hahn-Banach theorem it is known that there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Moreover, $x^*(x) \le \|x^*\|\|x\|$ for any $x \in X$. In virtue of the assumption that $x_n \stackrel{w}{\to} x$, we have also that $x^*(x_n) \to x^*(x) = \|x\|$. Hence

$$\begin{aligned} \|x\| = x^*(x) = x^*\left(\lim_{n\to\infty} x_n\right) &= \lim_{n\to\infty} x^*(x_n) = \liminf_{n\to\infty} x^*(x_n) \le \liminf_{n\to\infty} \|x^*\|\|x_n\| \\ &= \liminf_{n\to\infty} \|x_n\|, \end{aligned}$$

which finishes the proof. □

Remark 2.1. Notice that there is only a few spaces in which a norm is upper semi-continuous under the weak topology, that is,

$$\|x\| \ge \limsup_{n\to\infty} \|x_n\|$$

if the sequence $(x_n)_{n=1}^{\infty}$ in X is weakly convergent to $x \in X$. The spaces possessing this property have, the so called Schur property, that is, weak convergence of sequences coincides with the strong convergence. Indeed, if $x_n \stackrel{w}{\to} x$, then $x_n - x \stackrel{w}{\to} 0$, so the upper semi-continuity of the norm under the weak topology gives

$$\limsup_{n\to\infty} \|x_n - x\| \le \|0\| \le \liminf_{n\to\infty} \|x_n - x\|,$$

which means that $\|x_n - x\| \to 0$. It is known that ℓ^1 space is an example of the space with the Schur property.

It is obvious that every set $A \subset X$, which is weakly closed in X, is also closed in X under the norm topology (norm convergence implies weak convergence). Although the opposite relation does not hold in general, the following theorem is true.

Theorem 2.2. *If A is a convex and norm closed set in X, then A is also weakly closed in X.*

Proof. Assume that $x\in X$, $(x_n)_{n=1}^{\infty}$ is a sequence of elements of the set A and $x_n \stackrel{w}{\to} x$. By the Mazur theorem (Lemma 2.2), there exists a sequence $(y_n)_{n=1}^{\infty}$ of a convex combinations of elements of the sequence $(x_n)_{n=1}^{\infty}$ such that $\|y_n - x\| \to 0$ (see [23]). In virtue of the assumption that A is norm closed, we get that $x \in A$, which finishes the proof. □

Although Theorems 2.5, 2.6 and Corollaries 2.3 and 2.4 from this subsection were already presented together with their proofs in [37], they are also recalled below with proofs for the sake of completeness.

Theorem 2.3. *Let us assume that A is a non-empty subset of a Banach space X such that its intersection with an arbitrary closed ball in X having its center at the point 0, is a weakly compact set. Then $P_A(x) \neq \emptyset$ for any $x \in X$.*

Proof. It is enough to show only that $P_A(x) \neq \emptyset$ for an arbitrary $x \in X \backslash A$. Assume that $x \in X \backslash A$ and let $(x_n)_{n=1}^{\infty}$ be a minimizing sequence in A, that is, such a sequence that $d(x,A) = \lim_{n\to\infty} \|x - x_n\|$. Since

$$\begin{aligned}\|x_n\| &= \|x_n - x + x\| \leq \|x_n - x\| + \|x\| \\ &\leq d(x,A) + 1 + \|x\|,\end{aligned}$$

for n large enough, so the sequence $(\|x_n\|)_{n=1}^{\infty}$ is bounded, that is, there exists $K > 0$ such that $\|x_n\| \leq K$ for all $n \in \mathbb{N}$. Therefore, the sequence $(x_n)_{n=1}^{\infty}$ is contained in the set $A \cap B(0,K)$, where $B(0,K)$ is a norm-closed ball of radius K and centered at 0. By the assumption, we have that $A \cap B(0,K)$ is a weakly compact set, so there exist a subsequence of the sequence $(x_n)_{n=1}^{\infty}$ and an element $y \in A \cap B(0,K)$ such that $x_{n_k} \xrightarrow{w} y$, whence $x - x_{n_k} \xrightarrow{w} x - y$. By the lower semi-continuity of the norm under the weak topology, we have

$$\|x - y\| \leq \liminf_{k\to\infty} \|x - x_{n_k}\| = \lim_{n\to\infty} \|x - x_n\| = d(x,A).$$

Since $y \in A$, then the inequality $d(x,A) \leq \|x - y\|$ is obvious, whence $d(x,A) = \|x - y\|$, that is, $y \in P_A(x)$. This finishes the proof. □

From the last theorem, we can conclude the following three corollaries.

Corollary 2.1. *If X is a Banach space and A is a finite dimensional subspace of X, then $P_A(x) \neq \emptyset$ for any $x \in X$.*

Proof. For any $K > 0$ the set $A \cap B(0,K)$ is bounded closed and convex. In virtue of Theorem 2.2, we conclude that this set is weakly closed. Since the set $A \cap B(0,K)$ is compact under the norm topology, it is also (in virtue of the weak closedness showed in Theorem 2.2) a weakly compact set. Now it is enough to apply Theorem 2.3. □

Corollary 2.2. *If X is a Banach space and A is a non-empty convex and compact subset of X, then $P_A(x) \neq \emptyset$ for any $x \in X$.*

Proof. Notice that the set $A \cap B(0,K)$ is convex and compact for any ball $B(0,K)$ with $0 < k < \infty$. In virtue of Theorem 2.2 we conclude that this set is weakly compact. In order to finish the proof, it is enough to apply Theorem 2.3. □

Corollary 2.3. [37] *If X is a reflexive Banach space and A is a non-empty convex and closed subset of X, then $P_A(x) \neq \emptyset$ for any $x \in X$.*

Proof. Notice that a minimizing sequence $(x_n)_{n=1}^{\infty}$ for the element x, that is, a sequence of the elements of the set A satisfying condition $d(x,A)=\lim_{n\to\infty}\|x-x_n\|$ is contained in some norm closed ball $B(0,K)$, $K>0$. Hence, $(x_n)_{n=1}^{\infty}\subset A\cap B(0,K)$. By the reflexivity of the space X as well as by the convexity and closedness of the set $A\cap B(0,K)$, we conclude that this set is weakly compact. Next the proof can be proceeded in the same way as the suitable part of the proof of Theorem 2.3. $\square$

Lemma 2.3. *If $f^*\in X^*$, $\|f^*\|=1$, $V=\mathrm{Ker}(f^*):=\{x\in X: f^*(x)=0\}$, then for any $x_0\in X$, we have*

$$\mathrm{dist}(x_0,V)=|f^*(x_0)|. \tag{2.1}$$

Proof. If $x_0\in V$, then $|f^*(x_0)|=\mathrm{dist}(x_0,V)=0$, so inequality (2.1) holds. Assume now that $x_0\in X\setminus V$. Then

$$\|x_0-v\|\geq|f^*(x_0-v)|=|f^*(x_0)-f^*(v)|=|f^*(x_0)|$$

for any $v\in V$, whence

$$d(x_0,V)=\inf_{v\in V}\|x_0-v\|\geq|f^*(x_0)|. \tag{2.2}$$

On the other hand, there exists a sequence $(x_n)_{n=1}^{\infty}$ in $S(X)$ such that $f^*(x_n)\to\|f^*\|=1$. Define a new sequence $(v_n)_{n=1}^{\infty}$ by

$$v_n=x_0-\frac{f^*(x_0)x_n}{f^*(x_n)}.$$

We will show that $v_n\in\mathrm{Ker}(f^*)$ for any $n\in\mathbb{N}$. We have

$$f^*(v_n)=f^*(x_0)-f^*(x_0)\frac{f^*(x_n)}{f^*(x_n)}=f^*(x_0)-f^*(x_0)=0,$$

so the condition $v_n\in\mathrm{Ker}(f^*)$ for any $n\in\mathbb{N}$ is proved. Moreover,

$$\begin{aligned}\|x_0-v_n\|&=\left\|x_0-x_0+\frac{f^*(x_0)x_n}{f^*(x_n)}\right\|=\left\|\frac{f^*(x_0)x_n}{f^*(x_n)}\right\|=|f^*(x_0)|\frac{\|x_n\|}{|f^*(x_n)|}\\&=\frac{|f^*(x_0)|}{|f^*(x_n)|}\to|f^*(x_0)|\end{aligned}$$

as $n\to\infty$. Consequently,

$$d(x,V)\leq|f^*(x_0)|,$$

which together with inequality (2.2) gives the equality $d(x,V)=|f^*(x_0)|$. $\square$

Theorem 2.4. *If X is a nonreflexive Banach space, then there exists in X a convex and closed subset V such that $d(x,V)<\|x-v\|$ for any $x\in X\setminus V$ and $v\in V$.*

Proof. By the James theorem there exists a functional $f^*\in S(X^*)$ such that f^* does not attain its norm on the unit sphere $S(X)$. Let us define $V=\mathrm{Ker}(f^*)$ and let us assume for the contrary that there are $x_0\in X\backslash V$ and $v_0\in V$ such that

$$d(x_0,V)=\|x_0-v_0\|.$$

We have by Lemma 2.3 that

$$\|x-v_0\|=d(x_0,V)=|f^*(x_0)|=|f^*(x_0-v_0)|,$$

whence

$$\left|f^*\left(\frac{x_0-v_0}{\|x_0-v_0\|}\right)\right|=1,$$

which means that f^* attains its norm on the element $\frac{x_0-v_0}{\|x_0-v_0\|}\in S(X)$, a contradiction, which finishes the proof. $\square$

Let us consider now the problem of the uniqueness of the best approximation element. We will need the concept of a rotund (or strictly convex) Banach space.

A Banach space $(X,\|.\|)$ is said to be rotund if for any $x,y\in X$ such that $\|x\|=\|y\|=1$ and $x\neq y$, the inequality $\left\|\frac{x+y}{2}\right\|<1$ holds (see [9] and [23]). Since any norm $\|.\|$ is a convex function of X, it is easy to see that if for any $x,y\in S(X)$ condition $\|\lambda x+(1-\lambda)y\|=1$ holds for some $\lambda\in(0,1)$, then $\|\lambda x+(1-\lambda)y\|=1$ holds for any $\lambda\in(0,1)$, that is, the whole segment $[x,y]$ belongs to the unit sphere of X. This means that rotund Banach spaces are exactly these spaces which do not contain on their unit spheres segments of positive length.

Theorem 2.5. ([37], Theorem 1) *For any non-empty convex and closed set A contained in a Banach space X and for any $x\in X$, we have that* $\mathrm{Card}(P_A(x))\leq 1$ *if and only if X is rotund.*

Proof. Sufficiency. Assume that X is rotund and that for some non-empty closed and convex set $A\subset X$ and for some $x\in X$, we have that $y,z\in P_A(x)$. We need to show that $y=z$. We have

$$\|x-y\|=\|x-z\|=d(x,A).$$

Moreover, denoting $w=\frac{y+z}{2}$, we have

$$\begin{aligned}\|x-w\|&=\left\|x-\frac{y+z}{2}\right\|=\left\|\frac{x+x}{2}-\frac{y+z}{2}\right\|\\&=\frac{1}{2}\|(x-y)+(x-z)\|\leq\frac{1}{2}\Big(\|x-y\|+\|x-z\|\Big)\\&=\frac{1}{2}\Big(d(x,A)+d(x,A)\Big)=d(x,A).\end{aligned}$$

Denoting

$$\bar{y}=\frac{x-y}{d(x,A)},\qquad \bar{z}=\frac{x-z}{d(x,A)},\qquad \bar{w}=\frac{x-w}{d(x,A)},$$

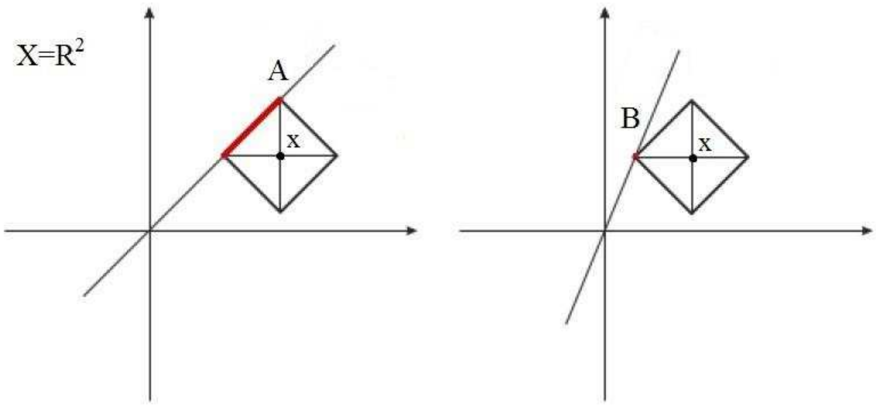

Figure 1: $P_A(x)$ and $P_B(x)$.

we have $\bar{y},\bar{z},\bar{w}\in S(X)$ and $\frac{\bar{y}+\bar{z}}{2}=\bar{w}$. Hence, by the assumption that X is rotund, we have $\bar{w}=\bar{y}=\bar{z}$, whence we get $w=y=z$. This finishes the proof of the sufficiency.

Necessity. Suppose that X is not rotund. Then there exist two elements $x,y\in S(X)$ such that $x\neq y$ and $\frac{x+y}{2}\in S(X)$. In virtue of the remark preceding Theorem 2.5, we have that $\|\lambda x+(1-\lambda)y\|=1$ for any $\lambda\in[0,1]$, which means that the whole segment $[x,y]$ belongs to the unit sphere $S(X)$. The segment is a non-empty convex and closed set. Denoting it by A, we have

$$1=\|z\|=\|z-0\|=d(0,A)$$

for all $z\in A$. Consequently, the set $P_A(0)$ contains a continuum of elements, which ends the proof. □

Remark 2.2. It may happen that even if a Banach space X is not rotund, the set $P_A(x)$ can be a singleton for any $x\in X$ and for some (not for all) sets A of X.
Indeed, $P_A(x)$ is the intersection of a subspace A and a ball with radius x tangent to this subspace. Let us take $X=\mathbb{R}^2$ equipped with the taxi norm $\|x\|=|x_1|+|x_2|$ for $x=(x_1,x_2)\in X$. We will consider two subspaces of X:

$$A=\{(u,u):u\in\mathbb{R}\},\qquad B=\{(u,2u):u\in\mathbb{R}\}.$$

Then, $P_A(x)$ for any $x\in X\backslash A$ is a segment with different endpoints and $P_B(x)$ is a singleton for any $x\in X$ (see Figure 1).

We will show now an example of a Banach space X and its subspace M which is not proximinal in X, that is, there exists $x\in X\backslash M$ such that $P_A(x)=\emptyset$.

Example 2.1. (See [91] and [18]). Let $C[0,1]$ be the space of continuous functions on the compact segment $[0,1]$. Let us define

$$X=\left\{x\in C[0,1]:\ x(0)=0\right\}\text{ and }M=\left\{x\in X:\ \int_0^1 x(t)dt=0\right\},$$

and let X be endowed with the norm induced from the space $C[0,1]$, that is, $\|x\|_\infty = \sup_{0\le t\le 1}|x(t)|$. Then X is the closed subspace of $C[0,1]$, so it also is a Banach space. However, M is a closed subspace of X (because for any $x\in C[0,1]$ and $(x_n)_{n=1}^\infty \subset C[0,1]$, the uniform convergence $x_n \rightrightarrows x$ implies that $\int_0^1 x_n(t)dt \to \int_0^1 x(t)dt$). Therefore, if $x_n\in M$ for any $n\in\mathbb{N}$ and $x_n\to x$ in X, that is, $x_n\rightrightarrows x$, then $x\in M$, which means the closedness of the subspace M in X.

Suppose for the contrary that M is proximinal in X. By Lemma 2.1 this means that there exists an element $x\in S(X)$ such that $d(x,M)=1$. Hence, there exists $x_1\in S(X)$ such that $\|x_1-y\|_\infty\ge 1$ for every $y\in M$. Let us take any element $x\in X\backslash M$ and define

$$\alpha = \int_0^1 x_1(t)dt \Big/ \int_0^1 x(t)dt.$$

Then

$$\begin{aligned}&\int_0^1 (x_1(t)-\alpha x(t))dt\\ =&\int_0^1 x_1(t)dt - \alpha\int_0^1 x(t))dt = \int_0^1 x_1(t)dt - \int_0^1 x_1(t)dt = 0.\end{aligned}$$

Hence, $x_1-\alpha x\in M$. Therefore, in virtue of the earlier observation, we have

$$\|x_1-(x_1-\alpha x)\|_\infty = \|\alpha x\|_\infty = |\alpha|\|x\|_\infty \ge 1,$$

whence

$$\Big|\int_0^1 x_1(t)dt\Big| \ge \frac{1}{\|x\|_\infty}\Big|\int_0^1 x(t)dt\Big| \tag{2.3}$$

for all $x\in X\backslash M$. Let us define $y_n(t)=t^{1/n}$ for $n\in\mathbb{N}$. Then $\|y_n\|_\infty=1$ and $\int_0^1 y_n(t)dt=\frac{n}{n+1}$ for all $n\in\mathbb{N}$, which means that $y_n\in X\backslash M$ for all $n\in\mathbb{N}$. Inequality (2.3) gives us (by its application to the function y_n instead of x) that $|\int_0^1 x_1(t)dt|\ge\frac{n}{n+1}$ for all $n\in\mathbb{N}$, whence

$$\Big|\int_0^1 x_1(t)dt\Big|\ge 1. \tag{2.4}$$

Since $|x_1(t)|\le\|x_1\|_\infty=1$ for any t, $x_1(0)=0$ and x_1 is continuous, we get $\int_0^1|x_1(t)|dt<1$. Consequently, $|\int_0^1 x_1(t)dt|<1$, which contradicts inequality (2.4), ending the proof. □

Remark 2.3. The problem of proximinality in Orlicz spaces and other modular spaces (for the information about the modular spaces we refere for example to [70]) is considered in [36].

An important geometrical property in Banach spaces, which implies both rotundity and reflexivity (see [67]) and therefore, plays an important and crucial role in the approximation theory, is uniform rotundity (called also uniform convexity), introduced by Clarkson in [15].

A normed space $(X,\|.\|)$ is said to be uniformly rotund if $\delta_X(\varepsilon)>0$ for any $\varepsilon\in(0,2]$, where $\delta_X(.)$ is the modulus of convexity in a Banach space X defined as (see [23])

$$\delta_X(\varepsilon):=\inf\left\{1-\left\|\frac{x+y}{2}\right\|:\|x\|\le 1,\|y\|\le 1\wedge\|x-y\|\ge\varepsilon\right\}.$$

It is easy to see that a Banach space $(X,\|.\|)$ is uniformly rotund if and only if for any $\varepsilon\in(0,2]$, we can find $\delta(\varepsilon)\in(0,1)$ such that if $x,y\in B(X)$ and $\|x-y\|\ge\varepsilon$, then $\left\|\frac{x+y}{2}\right\|\le 1-\delta(\varepsilon)$. Uniform rotundity can also be characterized in terms of sequences. Namely, a Banach space X is uniformly rotund if and only if for any two sequences $(x_n)_{n=1}^{\infty},(y_n)_{n=1}^{\infty}$ from the unit ball $B(X)$ such that $\left\|\frac{x_n+y_n}{2}\right\|\to 1$, the condition $\|x_n-y_n\|\to 0$ holds.

Theorem 2.6. ([67]) *Every uniformly rotund Banach space is rotund and reflexive.*

Proof. Assume that $x,y\in S(X)$ and $x\neq y$. Then $\|x-y\|>0$. By virtue of the definition of the modulus of convexity $\delta_X(.)$ as well as by the assumption that X is uniformly rotund, we get

$$\left\|\frac{x+y}{2}\right\|\le 1-\delta(\|x-y\|),$$

which finishes the proof of rotundity of the space X. In order to prove the reflexivity of X, we will use the theorem of R. C. James, which says that X is reflexive if and only if for any functional $x^*\in X^*\backslash\{0\}$ there exists an element $x\in S(X)$ such that $x^*(x)=\|x\|$ (see [23]). It is clear that it is enough to know that every functional $x^*\in S(X^*)$ attains his norm on $S(X)$. Let us take any $x^*\in S(X^*)$. We can find a sequence $(x_n)_{n=1}^{\infty}$ in $S(X)$ such that $\|x^*\|=\lim_{n\to\infty}x^*(x_n)$. Then, for any two subsequences (y_n) and (z_n) of the sequence (x_n) we have

$$x^*(y_n+z_n)=x^*(y_n)+x^*(z_n)\to 2\|x^*\|=2.$$

Hence, we have $\|y_n+z_n\|\to 2$ and, by uniform rotundity of X, we get $\|y_n-z_n\|\to 0$. We showed, in fact, that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. Therefore, there exists $x\in X$ such that $\|x_n-x\|\to 0$ as $n\to\infty$. By the inequalities $|\|x_n\|-\|x\||\le\|x_n-x\|$ and by the fact that $\|x_n\|=1$ for any $n\in\mathbb{N}$, we get $\|x\|=1$. Moreover, by continuity of the functional x^*, we have

$$x^*(x)=\lim_{n\to\infty}x^*(x_n)=\|x^*\|,$$

which finishes the proof. ◻

Corollary 2.4. ([37]) For any uniformly rotund Banach space X and for any non-empty convex and closed set $A\subset X$, the set $P_A(x)$ is a singleton for every $x\in X$.

Proof. By Corollary 2.3 we obtain that $P_A(x) \neq \emptyset$ for every $x \in X$. In virtue of Theorem 2.5, $P_A(x)$ is a singleton for every $x \in X$. $\square$

Recall that a Köthe space E (for the definition we refer, for example, to [41]) is called order continuous (see for instance [51]) if for any element $x \in E$ and any sequence (x_n) in E_+ with $0 \leq x_n \leq |x|$ for any $n \in \mathbb{N}$ and $x_n \to 0$ μ-a.e., there holds $\|x_n\|_E \to 0$.

Remark 2.4. If we restrict the problem of the best approximation to the special class of Banach spaces, namely, to Banach lattices and to the special subsets A of lattices X, namely, to the sets closed under the finite infima and suprema, then the role of reflexivity will be played by another property, called order continuity and the notion of rotundity will be replaced in this case by strict monotonicity of Banach lattices. For the general lattice theory we refer to the monograph [3], while for the best approximation problems in Banach lattices we refer to the papers [43, 51].

2.2 Distance to some subspaces in Köthe spaces

Let E be a Köthe space (for the definition we refer, for example, to [41]) over a $\sigma-$finite complete measure space (Ω,Σ,μ) endowed with a norm $\|.\|$ which is not order continuous. Denote by E_a the subspace of order continuous elements of E, that is,

$$E_a = \Big\{ x \in E : \text{for any } 0 \leq x_n \leq |x|, x_n \uparrow |x|, \text{ we have } \||x| - x_n\| \to 0 \Big\}.$$

Let us assume that $\operatorname{supp} E_a = \Omega$. We can find an ascending sequence of sets $(A_n)_{n=1}^{\infty}$ such that $\bigcup_n A_n = \Omega$ and $L^\infty(A_n) \hookrightarrow E(A_n) \hookrightarrow L^1(A_n)$ and $L^\infty(A_n) \hookrightarrow E_a(A_n)$ for any $n \in \mathbb{N}$ (see [47]). Define $d(x,E_a) = \inf\{\|x-y\| : y \in E_a\}$. It is obvious that $d(x,E_a) = 0$ if and only if $x \in E_a$, because E_a is the closed subset of Köthe space E.

The result of the next Theorem is certainly known in the literature, however the presented below proof is a joint unpublished proof of the second name author and Professor A. S. Granero.

Theorem 2.7. *Let E be a Köthe space over $\sigma-$finite measure space (Ω,Σ,μ) endowed with the norm $\|.\|$ which is not order continuous. Then*

$$d(x,E_a) = \lim_{n\to\infty} \|x - x_n\| \tag{2.5}$$

for any $x \in E$ such that $\operatorname{supp} E_a = \Omega$*, where*

$$x_n(t) = \begin{cases} x(t), & \text{when } |x(t)| \leq n \text{ and } t \in A_n \\ 0, & \text{in the opposite case,} \end{cases}$$

and where A_n, $n \in \mathbb{N}$ are the sets described in the paragraph proceeding this theorem.

Proof. Notice that it is enough to show the proof of formula (2.5) only in the case when $x\in E\backslash E_a$. Since $x_n\in E_a$ for any $n\in\mathbb{N}$, we get $\lim_{n\to\infty}\|x-x_n\|\ge d(x,E_a)$.

We will show now the opposite inequality. Without loss of generality, we can assume that $x\ge 0$. Hence, in the definition of $d(x,E_a)$ we can restrict ourselves to the functions y such that $0\le y\le x$, that is, $d(x,E_a)=\inf\{\|x-y\|: 0\le y\le x, y\in E_a\}$. Indeed, defining $\bar{y}=(y\vee 0)\wedge x$ for any $y\in E_a$, we get that $\bar{y}\in E_a$, $0\le\bar{y}\le x$ and $|x-\bar{y}|\le|x-y|$. Therefore, by the definition of $d(x,E_a)$, for any $\varepsilon>0$ we can find $y\in E_a$ such that $0\le y\le x$ and $\|x-y\|\le d(x,E_a)+\frac{\varepsilon}{2}$. Let us define $y_n(t)$ in the similar way as $x_n(t)$ were defined. We have $y_n\uparrow y$, so under $y\in E_a$ we obtain $\|y-y_n\|\searrow 0$. Since $0\le y_n\le x_n\le x$, so $0\le x-x_n\le x-y_n$ and, consequently,

$$\|x-x_n\|\le\|x-y_n\|\le\|x-y\|+\|y-y_n\|\le d(x,E_a)+\frac{\varepsilon}{2}+\|y-y_n\|,$$

whence $\|x-x_n\|\le d(x,E_a)+\varepsilon$ whenever $n\ge n_0$ and n_0 is large enough. This shows that the inequality

$$\lim_{n\to\infty}\|x-x_n\|\le d(x,E_a)$$

is true. Combining this inequality with $\lim_{n\to\infty}\|x-x_n\|\ge d(x,E_a)$, we get that

$$\lim_{n\to\infty}\|x-x_n\|=d(x,E_a),$$

which ends the proof of the theorem. □

Remark 2.5. In the case of modular function spaces the formula for the distance d can be simplified and calculated in another easier way (see [36]). We will present below how to get this formula in the case of Orlicz spaces only.

Denote by (Ω,Σ,μ) a positive, complete and $\sigma-$finite measure space and by $L^0=L^0(\Omega,\Sigma,\mu)$ the space of all (equivalence classes of) real-valued and $\Sigma-$measurable functions defined on Ω.

In this subsection we will use the following definition of an Orlicz function. Namely, a function $\Phi:\mathbb{R}\to[0,\infty)$ which is even, convex, $\Phi(0)=0$ and Φ is not identically equal to zero is said to be an Orlicz function (see [50, 60, 69, 81]).

Given any Orlicz function Φ we define on L^0 a convex semimodular (see [9, 50, 60, 66, 69, 81]) I_Φ by

$$I_\Phi(x)=\int_\Omega\Phi(x(\omega))d\mu.$$

The Orlicz space $L^\Phi=L^\Phi(\Omega,\Sigma,\mu)$ generated by an Orlicz function Φ is defined as

$$L^\Phi=\left\{x\in L^0: I_\Phi(\lambda x)<+\infty \ \text{ for some } \ \lambda>0\right\}.$$

We will consider Orlicz spaces L^Φ equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{\lambda>0: I_\Phi\left(\frac{x}{\lambda}\right)\leq 1\right\}.$$

Although the result, which we will present below, can be easily deduced from Theorem 2.1 from [36], however for the sake of completeness we present its another easy proof.

Theorem 2.8. *Let $(L^\Phi,\|.\|_\Phi)$ be an Orlicz space over a non-atomic and $\sigma-$finite measure space (Ω,Σ,μ) generated by the Orlicz function Φ having finite values only and let $(L^\Phi,\|.\|_\Phi)$ be equipped with the Luxemburg norm $\|.\|_\Phi$. Let E^Φ be the subspace of L^Φ of all order continuous elements from L^Φ. Then, for any $x\in L^\Phi$, the distance $d(x,E^\Phi)$ of the element x from L^Φ to its subspace E^Φ is defined by the formula*

$$d(x,E^\Phi):=\inf\left\{\lambda>0: I_\Phi\left(\frac{x}{\lambda}\right)<\infty\right\}.$$

Proof. We have by definition $d(x,E^\Phi)=\inf\{\|x-y\|_\Phi : y\in E^\Phi\}$ for any $x\in L^\Phi$. Let us denote

$$\lambda(x)=\left\{\lambda>0: I_\Phi\left(\frac{x}{\lambda}\right)<\infty\right\}$$

for any $x\in L^\Phi$. First, we will show that $d(x,E^\Phi)\leq\lambda(x)$ for any $x\in L^\Phi$. Taking any $x\in L^\Phi$ and $\varepsilon>0$, we get $I_\Phi\left(\frac{x}{\lambda(x)+\varepsilon}\right)<\infty$. Let $(T_n)_{n=1}^\infty$ be a sequence in Σ such that $0<\mu(T_n)<\infty$ for any $n\in\mathbb{N}$, $T_1\subset T_2\subset ...\subset T_n\subset T_{n+1}\subset ...$ and $\bigcup_{n=1}^\infty T_n=\Omega$. Let us take any $x\in L^\Phi$ and define the sequence $(x_n)_{n=1}^\infty$ in E^Φ by the formula

$$x_n(t)=\begin{cases} x(t), & \text{when } t\in T_n \text{ and } |x_n(t)|\leq n\\ 0, & \text{in the opposite case.}\end{cases}$$

Then $(|x_n|)_{n=1}^\infty$ is a non-decreasing sequence in E^Φ and $(|x-x_n|)_{n=1}^\infty$ is a non-increasing sequence in E^Φ which is convergent $\mu-$a.e. in Ω to the function $\theta(t)=0$ for all $t\in\Omega$. Moreover, $|x(t)-x_n(t)|\leq|x(t)|$ for $\mu-$a.e. $t\in\Omega$, whence

$$\frac{|x(t)-x_n(t)|}{\lambda(x)+\varepsilon}\leq\frac{|x(t)|}{\lambda(x)+\varepsilon} \tag{2.6}$$

for $\mu-$a.e. $t\in\Omega$. Since the Orlicz function Φ is non-decreasing on $\mathbb{R}_+$ and $\Phi(u)\to 0$ as $u\to 0$, from inequality (2.6), we obtain that

$$\Phi\left(\frac{|x(t)-x_n(t)|}{\lambda(x)+\varepsilon}\right)\leq\Phi\left(\frac{|x(t)|}{\lambda(x)+\varepsilon}\right)$$

and $\Phi\left(\frac{|x(t)-x_n(t)|}{\lambda(x)+\varepsilon}\right)\to 0$ $\mu-$a.e. in Ω as $n\to\infty$. Hence, by the fact that $I_\Phi\left(\frac{|x|}{\lambda(x)+\varepsilon}\right)<\infty$ and by the Lebesgue dominated convergence theorem, we get the equality

$$\lim_{n\to\infty} I_\Phi\left(\frac{x-x_n}{\lambda(x)+\varepsilon}\right)=0.$$

Consequently, there exists $m\in\mathbb{N}$ such that the inequality $\|x-x_n\|\le\lambda(x)+\varepsilon$ holds for any $n\ge m$. By the fact that $x_m\in E^\Phi$, we conclude that

$$d(x,E^\Phi)\le\|x-x_m\|\le\lambda(x)+\varepsilon.$$

By the arbitrariness of $\varepsilon>0$, we obtain that

$$d(x,E^\Phi)\le\lambda(x) \tag{2.7}$$

for any $x\in L^\Phi$.

Now we will prove the opposite inequality. In order to do this, we will show first that for any $x\in L^\Phi\backslash E^\Phi$, any $y\in E^\Phi$ and any $\varepsilon\in(0,\lambda(x))$, we have

$$I_\Phi\left(\frac{x-y}{\lambda(x)-\varepsilon}\right)=\infty. \tag{2.8}$$

Indeed, if there existed $x\in L^\Phi\backslash E^\Phi$, $y\in E^\Phi$ and $\varepsilon\in(0,\lambda(x))$ such that $I_\Phi\left(\frac{x-y}{\lambda(x)-\varepsilon}\right)<\infty$, then taking $\alpha=\frac{\lambda(x)-\varepsilon}{\lambda(x)-\frac{\varepsilon}{2}}$, we would have

$$\begin{aligned}\infty= I_\Phi\left(\frac{x}{\lambda(x)-\frac{\varepsilon}{2}}\right)&=I_\Phi\left(\alpha\frac{x-y}{\lambda(x)-\varepsilon}+(1-\alpha)\frac{\alpha}{(1-\alpha)(\lambda(x)-\varepsilon)}y\right)\\ &\le \alpha I_\Phi\left(\frac{x-y}{\lambda(x)-\varepsilon}\right)+(1-\alpha)I_\Phi\left(\frac{\alpha}{(1-\alpha)(\lambda(x)-\varepsilon)}y\right)<\infty,\end{aligned}$$

a contradiction, which finishes the proof of (2.8). Condition (2.8) implies that

$$\left\|\frac{x-y}{\lambda(x)-\varepsilon}\right\|_\Phi\ge 1,$$

that is, $\|x-y\|\ge\lambda(x)-\varepsilon$ for any $x\in L^\Phi\backslash E^\Phi$, $y\in E^\Phi$ and $\varepsilon\in(0,\lambda(x))$. Fixing $x\in L^\Phi\backslash E^\Phi$, we obtain

$$d(x,E^\Phi)=\inf\{\|x-y\|:y\in E^\Phi\}\ge\lambda(x)-\varepsilon.$$

By the arbitrariness of $\varepsilon\in(0,\lambda(x))$, we get $d(x,y)\ge\lambda(x)$. Combining this inequality with inequality (2.7), we get $d(x,y)=\lambda(x)$, which finishes the proof. □

2.3 Approximative compactness in general Banach spaces

Let us now present another important property called approximative compactness. A nonempty set $C\subset X$, where X denotes a real Banach space, is said to be approximatively compact if for any sequence $(x_n)_{n=1}^{\infty}$ in C and any $y\in X$ such that $\|x_n-y\|\to d(y,C)$, it follows that $(x_n)_{n=1}^{\infty}$ has a Cauchy subsequence. X is called approximatively compact if any nonempty closed and convex set in X is approximatively compact (see [26]).

Recall that approximative compactness was introduced by Efimov and Stečkin in [26]. This property for a Banach space X is strongly related to the approximation theory (see [4]). Namely, approximative compactness implies that any element in $x\in X$ has the best approximant in any nonempty closed and convex subset A of X. Recall that $y\in A$ is the best approximant for $x\in X$ if $\|x-y\|=d(x,A)$. Moreover, approximative compactness of a rotund Banach space X guarantees continuity of the function $x\to P_A(x)$, called the metric projection onto A, for any nonempty convex and closed subset A of X and any $x\in X$.

It is worth mentioning that every uniformly rotund Banach space is approximatively compact and that approximative compactness implies reflexivity (see [4]).

A Banach space X is said to have the Kadec-Klee property (or property H for short) if for any sequence $(x_n)\subset X$ and $x\in X$ such that $\|x_n\|=\|x\|=1$, we have $\|x_n-x\|\to 0$ provided $x_n\to x$ weakly. Recall that this property was oryginally considered by Radon [80] and next by Riesz ([83], [84]), where it has been proven that L^p-spaces $(1<p<\infty)$ have property H, although $L^1[0,1]$ has not.

A Banach space X is said to be fully k-rotund ($k\geq 2$, $k\in\mathbb{N}$) if every sequence (x_n) in $S(X)$ such that

$$\|x_n^{(1)}+x_n^{(2)}+...+x_n^{(k)}\|\to k \text{ as } n\to\infty$$

for all its subsequences $(x_n^{(1)})$, $(x_n^{(2)})$,...,$(x_n^{(k)})$, is a Cauchy sequence. Moreover, 2-fully rotund Banach spaces are called simply fully rotund spaces. Recall that the notion of full rotundity was introduced by K. Fan and I. Glicksberg [29]. It is known that k-fully rotund Banach spaces ($k\geq 2$) are approximatively compact (see [45], Corollary 1).

Let us now present an important theorem giving a characterization of approximative compactness in Banach spaces.

Theorem 2.9. ([42], Theorem 3) A Banach space X is approximatively compact if and only if X is reflexive and X has the Kadec-Klee property.

Although the proof of this theorem can be found in [42], for the convenience of the reader we will present it below.

Proof. It is well known (see [89], Corollary 2.4, p. 99) that if all closed subspaces are proximinal, then all linear functionals attain their norm (so X is reflexive). Since approximative compactness of X implies that all closed subspaces of X are proximinal, the necessity of reflexivity follows.

Now we will prove the necessity of the Kadec-Klee property. Suppose that X is approximatively compact and X has not the Kadec-Klee property. Then there is a sequence (x_n) in X and $x \in X$ such that $\|x_n\| = \|x\| = 1$, $x_n \to x$ weakly and (x_n) does not converge to x. Passing to a subsequence, if necessary, we can assume that there exists $d > 0$ such that $\|x_n - x\| \geq d$ for any natural number n. Let $f \in X^*$ be a norming functional for x, that is, $1 = f(x) = \|f\|$. Set $C = \{z \in X : f(z) \geq 1\}$. Obviously C is a nonempty closed and convex subset of X. Since $\|f\| = 1$, $\|z\| \geq 1$ for any $z \in C$. Hence $d(0,C) = 1 = \|x - 0\|$. Since $x_n \to x$ weakly, $f(x_n) \to f(x) = 1$. Setting $z_n = x_n / f(x_n)$, we have that $z_n \in C$, because $f(z_n) = 1$ for any $n \in \mathbb{N}$. Moreover, since $f(x_n) \to 1$ and $\|x_n\| = 1$, we have

$$\|z_n\| = \|z_n - 0\| \to 1 = d(0,C).$$

Since X is approximatively compact, (z_n) has a Cauchy subsequence (we will denote it again as (z_n)). Since X is a Banach space, $\|z_n - z\| \to 0$ for some $z \in X$. Hence $z_n \to z$ weakly. But $z_n \to x$ weakly, since $x_n \to x$ weakly and $f(x_n) \to 1$. Consequently, $z = x$. Hence $\|z_n - x\| \to 0$, which gives immediately that $\|x_n - x\| \to 0$, a contradiction. This shows that approximative compactness implies the Kadec-Klee property.

Sufficiency. Suppose that X is reflexive and X has the Kadec-Klee property. Let $C \subset X$ be a nonempty, closed and convex set. Assume $y \in X$ and $(x_n) \subset C$ is chosen in such a way that $\|x_n - y\| \to d(y,C)$. If $d(y,C) = 0$, then $\|x_n - y\| \to 0$ and (x_n) is a Cauchy sequence. So suppose that $d(y,C) = d > 0$. Since X is reflexive, passing to a subsequence, if necessary, we can assume that (x_n) converges weakly to some $x \in X$. Since C is closed and convex, C is weakly closed. Hence $x \in C$. Moreover,

$$d = d(y,C) \leq \|x - y\| \leq \liminf_{n \to \infty} \|x_n - y\| = d,$$

which shows that $\|x - y\| = d$. Set $z_n = (x_n - y)/\|x_n - y\|$ and $z = (x - y)/d$. Then $\|z_n\| = \|z\| = 1$ and $z_n \to z$ weakly, since $\|x_n - y\| \to d$ and $x_n \to x$ weakly. By the Kadec-Klee property of X, $\|z_n - z\| \to 0$ and consequently, $\|x_n - x\| \to 0$. Hence (x_n) is a Cauchy sequence as required. $\square$

Remark 2.6. Let us note here that Rolewicz [85] introduced the notions of the drop in a Banach space and the drop property for Banach spaces (for any $x \in X \backslash B(X)$ the drop determined by x is the set $D(x,B(X)) = \mathrm{conv}(\{x\} \cup B(X))$ and X is said to have the drop property if for any closed set C, disjoint with $B(X)$, there exists $x \in C$ such that $D(x,B(X)) \cap C = \{x\}$). Montesinos showed in [68] that a Banach space X has the drop property if and only if X is reflexive and it has property H. Hence, and by virtue of Thereom 2.9, the drop property and approximative compactness coincide.

Remark 2.7. ([42], Remark 1) Let X be an approximatively compact Banach space and V be a nonempty, closed and convex subset of X. Suppose $x \in X$ and $\mathrm{card}(P_V(x)) = 1$, where $P_V(x) = \{v \in V : \|x - v\| = d(x,V)\}$. Then for any $v_n \in V$ with $\|x - v_n\| \to d(x,V)$, we have $\|v_n - v\| \to 0$, where $\{v\} = P_V(x)$.

Proof. (See also [42]) Suppose for the contrary that $\|v_{n_k}-v\|\geq d>0$ for some subsequence (v_{n_k}) of (v_n). Since X is approximatively compact, there exist $z\in P_V(x)$ and a subsequence of (v_{n_k}) (we will also denote it again by (v_{n_k})) such that $\|v_{n_k}-z\|\to 0$. Since $\operatorname{card}P_V(x)=1$, we have $z=v$, which leads to a contradiction. □

2.4 Approximative compactness and full rotundity of Musielak-Orlicz spaces

We will present here the criteria for approximative compactness in the class of Musielak-Orlicz function as well as sequence spaces for the Luxemburg and for the Amemiya norm. Moreover, we will present criteria for full k-rotundity of Musielak-Orlicz spaces equipped with the Luxemburg norm in the case of a non-atomic finite measure space.

Let us start with some notations and definitions. Let (Ω,Σ,μ) be a measure space with a non-atomic and $\sigma-$finite measure μ. A function $\Phi:\Omega\times\mathbb{R}\to\mathbb{R}_+\cup\{+\infty\}$ is called a Musielak-Orlicz function if

(a) $\Phi(.,u)$ is a $\Sigma-$measurable function for any $u\in\mathbb{R}$,

(b) the function $\Phi(t,.)$ is convex, even, continuous at zero and left-continuous on $(0,+\infty)$ for $\mu-$almost all $t\in\Omega$, which means that $\lim_{u\to(b_\Phi(t))_-\Phi(t,u)}=\Phi(t,b_\Phi(t))$ for $\mu-$almost all $t\in\Omega$, where $b_\Phi(t)=\sup\{u>0:\Phi(t,u)<\infty\}$,

(c) $\Phi(t,0)=0$, $\Phi(t,u_t)<+\infty$ for some $u_t\in(0,+\infty)$ and $\Phi(t,u)\to\infty$ as $u\to\infty$ for almost all $t\in\Omega$.

In the case when $\Omega=\mathbb{N}$ and μ is the counting measure on $2^{\mathbb{N}}$, we can state that a function $\Phi=(\Phi_i)_{i=1}^\infty$ is called a Musielak-Orlicz function if

(a) $\Phi_i:\mathbb{R}\to\mathbb{R}_+\cup\{+\infty\}$ is convex, even, continuous at zero and left-continuous on $(0,+\infty)$ for all $i\in\mathbb{N}$,

(b) $\Phi_i(0)=0$, $\Phi_i(u_i)<+\infty$ for some $u_i\in(0,+\infty)$ and $\Phi_i(u)\to\infty$ as $u\to\infty$ for all $i\in\mathbb{N}$.

Given a Musielak-Orlicz function Φ, we define $\rho_\Phi:L^0\to\mathbb{R}_+\cup\{+\infty\}$ by

$$\rho_\Phi(f)=\int_\Omega\Phi(t,|f(t)|)d\mu(t).$$

Then ρ_Φ is called the modular generated by Φ and the space

$$L_\Phi=\left\{f\in L^0:\rho_\Phi(\lambda f)<+\infty \text{ for some } \lambda>0\right\}$$

is called the Musielak-Orlicz space generated by Φ. Analogously, for any real sequence $x=(x_i)_{i=1}^\infty$ (the space of such sequences is denoted by ℓ^0) the modular ρ_Φ at x has the form

$$\rho_\Phi(x)=\sum_{i=1}^\infty\Phi_i(|x_i|),$$

and then the space

$$\ell_\Phi = \left\{ x \in \ell^0 : \rho_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0 \right\}$$

is called the Musielak-Orlicz sequence space.

We consider two classical norms in Musielak-Orlicz spaces L_Φ (resp. ℓ_Φ): the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{ \lambda > 0 : \rho_\Phi\left(\frac{x}{\lambda}\right) \le 1 \right\}$$

and the Amemiya norm

$$\|x\|_\Phi^A = \inf\left\{ k > 0 : \frac{1+\rho_\Phi(kx)}{k} \right\}$$

(see [9, 69]). In the first case the Musielak-Orlicz space is denoted by L_Φ and in the second case by L_Φ^A. Analogously, the respective sequence spaces are denoted by ℓ_Φ and ℓ_Φ^A.

A Musielak-Orlicz function $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is said to satisfy the Δ_2-condition ($\Phi \in \Delta_2$) if for any $d > 1$ there exist $k > 1$ and $c \in L^1(\Omega)$, $c \ge 0$, such that for any $u \in \mathbb{R}_+$ and $\mu-$a.e. $t \in \Omega$, we have

$$\Phi(t, du) \le k\Phi(t,u) + c(t).$$

A Musielak-Orlicz function $\Phi = (\Phi_i)_{i=1}^\infty$ is said to satisfy the δ_2-condition ($\Phi \in \delta_2$) if for any $d > 1$ there exist $a > 0$, $k > 1$, $i_0 \in \mathbb{N}$ and a non-negative sequence $(c_i) \in \ell^1$ such that the inequality

$$\Phi_i(du) \le k\Phi_i(u) + c_i$$

holds for all $i > i_0$ and $u \in \mathbb{R}_+$ satisfying $\Phi_i(u) \le a$. For some equivalent forms of the Δ_2-condition and the δ_2-condition see [40] and [69].

Let us denote by Φ^* the function complementary to Φ in the sense of Young, i.e., $\Phi^*(t,u) = \sup_{v>0}\{vu - \Phi(t,v)\}$ for any $u \ge 0$ and $\mu-$a.e. $t \in \Omega$ (analogously $\Phi^* = (\Phi_n^*)$, where $\Phi_n^*(u) = \sup_{v>0}\{vu - \Phi_n(v)\}$ for any $u \ge 0$ and all $n \in \mathbb{N}$ in the sequence case).

For any non-zero $x \in L_\Phi^A$ or $x \in \ell_\Phi^A$, we define

$$k^*(x) = \inf\{k \ge 0 : I_{\Phi^*}(p \circ k|x|) \ge 1\}$$
$$k^{**}(x) = \sup\{k \ge 0 : I_{\Phi^*}(p \circ k|x|) \le 1\},$$

where $p(t,.)$ is the right hand side derivative of $\Phi(t,.)$ on $\mathbb{R}_+$ and $p \circ k|x|(t) := p(t, k|x(t)|)$ for $\mu-$a.e. $t \in \Omega$. Next we define $\overline{K}(x) = [k^*(x), k^{**}(x)]$, if $k^{**}(x) < \infty$. Recall that this interval has the property that $\|x\|_\Phi^A = \frac{1}{k}(1+\rho_\Phi(kx))$ if and only if $k \in \overline{K}(x)$. If $k^*(x) < \infty$ and $k^{**}(x) = \infty$, we have $\|x\|_\Phi^A = \frac{1}{k}(1+\rho_\Phi(kx))$ for any $k \in [k^*(x), k^{**}(x))$ and also

$$\|x\|_\Phi^A = \lim_{k\to\infty} \frac{1}{k}(1+\rho_\Phi(kx)) = \int_\Omega A(t)|x(t)|d\mu,$$

where $A(t) = \lim\limits_{k\to\infty} \frac{1}{u}\Phi(t,u)$.

Since, by Theorem 2.9, in Banach spaces reflexivity and the Kadec-Klee property are strictly connected with an approximative compactness, let us start from recalling the following results:

Lemma 2.4. ([42], Theorem 4)

(i) A Musielak-Orlicz space L_Φ is reflexive (with respect to the Luxemburg and Amemiya norms) if and only if $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$.

(ii) A Musielak-Orlicz sequence space ℓ_Φ is reflexive (with respect to the Luxemburg and Amemiya norms) if and only if $\Phi \in \delta_2$ and $\Phi^* = \{\Phi_i^* : i \in \mathbb{N}\} \in \delta_2$.

Lemma 2.5. ([22]) Let $\Phi = (\Phi_i)$ be a Musielak-Orlicz function. Set, for any $i \in \mathbb{N}$, $b_i = \sup\{u > 0 : \Phi_i(u) < +\infty\}$. Then the Musielak-Orlicz space $(l_\Phi, \|.\|_\Phi^A)$ has the Kadec-Klee property if and only if $\Phi \in \delta_2$ or $\sum_{i=1}^{\infty} \Phi_i^*(c_i) \leq 1$, where for any $i \in \mathbb{N}$,

$$c_i = \begin{cases} b_i, & \text{if } \Phi_i^*(b_i) < 1 \\ (\Phi_i^*)^{-1}(1), & \text{if } \Phi_i^*(b_i) \geq 1. \end{cases}$$

Lemma 2.6. ([19]) The Orlicz space L_Φ^A is rotund if and only if

(i) Φ is strictly convex,

(ii) $\lim_{u\to\infty} R(u) = \infty$, where $R(u) = A|u| - \Phi(u)$ and $A = \lim_{u\to\infty} \frac{\Phi(u)}{u}$.

Theorem 2.10. ([42], Theorem 9) Let $\Phi = (\Phi_i)_{i=1}^{\infty}$ be a Musielak-Orlicz function. Then the Musielak-Orlicz space ℓ_Φ endowed with the Luxemburg norm has the Kadec-Klee property with respect to the coordinatewise convergence if and only if Φ satisfies the δ_2-condition and for every $i \in \mathbb{N}$ there is $u_i > 0$ such that $\Phi_i(u_i) = 1$.

Let us now present the main theorems.

Corollary 2.5. ([42], Corollary 2) Suppose that $\Phi = (\Phi_i)_{i=1}^{\infty}$ be a Musielak-Orlicz function such that for any $i \in \mathbb{N}$ there exists $u_i > 0$ with $\Phi_i(u_i) = 1$. Then ℓ_Φ is approximatively compact if and only if ℓ_Φ is reflexive.

Proof. The proof follows immediately from Theorems 2.9 and 2.10. □

Theorem 2.11. ([42], Theorem 10) *The Musielak-Orlicz space ℓ_Φ^A equipped with the Amemiya norm is approximatively compact if and only if it is reflexive, that is, if and only if $\Phi \in \delta_2$ and $\Phi^* \in \delta_2$.*

Theorem 2.12. ([42], Theorem 11) *Let $X = L_\Phi^A$. The following conditions are equivalent:*

(i) X is approximatively compact,

(ii) *X is reflexive and rotund,*

(iii) $\Phi,\Phi^*\in\Delta_2$ *and* $\Phi(t,.)$ *is strictly convex on* $\mathbb{R}$ *for* $\mu-$*a.e.* $t\in\Omega$.

Theorem 2.13. ([42], Theorem 12) *Let* $X=L_\Phi$, *where* μ *is* $\sigma-$*finite, atomless measure. Then X is approximatively compact if and only if X is reflexive and rotund.*

Theorem 2.14. ([42], Theorem 13) *Let* μ *be non-atomic,* $\mu(\Omega)<\infty$ *and* Φ *be a Musielak-Orlicz function such that* $\frac{\Phi(t,u)}{u}\to 0$ *as* $u\to 0$ *for* $\mu-$*a.e.* $t\in\Omega$. *Then the Musielak-Orlicz space* L_Φ *is fully k-rotund if and only if* $\Phi(t,.)$ *are strictly convex functions for* $\mu-$*a.e.* $t\in\Omega$ *and* $\Phi\in\Delta_2$, $\Phi^*\in\Delta_2$.

2.5 Approximative compactness of Orlicz spaces

Now we will present some criteria for approximative compactness of Orlicz function spaces and Orlicz sequence spaces generated by *N*-functions Φ for both: the Luxemburg $\|.\|_\Phi$ and the Amemiya norm $\|.\|^o_\Phi$. Recall that Orlicz spaces are the special case of Musielak-Orlicz spaces.

In what follows, let us take the Lebesgue measure space (Ω,Σ,m), where $\Omega\subset\mathbb{R}$ and $m(\Omega)<\infty$ and let us define, for any $t\in\Omega$ and $u\in\mathbb{R}$, the Musielak-Orlicz function $\Psi(t,u):=\Phi(u)$, where Φ is an *N*-function on $\mathbb{R}$ (for the definition of the *N*-function we refer to [50]). Then $L_\Psi=L^\Phi$ or $L^A_\Psi=L^\Phi_A=(L^\Phi,\|.\|^o_\Phi)$ is, in fact, an Orlicz space equipped with the Luxemburg or the Amemiya norm (which is equal to the Orlicz norm in general, that is, not only for the *N*-functions -see [44]), respectively.

Recall that $\Phi\in\Delta_2(0)$ [resp. $\Phi\in\Delta_2(\infty)$] iff $\limsup\frac{\Phi(2u)}{\Phi(u)}<\infty$ as $u\to 0$ [resp. $u\to\infty$]. $\Phi\in\Delta_2(\mathbb{R}_+)$ if $\Phi\in\Delta_2(0)$ and $\Phi\in\Delta_2(\infty)$. Φ is said to be strictly convex if for all $u,v\in\mathbb{R}$ with $u\neq v$ it holds that

$$\Phi\left(\frac{u+v}{2}\right)<\frac{\Phi(u)+\Phi(v)}{2}.$$

Let us denote by Φ^* the function complementary to Φ in the sense of Young.

Theorem 2.15. ([45], Theorem 1) *The Orlicz sequence space* $(\ell^\Phi,\|.\|_\Phi)$ *is approximatively compact if and only if* $\Phi\in\Delta_2(0)$ *and* $\Phi^*\in\Delta_2(0)$.

Theorem 2.16. ([45], Theorem 2 and Remark 1) *The Orlicz function space* $(L^\Phi,\|.\|_\Phi)$ *equipped with the Luxemburg norm is approximatively compact if and only if* Φ *is strictly convex on* $\mathbb{R}$ *and* $\Phi\in\Delta_2(\infty)$ *and* $\Phi^*\in\Delta_2(\infty)$ *in the case of a non-atomic finite Lebesgue measure space while* $\Phi\in\Delta_2(\mathbb{R}_+)$ *and* $\Phi^*\in\Delta_2(\mathbb{R}_+)$ *in the case of a non-atomic infinite Lebesgue measure space.*

Theorem 2.17. ([45], Theorem 3 and Remark 1) *The Orlicz function space* $(L^\Phi,\|.\|^o_\Phi)$ *equipped with the Amemiya norm is approximatively compact if and only if* Φ *is strictly convex on* $\mathbb{R}$ *and* $\Phi\in\Delta_2(\infty)$ *and* $\Phi^*\in\Delta_2(\infty)$ *in the case of a non-atomic finite Lebesgue measure space while* $\Phi\in\Delta_2(\mathbb{R}_+)$ *and* $\Phi^*\in\Delta_2(\mathbb{R}_+)$ *in the case of a non-atomic infinite Lebesgue measure space.*

2.6 Approximative compactness of Orlicz-Lorentz spaces

Let (I,Σ,m) be the Lebesgue measure space with $I=(0,1)$ or $I=(0,\infty)$. Let $\Phi:[0,\infty]\to[0,\infty]$ be an Orlicz function (i.e. a Musielak-Orlicz function which does not depend on the parameter t) and $\omega: I\to(0,\infty)$ be a weight function (i.e. a nonincreasing and locally integrable function with respect to the measure m and such that $\int_0^\infty \omega dm=\infty$ if $I=(0,\infty)$). For any $x\in L^0$, x^* denotes the nonincreasing rearrangement of $|x|$ defined by

$$x^*(t)=\inf\{\lambda>0:\mu_x(\lambda)\le t\}$$

for any $t>0$. (by convention $\inf(\emptyset)=\infty$), where $\mu_x(\lambda)=\mu(\{s\in\Omega:|x(s)|>\lambda\})$ for any $\lambda>0$. The Orlicz-Lorentz function space $\Lambda_{\Phi,\omega}$ is defined by

$$\Lambda_{\Phi,\omega}=\left\{x\in L^0(m):\int_I \Phi(\lambda x^*)\omega dm<\infty \text{ for some } \lambda>0\right\}.$$

In the case of counting measure on $2^{\mathbb{N}}$ the Orlicz-Lorentz sequence space $\lambda_{\Phi,\omega}$ is defined by

$$\lambda_{\Phi,\omega}=\left\{x=(x(k))\in\ell^0:\sum_{k=1}^{\infty}\Phi(\lambda x^*(k))\omega(k)<\infty \text{ for some } \lambda>0\right\}.$$

Here $\omega=(\omega(k))$ is a weight sequence, that is, a nonincreasing sequence of positive reals such that $\Sigma_{k=1}^{\infty}\omega(k)=\infty$. In this case x^* is nothing else but the permutation of $|x|$ such that x^* is nondecreasing sequence if $\operatorname{supp}x=\mathbb{N}$.

The Orlicz-Lorentz function (resp. sequence) space is a symmetric function (resp. sequence) space with the Fatou property, if it is equipped with with the norm

$$\|x\|_\Phi^\omega=\inf\left\{\lambda>0:\rho_\Phi^\omega\left(\frac{x}{\lambda}\right)\le 1\right\},$$

where $\rho_\Phi^\omega(x)=\int_\Omega \Phi(x^*(t))\omega(t)d\mu$ (resp. $\rho_\Phi^\omega(x)=\sum\limits_{n=1}^{\infty}\Phi(x^*(n))\omega(n)$ in the sequence case). The symmetry of the space means the fact that if x and y are equi-measurable, that is, $\mu_x=\mu_y$, then $\|x\|_{\Phi,\omega}=\|y\|_{\Phi,\omega}$.

We say that an Orlicz function Φ satisfies condition $\Delta_2(\mathbb{R}_+)$ ($\Phi\in\Delta_2(\mathbb{R}_+)$ for short) if there is a positive constant K such that the inequality $\Phi(2u)\le K\Phi(u)$ holds for all $u>0$. We say that an Orlicz function Φ satisfies condition Δ_2 at zero ($\Phi\in\Delta_2(0)$ for short) [resp. Δ_2 at infinity ($\Phi\in\Delta_2(\infty)$ for short)] if there are positive constants K and a such that $\Phi(a)>0$ [resp. $\Phi(a)<\infty$] and the inequality $\Phi(2u)\le K\Phi(u)$ holds for all $u\in[0,a]$ [resp. for all $u\ge a$].

Theorem 2.18. ([42], Theorem 15)

(i) If Φ is an Orlicz function vanishing only at zero and $\omega: I\to\mathbb{R}_+$ is a weighted function that is strictly decreasing on I, then the Orlicz-Lorentz space $\Lambda_{\Phi,\omega}$ is approximatively compact if and only if $\Lambda_{\Phi,\omega}$ is reflexive, that is, Φ and Φ^* satisfy condition $\Delta_2(\infty)$ if $m(I)<\infty$ and condition $\Delta_2(\mathbb{R}_+)$ if $m(I)=\infty$, and $\int_I\omega(t)dm=\infty$ if $m(I)=\infty$.

(ii) If Φ is an Orlicz function vanishing only at zero and (ω_n) is a weighted sequence from c_0, then the Orlicz-Lorentz space $\lambda_{\Phi,\omega}$ is approximatively compact if and only if Φ and Φ^* satisfy condition $\Delta_2(0)$ and $\sum\limits_{n=1}^{\infty}\omega_n=\infty$.

Recall also the following characterization of the Kadec-Klee property in the Orlicz-Lorentz sequence spaces.

Theorem 2.19. ([42], Theorem 14) *Suppose that $\Phi:\mathbb{R}_+\to\mathbb{R}_+$ is a convex function. Then the Orlicz-Lorentz sequence space $\lambda_{\Phi,\omega}$ has the Kadec-Klee property if and only if*

$$a(\Phi):=\sup\{u>0:\Phi(u)=0\}=0,\ \Phi\in\Delta_2(0),\ \sum_{n=1}^{\infty}\omega_n=+\infty.$$

2.7 Monotonicity properties of Banach lattices and their relationships to dominated best approximation problems

Let us come back to the best approximation problems from the beginning of Section 2. If $X=(X,\|.\|)$ is a Banach lattice, $A\subset X$, $A\neq\emptyset$, $x\in X$ and $x\leq A$ (that is $x\leq y$ for any $y\in A$) or $x\geq A$, then the projection $P_A:X\to A$ can be called the dominated projection and the problems of the existence of $P_A(x)$ for any $x\in X$, its non-emptiness as well as its uniqueness can be called dominated best approximation problems.

It is worth noticing that in this special case, the role of rotundity of Banach spaces plays the notion of strict monotonicity of Banach lattices X and the role of reflexivity of Banach spaces plays the notion of order continuity of Banach lattices of X.

Also uniform rotundity and local uniform rotundity of Banach spaces have their counterparts in Banach lattices. The counterpart of uniform rotundity is the uniform monotonicity and local uniform rotundity have two different counterparts called lower local uniform monotonicity and upper local uniform monotonicity. Also these notions are useful in the dominated best approximation problems.

We refer the readers interested in these problems to the original papers, like for example: [10,11,13,14,20,30,31,38,39,41,43,49,51] and to the survey paper [32].

2.8 Proximinality of some subspaces of Calderón-Lozanovskiĭ spaces E_φ

In the sequel of this subsection, (Ω,Σ,μ) is a complete $\sigma-$finite measure space and $L^0=L^0(\Omega,\Sigma,\mu)$ denotes the space of all (equivalence classes of) $\Sigma-$measurable real functions defined on Ω. Recall that L^0 is a complete vector lattice with the order $x\leq y$ if and only if $x(t)\leq y(t)$, $\mu-$a.e. in Ω.

In this subsection, by a Banach function space (Köthe space) over (Ω,Σ,μ) we understand a Banach space $(E,\|.\|)$ with $E\subset L^0(\Omega,\Sigma,\mu)$ satisfying the following two conditions:

1. $E\subseteqq L^0$ and if $x\in E$, $y\in L^0$ and $|y|\leq|x|$ $\mu-$a.e., then $y\in E$ and $\|y\|_E\leq\|x\|_E$.

2. If $A \in \Sigma$ with $0 < \mu(A) < +\infty$ then $\mathbf{1}_A \in E \setminus \{0\}$.

We say that $x \in E$ is order continuous (o-continuous for short) if for every downward directed set $\{x_i\}_{i\in I}$ in E such that $x_i \downarrow 0$ and $x_i \le |x|$ $\mu-$a.e., we have $\|x_i\|_E \downarrow 0$. Denote by E_a the closed ideal of o-continuous elements of E. If $E = E_a$, then E is called o-continuous. We say that E has the Fatou property if $x_n \in E$, $x \in L^0$, $0 \le x_n \uparrow x$, in order, and $\sup_n \|x_n\|_E < \infty$ imply $x \in E$ and $\|x\|_E = \lim_{n\to\infty} \|x_n\|_E$. In the sequel we assume that E is order continuous and that it has the Fatou property.

Although considerations in [36] were led not only for convex functions φ, we restrict ourselves within this and the next subsection to convex Orlicz functions φ only (i.e. $\varphi: \mathbb{R} \to [0,+\infty]$ is called in this subsection and the next one an Orlicz function if φ is even, convex, non-decreasing and left continuous for $x > 0$, $\Phi(0) = 0$ and $\Phi(x) \to \infty$ as $x \to \infty$). Since on any open interval every convex and finite-valued function is continuous, the left continuity of the Orlicz function φ means that $\lim_{u \to b(\varphi)_-} \varphi(u) = \varphi(b(\varphi))$.

For an Orlicz function φ, let us define

$$a(\varphi) := \sup\{t \ge 0 : \varphi(t) = 0\}, \ \ b(\varphi) := \sup\{t \ge 0 : \varphi(t) < \infty\}.$$

If $b(\varphi) = 0$, we say that φ is degenerated. Unless stated otherwise, we assume that φ is not degenerated. Consider on L^0 the modular

$$\rho(x) := \begin{cases} \|\varphi(x)\|_E, & \text{if } \varphi(x) \in E \\ \infty, & \text{otherwise} \end{cases}$$

(see [36]) and define

$$E_\varphi = \{x \in L^0 : \varphi(\lambda x) \in E \text{ for some } \lambda > 0\}.$$

Let us notice that if $b(\varphi) = 0$, then $E_\varphi = \{0\}$. The space E_φ is considered as a Banach function lattice under the Luxemburg norm $\|.\|_L$ as well as under the Amemiya norm, denoted in this subsection, following [36], by $\|.\|_o$. These norms are defined analogously as in the previous subsections of this section.

Remark 2.8. It is known that the space $(E_\varphi, \|.\|_L)$ is isometric to the Calderón-Lozanovskiĭ space $\Psi(L^\infty(\Omega,\Sigma,\mu), E)$, with $\Psi(s,t) = s\varphi^{-1}(t/s)$, $s,t \in \mathbb{R}_+$, φ^{-1} being the generalized inverse of φ. Recall that, if $\Psi : [0,+\infty)^2 \to [0,+\infty)$ is a homogeneous function, separately concave and such that both functions $\Psi(x,.)$, $\Psi(.,y)$ vanish only at 0, to any couple (E_0, E_1) of Banach function lattices on Ω we can associate the Calderón-Lozanovskiĭ space $\Psi(E_0, E_1)$ (see [6], [59]). It is the Banach function lattice that consists of all $x \in L^0$ such that

$$|x| \le \lambda \Psi(|x_0|, |x_1|) \tag{2.9}$$

holds $\mu-$a.e. for some $\lambda > 0$ and $x_0 \in B_{E_0}$, $x_1 \in B_{E_1}$ with the norm $\|x\|_{\Psi(E_0,E_1)} =$ infimum of the set of $\lambda > 0$ for which there exists $x_i \in B_{E_i}$, $i = 0,1$, such that the inequality (2.9) holds true. In the case when $E_0 = L^\infty(\Omega,\Sigma,\mu)$ and $E_1 = E$, this norm coincides with the Luxemburg norm generated by the modular ρ.

For the Calderón-Lozanovskiĭ space E_φ and an ideal S in E_φ let us define

$$H(S):=\left\{x\in E_\varphi:\forall\lambda>0\,\exists s\in S \text{ such that } \rho\left(\frac{x-s}{\lambda}\right)<+\infty\right\}.$$

A function $f\in L^0$ is said to be a real simple function if $f=\sum_{i=1}^n x_i\cdot\mathbf{1}_{A_i}$, where $x_i\in\mathbb{R}$ and $A_i\in\Sigma$ with $\mu(A_i)<\infty$. Denote by S_o the ideal generated in L^0 by the real simple functions (which in the sequence case means that S_o consists of all sequences having the finite number of coordinates different from 0).

Theorem 2.20. ([36], Theorem 3.2) *Let (Ω,Σ,μ) be a σ-finite complete measure space, E a Banach function lattice over Ω which is order continuous and has the Fatou property, S_o the ideal in L^0 generated by the real simple functions from L^0 and let φ be an Orlicz function. Then*

1. *If φ is finite (i.e. $\varphi(\mathbb{R})\subset[0,\infty)$), then $(E_\varphi)_a=H(\{0\})$ and $H(\{0\})$ is proximinal in $(E_\varphi,\|.\|_L)$.*
2. *If φ is infinite, then $H(S_o)$ is proximinal in $(E_\varphi,\|.\|_L)$.*

If $f:\Omega\to\mathbb{R}$ is a function and $\varepsilon>0$, define for any $t\in\Omega$,

$$f_\varepsilon(t)=\begin{cases} f(t), & \text{if } f(t)\geq\varepsilon\\ 0, & \text{otherwise.}\end{cases}$$

If φ is an Orlicz function, then we say that $((\Omega,\Sigma,\mu),E,\varphi)$ satisfies the $\Delta_2^\infty S_o$-condition whenever for any $f\in E_\varphi$ and any $\varepsilon>0$, we have that $f_\varepsilon\in H_\varphi(S_o):=\{x\in E_\varphi:\forall\lambda>0,\exists s\in S_o$ such that $\rho_\varphi\left(\frac{x-s}{\lambda}\right)<+\infty\}$. If φ is degenerated, then the $\Delta_2^\infty S_o$-condition is always satisfied.

In the following theorem we characterize the $\Delta_2^\infty S_o$-condition.

Theorem 2.21. ([36], Theorem 3.4) *Let (Ω,Σ,μ) be a σ-finite complete measure space, E a Banach function lattice over Ω which is order continuous and has the Fatou property, S_o the ideal in L^0 generated by the real simple functions from L^0 and let φ be a finite Orlicz function such that $a(\varphi)=0$. Then, we have*

(a) If μ is not purely atomic, the $\Delta_2^\infty S_o$-condition is satisfied if and only if $\varphi\in\Delta_2^\infty$.

(b) If μ is purely atomic and there exists $c>0$ such that, for each atom A, we have $\|\mathbf{1}_A\|_E\geq c$, the $\Delta_2^\infty S_o$-condition is always satisfied.

Theorem 2.22. ([36], Theorem 3.3) *Let (Ω,Σ,μ) be a σ-finite complete measure space, E a Banach function lattice over Ω which is order continuous and has the Fatou property, S_o the ideal in L^0 generated by the real simple functions from L^0 and let φ be an Orlicz function. Then the following are equivalent:*

1. *$H(S_o)$ is proximinal in $(E_\varphi,\|.\|_o)$.*

2. *One of the following two conditions holds:* $H(S_o)$ *is trivially proximinal (i.e.* $H(S_o)=E_\varphi$*) or* $a(\varphi)>0$ *and, if* ϕ *is the Orlicz function (possibly degenerated) such that* $\phi(t)=\varphi(t+a(\varphi))$*,* $t>0$*, then the triple* $((\Omega,\Sigma,\mu),E,\phi)$ *satisfies the* $\Delta_2^\infty S_o$*-condition.*

Corollary 2.6. ([36], Corollary 3.1) Let (Ω,Σ,μ) be a σ-finite complete measure space, E a Banach function lattice over Ω which is order continuous and has the Fatou property and let φ be a finite Orlicz function. Then, we have:

(a) If μ is not purely atomic, $H(S_o)$ is proximinal in $(E_\varphi,\|.\|_o)$ if and only if $H(S_o)$ is trivially proximinal or $a(\varphi)>0$ and $\varphi\in\Delta_2^\infty$.

(b) If μ is purely atomic and there exists $c>0$ such that for each atom A, we have $\|\mathbf{1}_A\|_E\geq c$, $H(S_o)$ is proximinal in $(E_\varphi,\|.\|_o)$ if and only if $H(S_o)$ is trivially proximinal or $a(\varphi)>0$.

Theorem 2.23. ([36], Theorem 3.5) *Let* (Ω,Σ,μ) *be a* σ*-finite complete measure space, E a Banach function lattice over* Ω *which is order continuous and has the Fatou property and let* φ *be an Orlicz function with* $0=a(\varphi)<b(\varphi)<\infty$*. The following are equivalent:*

1. *The condition* $\Delta_2^\infty S_o$ *is satisfied.*

2. *If* $A\in\Sigma$ *and* $\mathbf{1}_A\in E_\varphi$*, then* $\mu(A)<\infty$*.*

3. *For any* $\varepsilon>0$ *and any* $h\in E_\varphi$ *we have that* $h_\varepsilon\in S_o$*.*

Hence, if $\mu(\Omega)<\infty$*, the condition* $\Delta_2^\infty S_o$ *is satisfied.*

Corollary 2.7. ([36], Corollary 3.2) Let (Ω,Σ,μ) be a σ-finite complete measure space, E a Banach function lattice over Ω which is order continuous and has the Fatou property and let φ be an Orlicz function with $b(\varphi)<\infty$. Then the following are equivalent:

1. $H(S_0)$ is proximinal in $(E_\varphi,\|.\|_o)$.

2. One of the following two conditions are satisfied: (a) or (b), where

   (a) $H(S_o)$ is trivially proximinal.

   (b) $a(\varphi)>0$ and one of the following two conditions are satisfied: $\mathbf{1}_A\in E_\phi$, $A\in\Sigma$, implies that $\mu(A)<\infty$, or ϕ is degenerated, ϕ being the Orlicz function such that $\phi(t)=\varphi(t+a(\varphi))$, $t>0$.

Remark 2.9. In this subsection we have assumed that the Banach function lattice E satisfies the condition $1_A\in E\backslash\{0\}$ whenever $A\in\Sigma$ and $0<\mu(A)<\infty$. This condition makes the proofs presented in [36] easier, but it is not necessary. Namely, we can drop it and prove all the results using, instead of S_o, the ideal of L^0 generated by the real simple functions of E.

2.9 Proximinality of some subspaces of Orlicz spaces

In this subsection we will present the interpretation of main results of the previous subsection in the case when $E=L^1(\Omega,\Sigma,\mu)$. Recall that if φ is an Orlicz function and $E=L^1(\Omega,\Sigma,\mu)$, then the space E_φ is exactly the Orlicz space $L^\varphi(\Omega,\Sigma,\mu)$.

Theorem 2.24. ([36], Corollary 4.2) *Let (Ω,Σ,μ) be a $\sigma-$finite complete measure space and φ be an Orlicz function. Then:*

1. *If φ is finite, then $(L^\varphi)_a=H(\{0\})$ and $H(\{0\})$ is proximinal in $(L^\varphi,\|.\|_L)$.*
2. *If φ is infinite, then $H(S_o)$ is proximinal in $(L^\varphi,\|.\|_L)$.*

Proof. This is consequence of Theorem 2.20. □

Theorem 2.25. ([36], Corollary 4.3) *Let (Ω,Σ,μ) be a $\sigma-$finite complete measure space and φ be a finite Orlicz function. Then*

(a) If μ is not purely atomic, then $H(\{0\})$ is proximinal in $(L^\varphi,\|.\|_o)$ if and only if $H(\{0\})$ is trivially proximinal or $a(\varphi)>0$ and $\varphi\in\Delta_2^\infty$.

(b) If μ is purely atomic and there exists $c>0$ such that for each atom A we have $\mu(A)\geq c$, $H(\{0\})$ is proximinal in $(L^\varphi,\|.\|_o)$ if and only if $H(\{0\})$ is trivially proximinal or $a(\varphi)>0$.

Proof. This is a consequence of Corollary 2.6. □

Remark 2.10. Concerning Corollary 2.6 and Theorem 2.25, when μ is purely atomic and there exists a sequence of atoms $\{A_n\}_{n\geq 1}$ such that $\mu(A_n)\to 0$, it is difficult to express the condition $\Delta_2^\infty S_o$ in terms of a suitable requirement on φ. Of course, the condition $\varphi\in\Delta_2$ is sufficient but not necessary in general in order that the condition $\Delta_2^\infty S_o$ be satisfied.

Theorem 2.26. ([36], Corollary 4.4) *Let (Ω,Σ,μ) be a $\sigma-$finite complete measure space and φ be an Orlicz function with $b(\varphi)<\infty$. Then the following are equivalent:*

(A) $H(S_o)$ is proximinal in $(L^\varphi,\|.\|_o)$.

(B) $H(S_o)$ is trivially proximinal or $a(\varphi)>0$.

Proof. This follows from Corollary 2.7 and the fact that if $1_A\in L^1$ then $\mu(A)<\infty$. □

Consider the space L^1+L^∞ with the norm

$$\|x\|=\inf\left\{\|x_1\|_1+\|x_2\|_\infty : x=x_1+x_2,\ x_1\in L^1,\ x_2\in L^\infty\right\},$$

for any $x\in L^1+L^\infty$. It is known that if $\varphi:\mathbb{R}\to[0,\infty)$ is defined as

$$\varphi(t)=\begin{cases} 0, & \text{if } |t|\leq 1\\ |t|-1, & \text{if } |t|>1, \end{cases} \tag{2.10}$$

then $(L^1+L^\infty,\|.\|)$ is exactly the Orlicz space $(L^\varphi,\|.\|_o)$. Also, the space $L^1\cap L^\infty$ with the norm $\|x\|=\max\{\|x\|_1,\|x\|_\infty\}$ for any $x\in L^1\cap L^\infty$, coincides with the Orlicz space $(L_\psi,\|.\|_L)$, where

$$\psi(t)=\begin{cases} |t|, & \text{if } |t|\le 1\\ \infty, & \text{if } |t|>1. \end{cases} \tag{2.11}$$

Theorem 2.27. ([36], Corollary 4.5) *Let (Ω,Σ,μ) be a $\sigma-$finite complete measure space. Then*

(A) If φ is defined as in (2.10), *then:*

1. $H(\{0\})=\{x\in L^\varphi:\forall\lambda>0,\int_{|x|>\lambda}(|x|-\lambda)<\infty\}$.
2. *$H(\{0\})$ is proximinal in $(L^\varphi,\|.\|_L)$ and in $(L^\varphi,\|.\|_o)$.*

(B) If ψ is defined as in (2.11), *then $H(S_o)$ is trivially proximinal in L^ψ, i.e. $H(S_o)=L^\psi$.*

Theorem 2.28. ([36], Corollary 4.6) *Let (Ω,Σ,μ) be a $\sigma-$finite complete measure space. Then $H(S_o)=\overline{S_o}$ is proximinal in $(L^\infty,\|.\|_\infty)$.*

Proof. This is a consequence of Theorem 2.20 or Corollary 2.7 (because in $L^\infty=L^\psi$ with ψ defined in (2.11), and the norms $\|.\|_L$ and $\|.\|_o$ coincide in L^ψ). □

If φ is an Orlicz function and I a set, let us denote by $\ell_\varphi(I)$ the Orlicz space

$$\ell_\varphi(I)=\left\{x\in\mathbb{R}^I:\exists\lambda>0,\ \sum_{i\in I}\varphi(x_i/\lambda)<\infty\right\}.$$

Theorem 2.29. ([36], Corollary 4.8) *Let I be a set and φ be a convex Orlicz function. Then the Orlicz space $\ell_\varphi(I)$ satisfies:*

(A) $H(S_o)$ is proximinal in $(\ell_\varphi,\|.\|_L)$.

(B) $H(S_o)$ is proximinal in $(\ell_\varphi,\|.\|_o)$ if and only if $H(S_o)$ is trivially proximinal or $a(\varphi)>0$.

Proof. (A) is the consequence of Theorem 2.24 and (B) follows from Theorems 2.25 and 2.26. □

Remark 2.11. The main results of Subsection 2.8 can be also applied to other interesting classes of spaces, like for example, the Orlicz-Lorentz spaces and the Calderón-Lozanovskiĭ-Musielak spaces E_φ (where φ is now a Musielak-Orlicz function).

3 Applications of geometry of Banach spaces in some problems of generalized inverses

The observation that generalized inverses are like prose (Good Heavens! For more than forty years I have been speaking prose without knowing it - Molière, *Le Bourgois Gentilhomme*) is nowhere truer than in the literature of linear operators. Generalized inverses of integral and differential operators have been studied by Fredholm, Hilbert, Schmidt, Bounitzky, Hurwitz, and others, before E. H. Moore introduced formally the concept of generalized inverses in an algebraic setting, see, e.g., the historical survey by Reid [82].

The theory of generalized inverses has its genetic roots essentially in the context of so called "ill-posed" linear problems. It is well known that if A is a nonsingular (square) matrix, then there exists a unique matrix B, which is called the inverse of A, such that $AB=BA=I$, where I is the identity matrix. If A is a singular or a rectangular (but not square) matrix, no such matrix B exists. If A^{-1} exists, then the system of linear equations $Ax=b$ has the unique solution $x=A^{-1}b$ for each b. On the other hand, in many cases, solutions of a system of linear equations exist even when the inverse of the matrix defining these equations does not exist. Also in the case when the equations are inconsistent, we are often interested in a least-squares solution, i.e., vectors that minimize the sum of the squares of the residuals. These problems, along with many other problems in numerical analysis, numerical linear algebra, linear programming, optimization, optimal control, statistics, game theory, inverse problems, differential equations, and other areas of analysis and applied mathematics, are readily handled via the concept of a generalized inverse (or pseudo inverse) of a matrix or a linear operator [2].

3.1 Geometric properties of Banach spaces related to generalized inverses

The concept of the duality mapping is one of the most important geometric concepts in Banach spaces. Certain geometric or topological properties in some Banach spaces can be characterized by this mapping.

Definition 3.1. ([94]) *If X is a Banach space, then the set-valued mapping $F_X: X \rightrightarrows X^*$ defined by*

$$F_X(x)=\left\{x^*\in X^*:\langle x^*,x\rangle=\|x\|^2=\|x^*\|^2\right\},\quad \forall x\in X$$

is called the duality mapping of X.

It is well known that $F_X(x)\neq\emptyset$. Namely, it is well known that for any $x\in X$ there exists $x^*\in X^*$ such that $\|x^*\|=1$ and $x^*(x)=\|x\|$. Let us note that if $y^*:=\|x\|x^*$ then $\|y^*\|=\|x\|$ and $y^*(x)=\|x\|x^*(x)=\|x\|^2=\|y^*\|^2$, that is, $y^*\in F_X(x)$.

Remark 3.1. (1) F_X is a homogeneous set-valued mapping;

(2) F_X is injective if and only if X is strictly convex;

(3) F_X is surjective if and only if X is reflexive;

(4) F_X is single-valued if and only if X is smooth;
(5) F_X is additive if and only if X is Hilbert space.

Definition 3.2. ([89]) *Let X be a Banach space, $K \subset X$, the set-valued mapping $\mathcal{P}_K : X \rightrightarrows K$ defined by*

$$\mathcal{P}_K(x) = \{y \in K : \|x-y\| = d_K(x)\}, \ \forall x \in X,$$

where $d_K(x) = \inf_{y\in K}\|x-y\|$ is called the metric projection.

Let us recall the following definitions and fix some notations for any $K \subset X$, $K \neq \emptyset$:

1. K is said to be proximinal if $\mathcal{P}_K(x) \neq \emptyset$ for any $x \in X$;

2. K is said to be semi-Chebyshev if $\mathcal{P}_K(x)$ is at most a single point set for each $x \in X$;

3. K is called a Chebyshev set if it is both proximinal and semi-Chebyshev;

4. When K is a Chebyshev set, we denote $\mathcal{P}_K(x)$ by $\pi_K(x)$ for $x \in X$.

The metric projection from X onto $K \subset X$ has some relationships to the structure of the Banach spaces X.

Lemma 3.1. ([94]) *If X is a Banach space, then:*

(1) For any closed convex non-empty set $C \subset X$, we have that $\mathcal{P}_C(x) \neq \emptyset$ for each $x \in X$ if and only if X is reflexive.

(2) For any closed convex set $C \subset X$, the set $\mathcal{P}_C(x)$ is at most a singleton for each $x \in X$ if and only if X is strictly convex.

In 2001 and 2005, Y. W. Wang, H. Wang, H. Hudzik and W. Song generalized the famous Riesz orthogonal decomposition theorem in Hilbert spaces to generalized orthogonal decomposition theorem in Banach spaces, by applying the duality mapping and the metric projection. It is one of key tools for the research on generalized inverse of operators in Banach spaces (see [46], [90]).

Theorem 3.1. ([46]) *Let L be a proximinal subspace of X. Then for any $x \in X$, we have the decomposition*

$$x = x_1 + x_2,$$

where $x_1 \in L$ and $x_2 \in F_X^{-1}(L^\perp)$, where $L^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \forall x \in L\}$ and $F_X^{-1}(L^\perp)$ is the counter-image of $L^\perp$, so in this case we have $X = L + F_X^{-1}(L^\perp)$. If L is a Chebyshev subspace of X, then the decomposition is unique and

$$x = \pi_L(x) + x_2, \ \ x_2 \in F_X^{-1}(L^\perp).$$

In this case we have $X = \pi_L(x) \dot{+} F_X^{-1}(L^\perp)$.

Metric projection is a nonlinear and homogeneous operator in general, which have the following linear and quasi-linear characterizations.

Lemma 3.2. ([94]) *If L is a closed subspace of X, then the following statements are equivalent:*
(i) *π_L is a linear operator;*
(ii) *$\pi_L^{-1}(\theta)$ is a linear subspace of X;*
(iii) *$\pi_L^{-1}(y)$ is a linear manifold of X for every $y\in L$.*

Lemma 3.3. ([94]) *If X is a normed linear space, and L is a subspace of X, then:*
(i) *$\pi_L^2(x)=\pi_L(x)$ for all $x\in\mathcal{D}(\pi_L)$, i.e. π_L is idempotent;*
(ii) *$\|x-\pi_L(x)\|\leq\|x\|$ for all $x\in\mathcal{D}(\pi_L)$.*
Furthermore, if L is a semi-Chebyshev subspace, then:
(iii) *$\pi_L(\alpha x)=\alpha\pi_L(x)$ for all $x\in X$ and $\alpha\in R$, i.e. π_L is homogeneous;*
(iv) *$\pi_L(x+y)=\pi_L(x)+\pi_L(y)=\pi_L(x)+y$ for all $x\in\mathcal{D}(\pi_L)$ and $y\in L$, i.e. π_L is quasi-additive.*

Although single valued metric projection is, as we saw above, nonlinear, it is idempotent, bounded, homogeneous and quasi-additive. These linear characterizations motivate introducing the following definition.

Definition 3.3. ([57]) *Let V be a linear space. A mapping $S:V\to V$ is called a quasi-linear projector on V, if S satisfies the following conditions:*
(1) *S is homogeneous;*
(2) *S is idempotent, i.e. $S^2=S$;*
(3) *S is quasi-additive with respect to $L=R(S)$, where*

$$R(S)=\{v\in V: v=S(u),u\in V\}$$

is the range of S. In this case, we may denote $S=S_L$.

Definition 3.4. ([26]) *A nonempty subset C of X is said to be approximatively compact, if for any sequence $\{x_n\}$ in C and any $y\in X$ such that*

$$||x_n-y||\to dist(y,C):=\inf\{||y-z||:z\in C\},$$

we have that $\{x_n\}$ has a Cauchy subsequence. X is called approximatively compact if any nonempty closed and convex subset of X is approximatively compact.

If a semi-Chebyshev closed subset $C\subset X$ is approximatively compact, then C is a Chebyshev subset and the metric projector π_C is continuous (see [12]). This is the continuity characteristic, which plays a role in building selections of metric generalized inverses.

3.2 Generalized inverses in Banach spaces

The first result for a generalization from finite dimensional spaces to infinite-dimensional spaces for the theory of generalized inverses was done by Y. Y. Tseng. He studied the generalized inverses of unbounded linear operators in Hilbert spaces via orthogonal projector, which was later named the Tseng inverse (see [92]).

There are many research achievements regarding generalized inverses of operators in Hilbert spaces, such as the optimal approximation solution of two-point boundary value problems of linear differential equations, singular integral equations and the singular optimal control problems in Hilbert spaces, ect. ([2, 54, 58, 62]).

Throughout this section, X and Y will denote two real Banach spaces. Let $D(T)$, $R(T)$ and $N(T)$ be the domain, the range and the null space of an operator T, respectively. The space of all bounded linear operators from X to Y is denoted by $B(X,Y)$. We write $H(X,Y), L(X,Y)$ for the space of all bounded homogeneous operators and all linear operators from X to Y, respectively.

3.2.1 Linear generalized inverses of operators in Banach spaces

An operator $T^+ \in B(Y,X)$ is said to be a generalized inverse of an operator $T \in B(X,Y)$ provided that $TT^+T = T$ and $T^+TT^+ = T^+$, which is a generalization of the inverse T^{-1} of T. It is known that T^+ is a linear operator.

In general, an operator $T \in B(X,Y)$ has a generalized inverse in $B(Y,X)$ if and only if $N(T)$ and $R(T)$ are both split, that is, there exist linear subspaces $R^+ \subset X$ and $N^+ \subset Y$ such that the following decompositions of X and Y hold:

$$X = N(T) \oplus R^+, \; Y = R(T) \oplus N^+.$$

The linear subspaces R^+ and N^+ which appeared in the last equalities are called topological complements of $N(T)$ and $R(T)$, respectively.

Since 1970 it was natural and important to extend the notion of generalized inverses of linear operators from Hilbert spaces to Banach spaces due to the applications of the generalized inverses.

With this aim, an advanced seminar was sponsored by the Mathematics Research Center at the University of Wisconsin, Madison, October 8–10, 1973, edited by M. Z. Nashed. Nashed and Votruba [75] present the most comprehensive treatment to date of the general theory of generalized inverses of linear operators. A number of new concepts and results concerning generalized inverses in abstract spaces and related projectional, extremal and proximal properties are discussed themselves in details. A unique algebraic generalized inverse which depends on a choice of projectors with ranges, which are complementary to the range and null space of a given linear transformation, is introduced. Various types of generalized inverses are investigated when the domain space, the range space, or both of them are linear topological spaces. The discussion begins with some helpful advice on the sources and types of difficulties which may arise together with the introduction of topologies. For linear transformations which take a vector space into a topological vector space and which have a range whose closure is topologically complemented, the concept of a right-topological inner inverse is introduced and investigated. When the domain space supports a vector space topology and the range space is an algebraic vector space, a left-topological inner inverse is defined under the assumption that the domain of the transformation satisfies a certain "decomposability" condition. A topological generalized inverse of a linear transformation mapping one topological vector space into another is defined to be a linear transformation which is at the same time an algebraic outer inverse and a left- and right-topological inner inverse. Various types of algebraic generalized inverses, including the Moore-Penrose inverse, the oblique generalized inverse, the weighted generalized inverse and the group generalized inverse are then investigated. Next, generalized inverses of topological homomorphisms in linear topological spaces and generalized inverses in Banach and Hilbert spaces are discussed. Then follows a very nice exposition of generalized inverses in Hilbert spaces put forward by Tseng, Arghiriade, Hestenes and Erdélyi, all of which are subsumed by the unified approach to generalized inverses. The authors of [75] also presented a very general theory of generalized inverses of linear operators between two linear (or linear topological) spaces. New results include an explicit transformation of generalized inverses under change of projectors.

3.2.2 Nonlinear generalized inverses of operators in Banach spaces

For any $T\in L(X,Y)$, an element $x_0\in X$ is said to be an extremal solution of the equation $Tx=y$, if $x=x_0$ minimizes the functional $||T(x)-y||$ on X, that is, $\inf\{\|T(x)-y\|:x\in X\} = \|T(x_0)-y\|$. Any extremal solution with the minimal norm is called the best approximate solution (b.a.s. for short). In 1974, M. Z. Nashed and G. F. Votruba introduced the concept of the metric generalized inverse for linear operators between Banach spaces, which are set-valued operators in general.

Definition 3.5. ([76]) *Let $T\in L(X,Y)$ and consider $y\in Y$ such that $T(x)=y$ has a b.a.s. in $D(T)$. We define*

$$T^{\partial}(y)=\{x\in D(T): x \text{ is a b.a.s. to } T(x)=y\},$$

and call the set-valued mapping $y\mapsto T^{\partial}(y)$ the metric generalized inverse of T. Let

$$D(T^{\partial})=\{y\in Y: T(x)=y \text{ has a b.a.s. in } D(T)\}.$$

A function $T^{\sigma}:D(T^{\partial})\to D(T)$ (in general, nonlinear) such that

$$T^{\sigma}(y)\in T^{\partial}(y),\ \ \forall y\in D(T^{\partial}),$$

is called a selection for the metric generalized inverse.

It is well known that the metric generalized inverses themselves satisfy requirements of the approximate and semi-Chebyshev property which come from the research on the geometric properties of Banach spaces.

Since the nineteen eighties, deep and systematical research on the metric generalized inverses of linear operators in Banach spaces has been conducted by Y. W. Wang and his students. The metric generalized inverses, which are nonlinear generalized inverses, have been investigated, and many useful results were obtained (see [94–96]). For example, the existence of the Tseng metric generalized inverses of linear operators in Banach spaces, characteristic of Moore-Penrose metric generalized inverses and projective left and right inverses of bounded linear operators in Banach spaces were given.

M. Z. Nashed and G. F. Votruba in [76] said that obtaining selections with nice properties for metric generalized inverses merits further study. Bounded homogeneous selections for the set-valued metric generalized inverses of linear operators in Banach spaces were given by H. Hudzik, Y. W. Wang and W. J. Zheng in [46]. In 2012, some continuous homogeneous selections for the set-valued metric generalized inverses of linear operators in Banach spaces by using the methods of geometry of Banach spaces were investigated by H. F. Ma, H. Hudzik and Y. W. Wang in [61].

3.2.3 Perturbation analysis of generalized inverses in Banach spaces

Perturbation analysis of generalized inverses of linear operators in Banach spaces plays an important role in many applications, such as computational mathematics, control theory, optimization frame theory and nonlinear analysis and so on (see [1,8,64,65,74,86,94, 100,101]).

In the case of nonlinear generalized inverses different research methods are needed then in the case of linear generalized inverses. In 2013, perturbation analysis of bounded homogeneous generalized inverses in Banach spaces was introduced by J. B. Cao, Y. F. Xue in [7]. In 2014, using the continuity of the metric projection operators and quasi-additivity of metric generalized inverses, H. F. Ma and Y. W. Wang *et al.* in [63] gave a perturbation analysis of single-valued Moore-Penrose metric generalized inverses for operators between Banach spaces.

In order to give a characterization of the set of all extremal solutions or least–extremal solutions of the linear inclusions, the definition of Moore-Penrose bounded quasi-linear projection generalized inverses in Banach spaces was introduced by P. Liu and Y. W. Wang in [57]. There are bounded quasi-linear projection generalized inverses of bounded linear operators between two reflexive Banach space, which are not only linear but also metric generalized inverses.

In 2016, Z. Wang, B. Y. Wu and Y. Y. Wang obtained a perturbation analysis of Moore-Penrose quasi-linear projection generalized inverses of closed linear operators in Banach spaces (see [98]).

Definition 3.6. ([98]) *Let $T\in L(X,Y)$ be a linear operator from X into Y, and there exist two bounded quasi-linear projectors $S_{\overline{N(T)}}, S_{\overline{R(T)}}$ from X,Y onto $\overline{N(T)},\overline{R(T)}$, respectively. An operator $T^h\in H(Y,X)$, which is quasi-additive with respect to $R(T)\subset Y$, is called the Moore-Penrose bounded quasi-linear projection generalized inverse of T, if the following four equations hold:*

1. *$TT^hT=T$ on $D(T)$;*

2. $T^hTT^h = T^h$ *on* $D(T^h)$;

3. $T^hT = I_X - S_{\overline{N(T)}}$ *on* $D(T)$;

4. $TT^h = S_{\overline{R(T)}}$ *on* $D(T^h)$, *where* $D(T^h) = R(T) \dot{+} S^{-1}_{\overline{R(T)}}(\theta)$.

Theorem 3.2. ([98]) *Let* X,Y *be Banach spaces,* $T \in C(X,Y)$ *with* $\overline{D(T)} = X$ *be a closed linear operator with closed range* $R(T)$ *such that the Moore-Penrose bounded quasi-linear projection generalized inverse* T^h *of* T *exists,* $T^h \in H(Y,X)$, *and* $\delta T \in L(X,Y)$ *with* $D(T) = D(\delta T)$, $N(T) \subset N(\delta T)$, $R(\delta T) \subset R(T)$, δT *being* $T-$*bounded, i.e.*

$$||\delta Tu|| \le a||u|| + b||Tu|| \quad \forall u \in D(T),$$

with some nonnegative a,b. *Suppose also that the inequality*

$$a\|T^h\| + b\|S_{R(T)}\| < 1$$

holds, where $S_{R(T)}$ *is the bounded quasi-linear projection onto* $R(T)$. *Let* $\overline{T} = T + \delta T$ *be the perturbation operator. Then we have the following:*

1. *The perturbation is stable, i.e.,* $\overline{T} = T + \delta T$ *is closed, and* $R(\overline{T}) = R(T)$, $N(\overline{T}) = N(T)$;

2. *The bounded quasi-linear operator* $\Psi = T^h(I_Y + \delta TT^h)^{-1} = \left(I_{D(T)} + T^h\delta T\right)^{-1} T^h$ *is the Moore-Penrose bounded quasi-linear projection generalized inverse* $\overline{T}^h$ *of* $\overline{T}$,*i.e.,*

$$\overline{T}^h = T^h\left(I_Y + \delta TT^h\right)^{-1} = \left(I_{D(T)} + T^h\delta T\right)^{-1} T^h.$$

3. *There hold the following two inequalities*

$$||\overline{T}^h|| \le \frac{||T^h||}{1 - ||\delta TT^h||} \le \frac{||T^h||}{1 - a||T^h|| - b\|S_{R(T)}\|}, \tag{3.1}$$

$$||\overline{T}^h - T^h|| \le \frac{||T^h||||\delta TT^h|}{1 - \|\delta TT^h\|} \le \frac{||T^h||(a||T^h|| + b\|S_{R(T)}\|)}{1 - a||T^h|| - b\|S_{R(T)}\|}. \tag{3.2}$$

From the main perturbation theorem, we may get the perturbation theorem for the oblique projection generalized inverse under the conditions that δT is $T-$bounded with $a||T^+|| + b||Q|| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$:

Corollary 3.1. ([99]) *Let* X,Y *be Banach spaces, let* $T \in C(X,Y)$ *with* $\overline{D(T)} = X$, *be a closed linear operator such that the bounded oblique generalized inverse* T^+ *of* T *exists,* $T^+ \in B(Y,X)$, *and* $\delta T \in L(X,Y)$ *with* $D(T) = D(\delta T)$, $N(T) \subset N(\delta T)$, δT *be* $T-$*bounded, i.e.*

$$||\delta Tu|| \le a||u|| + b||Tu|| \qquad \forall u \in D(T)$$

with some nonnegative a,b. Under these assumptions, if

$$a\left|\left|T^+\right|\right|+b||Q||<1\ ,$$

where Q is the continuous linear projection of Y onto $R(T)$ and if $\overline{T}=T+\delta T$ is the perturbation operator, then

(i) $\overline{T}=T+\delta T$ *is closed, and* $R\left(\overline{T}\right)=R(T)$, $N\left(\overline{T}\right)=N(T)$;

(ii) $\overline{T}^+\in B(Y,X)$, *the oblique projection generalized inverse of* $\overline{T}$ *exists,* $\overline{T}^+=T^+\left(I_Y+\delta TT^+\right)^{-1}$, *as well as*

$$||\overline{T}^+||\leq\frac{||T^+||}{1-a||T^+||-b|Q|},$$
$$\frac{||\overline{T}^h-T^h||}{T^h}\leq\frac{a||T^+||+b||Q||}{1-a||T^+||-b||Q||}.$$

Definition 3.7. ([93]) *Let $T\in L(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If there exists a homogeneous operator $T^M:D(T^M)\to D(T)$ such that:*

1. $TT^MT=T$ *on* $D(T)$.
2. $T^MTT^M=T^M$ *on* $D(T^M)$.
3. $T^MT=I_{D(T)}-\pi_{\overline{N(T)}}$ *on* $D(T)$.
4. $TT^M=\pi_{\overline{R(T)}}$ *on* $D(T^M)$,

then T^M is called the Moore-Penrose metric generalized inverse of T, where $I_{D(T)}$ is the identity operator on $D(T)$ and $D(T^M)=R(T)\dot{+}F_Y^{-1}(R(T)^\perp)$.

Corollary 3.2. ([63]) *Let X,Y be Banach spaces, $T\in B(X,Y)$, $\delta T\in B(X,Y)$ and $\overline{T}=T+\delta T$. Assume that $N(T)$ is a Chebyshev subspace of X, $R(T)$ is a Chebyshev subspace of Y, if $||T^M||||\delta T||<1$, $N(T)\subset N(\delta T)$, $R(\delta T)\subset R(T)$. Then*

(1) $R\left(\overline{T}\right)=R(T)$, $N\left(\overline{T}\right)=N(T)$;

(2) $\overline{T}^M=T^M\left(I_Y+\delta TT^M\right)^{-1}=\left(I_{D(T)}+T^M\delta T\right)^{-1}T^M$;

(3) The following inequalities hold:

$$||\overline{T}^M||\leq\frac{||T^M||}{1-||\delta TT^M||}\leq\frac{||T^M||}{1-||\delta T||||T^M||},$$

$$||\overline{T}^M-T^M||\leq\frac{||T^M||||\delta TT^M||}{1-||\delta TT^M||}\leq\frac{||T^M||^2||\delta T||}{1-||\delta T||||T^M||}.$$

The main perturbation theorem extended the result of Ma and Wang [63]. Some well-known results from [101] can be obtained from the following Corollary.

Corollary 3.3. ([98]) Let $T\in B(Xs,Y)$ satisfy the equality $D(T)=X$ and let the oblique projection generalized inverse T^+ belong to $B(Y,X)$. Let $\delta T\in B(X,Y)$ satisfy the inequality $||\delta T||\cdot||T^+||<1$ and let $N(T)\subset N(\delta T)$. Then $R(\overline{T})$ is closed and the operator $B=T^+(I+\delta TT^+)^{-1}$ is an oblique projection generalized inverse of $\overline{T}$.

3.2.4 Extremal solutions and optimal approximation solutions of multi-valued linear operators

From the beginning of nineteen eighties to the end of the nineteen nineties of the previous century, linear operators and orthogonal generalized inverses of multi-valued operators in Hilbert spaces and some applications in nonlinear ill-posed operator equations and numerical approximation were systematically studied by Nashed, Lee, Chen, Engle and Craven *et al.* (see [16,27,28,72–74]).

In 1989, some structural properties of the solution sets of constrained minimization problems and a characteristic of their existence in Hilbert spaces were investigated by Lee and Nashed in [54]. However, these research could not be extended to Banach spaces. The main difficulty to do this in Banach spaces is the nonlinearity of the metric generalized inverses. In 2001, the extremal solution and the best-approximate solution of linear operator equations in Banach spaces were investigated by Wang and Wang in [93]. In 2005, Wang and Liu introduced the metric generalized inverses of multi-valued linear operators in Banach spaces. Next, the extremal solution of linear inclusions in Banach spaces was described in [96]. In 2012, criteria for the single-valued metric generalized inverses of multi-valued linear operators in Banach spaces were established by Wang and Zhang.

In 2016, extremal solutions of multi-valued linear inclusions in Banach spaces were first constrained by Wang, Wu and Wang in [97]. By using the extremal solution of some interrelated multi-valued linear inclusions in the same spaces, the constrained extremal solution of multi-valued linear inclusions could be given, which include a class of constrained extremal problems and optimal control problems subject to generalized boundary conditions.

Definition 3.8. ([96]) *Let X and Y be Banach spaces, $A\subset X\times Y$ be a linear manifold, $N(A)$ and $R(A)$ be Chebyshev subspaces of X and Y, respectively, $\pi_{N(A)}:X\to N(A)$ and $\pi_{R(A)}:Y\to R(A)$ be the metric projectors. The metric generalized inverse $A^\#$ of A is defined by*

$$A^\#=\Big\{\Big\{y,\Big(I_{D(A)}-\pi_{N(A)}\Big)(g)\Big\}:y\in Y \text{ and } \Big\{g,\pi_{R(A)}(y)\Big\}\in M\Big\}.$$

If both X and Y are Hilbert spaces, the metric generalized inverse $A^\#$ of A is just the orthogonal generalized inverse (see [52–55]). If X and Y are Banach spaces, $T:X\to Y$ is a linear operator, and $N(T)$, $R(T)$ are Chebyshev subspaces in X and Y, respectively, then $T^\#$ is just the Moore-Penrose metric generalized inverse of T, denoted by T^M.

Theorem 3.3. ([97]) *Let X and Y be Banach spaces, $L\subset X\times Y$ be a multi-valued linear operator from X to Y, N be a subspace of X, P be an algebraic projector from Y onto the subspace $L(\theta)$, $L_{S,P}$ be any fixed algebraic operator part of L with respect to the projector P. Let*

$$S=g+N \quad and \quad A:=L|_N,$$

where $g\in D(L)$. Suppose also that $N(A)$ and $R(A)$ are Chebyshev subspaces of X and Y, respectively. Then for any $y\in Y$, the set of all constrained extremal solutions of the linear inclusion $y\in L(x)$ with respect to S, denoted by Ω_y, is not empty and is given by

$$\Omega_y=\{g-A^{\#}[L_{S,P}(g)-y]\}\dot{+}N(A), \tag{3.3}$$

where $A^{\#}$ is the metric generalized inverse of the multi-valued linear operator A, $A=L|_N$ and $S=g+N$.

Corollary 3.4. ([54]) *Let H_1 and H_2 be Hilbert spaces and let $L\subset H_1\times H_2$ and $N\subset H_1$ be linear manifolds. Let $L_{S,P}$ be an arbitrary, but fixed algebraic operator part of L, corresponding to an algebraic projector P of H_1 onto $L(\theta)$ and let*

$$S:=g\dot{+}N \text{ and } M:=L|_N,$$

where $g\in D(L)$. Then we have the following

I. for a fixed $h\in H_2$, the following statements are equivalent:

(i) w is a restricted least-squares solution (LSS) of the linear inclusion $y\in L(x)$ with respect to S.

(ii) $k:=g-w$ is an LSS of

$$L_{S,P}(g)-h\in M(x).$$

(iii) $w\in S\cap D(L)$ and

$$L_{S,P}(g)-h\in L(\theta)+N(M^*),$$

where $M^*:=\{(x,y):(-y,x)\in M^{\perp}\}$ is the adjoint subspace of the linear manifold $M\subset H_1\times H_2$, and $M^{\perp}$ is the orthogonal complement of M in the Hilbert space $H_1\times H_2$.

(iv) $g\in D(L)$ is such that

$$L_{S,P}(g)-h\in R(M)\dot{+}N(M^*).$$

In particular, if $R(M)$ is closed, then for each $h\in H_2$ a restricted LSS exists.

II. the set of all restricted LSS of the linear inclusion $y\in L(x)$ with respect to S, denoted by Ω_y, is not empty and is given by

$$\Omega_y=\{g-M^{\#}[L_{S,P}(g)-y]\}\dot{+}N(M).$$

4 On the modulus of nearly uniform smoothness in Orlicz sequence spaces and application to fixed point property

Through the last century, the fixed point property has been studied by many scholars. Recall that a Banach space X is said to have the fixed point property (FPP, for short) if every nonexpansive mapping $T:C\to C$, i.e. a mapping T such that

$$\|Tx-Ty\|\leq\|x-y\|,\forall x,y\in C,$$

acting from a nonempty bounded closed and convex subset C of X into itself has a fixed point. A natural generalization of FPP is the weak fixed point property (WFPP, for short). A Banach space X is said to have the WFPP, whenever it satisfies the above condition from the definition of FPP with"weakly compact" in place of "bounded closed". In 1965, Browder [5] and Göhde [35] proved independently that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. This result was also obtained by Kirk [48], under a slightly weaker in a technical sense assumptions. Another, more geometric and elementary in its nature proof, has been given by Goebel [34]. In 1965, Kirk [48] proved that any reflexive Banach space with the normal structure has the FPP and he also asserts that a Banach space with the weak normal structure has the WFPP. In 1989, S. Prus [79] introduced the geometric property of a Banach space X, called nearly uniform smoothness and he proved that a Banach space is nearly uniformly convex if and only if its dual space X^* is nearly uniformly smooth. In 1992, S. Prus [78] also proved that weakly nearly uniformly smooth Banach spaces with the weak Opial property have the FPP. In [24], Domínguez-Benavides proved an important result on the existence of fixed points for nonexpansive mappings. In order to do this, for a Banach space X and a nonnegative real number a, he defined the parameter

$$R(a,X):=\sup\left\{\liminf_{n\to\infty}\|x_n+x\|\right\},$$

where the supremum is taken over all x $\in X$ with $\|x\|\leq a$ and all weakly null sequences (x_n) in $B(X)$ such that $[(x_n)]:=\limsup_{n\to\infty,m\to\infty}\|x_n-x_m\|\leq 1$. He also defined the coefficient

$$M(X):=\sup\left\{\frac{1+a}{R(a,X)}:a\geq 0\right\}.$$

The result that a Banach space X has the WFPP, whenever $M(X)>1$ was also obtained in [24], by using of an embedding of $l_\infty(X)/c_0(X)$.

4.1 Preliminaries

Throughout this section, X is a Banach space which is assumed not to have the Schur property because in such a space there exists a weakly convergent sequence that is not norm convergent. By $S(X)$ and $B(X)$ we denote the unit sphere and the unit ball of X, respectively, and l^0 denotes the set of all real sequences.

Inspired by the underlying ideas from the definition of nearly uniformly smooth Banach spaces given by Prus [79] as well as on the modulus of uniform smoothness, Domínguez-Benavides defined in [25] the modulus of nearly uniform smoothness.

We start with some definitions which are used in this section.

Definition 4.1. ([56]) *A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. A sequence $(x_n)_{n=1}^{\infty}$ which is a Schauder basis of its closed linear span is called a basic sequence.*

Definition 4.2. *The modulus of nearly uniform smoothness is defined by*

$$\Gamma_X(t) := \sup\left\{ \inf_{n>1} \left(\frac{\|x_1 + t x_n\| + \|x_1 - t x_n\|}{2} - 1 \right) \right\},$$

where the supremum is taken over all basic sequences (x_n) in $B(X)$.

It is shown also in [25] that a Banach space X is nearly uniformly smooth if and only if X is reflexive and

$$\Gamma'_X(0) = \lim_{t \to 0^+} \frac{\Gamma_X(t)}{t} = 0,$$

where

$$\Gamma_X(t) := \sup\left\{ \inf_{n>1} \left(\frac{\|x_1 + t x_n\| + \|x_1 - t x_n\|}{2} - 1 \right) \right\},$$

with the supremum taken over all weak null sequences (x_n) in $B(X)$.

In this section we will use Orlicz functions Φ that have only finite values and satisfy additionally the following conditions: Φ vanishes only at zero, i.e.,

$$\lim_{u \to 0} \frac{\Phi(u)}{u} = 0, \quad \lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty.$$

Such Orlicz functions are called N-functions (see [9, 50, 66, 69, 81]).

Definition 4.3. *For any Orlicz function Φ we define its complementary function $\Psi : \mathbb{R} \to [0, \infty)$ by the formula*

$$\Psi(v) := \sup_{u>0} \{ u|v| - \Phi(u) \}$$

for any $v \in \mathbb{R}$.

Definition 4.4. *We say an Orlicz function Φ satisfies the δ_2-condition ($\Phi \in \delta_2$ for short) if there exist $k>0$ and $u_0>0$ such that the inequality*

$$\Phi(2u) \leq k\Phi(u)$$

holds whenever $|u| \leq u_0$. We say that an Orlicz function Φ satisfies the $\overline{\delta}_2$-condition ($\Phi \in \overline{\delta}_2$ for short) if its complementary function Ψ satisfies the δ_2-condition.

The Orlicz sequence space l_Φ is defined in the following way

$$\ell_\Phi := \left\{ x \in \ell^0 : I_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda x(i)) < \infty \text{ for some } \lambda > 0 \right\}$$

and it will be assumed in this paper that it is equipped with the Amemiya-Orlicz norm

$$\|x\|_o := \inf\left\{ k>0 : \frac{1}{k}(1+I_\Phi(kx)) \right\}.$$

The space

$$h_\Phi = \left\{ x \in \ell^0 : \ I_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda x(i)) < \infty \ \text{ for all } \lambda > 0 \right\}$$

equipped with the Amemiya-Orlicz norm induced from ℓ_Φ, is also a Banach space and it is a closed subspace of ℓ_Φ.

It is well known that

$$K(x) = \left\{ k>0 : \frac{1}{k}(1+I_\Phi(kx)) = \|x\|_o \right\} \neq \emptyset$$

for any nonzero element $x \in \ell_\Phi$ (see [9]).

More basic information about Orlicz spaces can be found in [9,17,50,66,69,81].

Recall also that a Köthe sequence space X is said to have the semi-Fatou property if for every sequence (x_n) in X and $x \in X$ such that $0 \leq x_n \uparrow x$, we have $\|x_n\| \to \|x\|$.

4.2 Results

In what follows, the Amemiya-Orlicz norm in Orlicz sequence spaces will be denoted for simplicity by $\|.\|$ instead of $\|.\|_o$. Now, we start to present some results.

Theorem 4.1. *If $\Phi \notin \delta_2$ and ℓ_Φ is equipped with the Amemiya-Orlicz norm, then $\Gamma_{\ell_\Phi}(t) = t$ for any $t>0$.*

Proof. If $\Phi \notin \delta_2$ then h_Φ is a proper closed subspace of ℓ_Φ. By the Riesz Lemma, for any $\varepsilon \in (0,1)$ there exists $x \in S(\ell_\Phi)$ such that

$$d(x,h_\Phi) = \inf\{\|x-y\| : y \in h_\Phi\} > 1-\varepsilon.$$

Using Theorem 1.43 in [9] (see also Theorem 2.7 in Section 2), we have

$$d(x,h_\Phi) = \lim \|(0,\cdots,0,x(n+1),x(n+2),\cdots)\|,$$

so by the semi-Fatou property, there are natural numbers $n_1 < n_2 < \cdots$, such that

$$\left\|\sum_{i=1}^{n_{j+1}-n_j} x(n_j+i)e_i\right\| \geq 1-\varepsilon$$

for each $j \in N$. Put

$$x_j = \sum_{i=1}^{n_{j+1}-n_j} x(n_j+i)e_i - \sum_{i=1}^{n_{j+2}-n_{j+1}} x(n_j+i)e_i.$$

Then for any $f \in l_{(\Psi)}$, we have

$$f(x_j) = \sum_{i=1}^{n_{j+1}-n_j} x(n_j+i)f(i) \to 0$$

thanks to

$$f(x) = \sum_{i=1}^{\infty} x(i)f(i) < \infty$$

as well as to the fact that

$$\varphi(x_j) = 0$$

for any singular functional $\varphi \in (\ell_\Phi)^*$ because singular functionals from $(\ell_\Phi)^*$ vanish on the subspace h_Φ and $x_j \in h_\Phi$ for any $j \in \mathbb{N}$. This shows that (x_j) is a weakly null sequence in l_Φ. Moreover,

$$\|x+tx_j\| \geq (1+t)\left\|\sum_{i=1}^{n_{j+1}-n_j} x(n_j+i)e_i\right\| \geq (1+t)(1-\varepsilon),$$

$$\|x-tx_j\| \geq (1+t)\left\|\sum_{i=1}^{n_{j+2}-n_{j+1}} x(n_j+i)e_i\right\| \geq (1+t)(1-\varepsilon),$$

whence

$$\Gamma_{\ell_\Phi}(t) = t$$

for any $t > 0$. □

Now, we will define the numbers $c_x(t)$, which will play a crucial role in Theorem 4.2. For any $x,y\in S(\ell_\Phi)$ with $supp(x)\cap supp(y)=\emptyset$, where $supp(x)=\{i\in N: x(i)\neq 0\}$ and any $t>0$, there exits $k>0$ such that

$$\|x+ty\|=\frac{1}{k}(1+I_\Phi(k(x+ty))).$$

Assuming additionally that $x,y\in h_\Phi$, by continuity of the function

$$f(c)=I_\Phi\left(\frac{kx}{c}\right)+I_\Phi\left(\frac{kty}{c}\right),$$

for any $k>1$, there exists $c_{x,y,k,t}>0$, such that $f(c_{x,y,t,k})=k-1$. Put

$$c_{x,y,t}=\inf\{c_{x,y,t,k}: k>1\}.$$

From the equality

$$k-1=I_\Phi\left(\frac{kx}{c_{x,y,t,k}}\right)+I_\Phi\left(\frac{kty}{c_{x,y,t,k}}\right),$$

we obtain

$$\begin{aligned}1&=\frac{1}{k}\left(1+I_\Phi\left(\frac{kx}{c_{x,y,t,k}}\right)+I_\Phi\left(\frac{kty}{c_{x,y,t,k}}\right)\right)\\&\geq\inf_{k>0}\frac{1}{k}\left(1+I_\Phi\left(\frac{kx}{c_{x,y,t,k}}\right)+I_\Phi\left(\frac{kty}{c_{x,y,t,k}}\right)\right)=\left\|\frac{x+ty}{c_{x,y,t,k}}\right\|,\end{aligned}$$

whence $c_{x,y,t,k}\geq\|x+ty\|$. Using $m\leq 1$, there is a $k_{x,y,t}>1$ such that

$$\begin{aligned}1&=\frac{1}{k_{x,y,t}}\left(1+I_\Phi\left(\frac{k_{x,y,t}(x+ty)}{\|x+ty\|}\right)\right)\\&=\frac{1}{k_{x,y,t}}\left(1+I_\Phi\left(\frac{k_{x,y,t}x}{\|x+ty\|}\right)+I_\Phi\left(\frac{k_{x,y,t}ty}{\|x+ty\|}\right)\right)\end{aligned}$$

or equivalently

$$k_{x,y,t}-1=I_\Phi\left(\frac{k_{x,y,t}x}{\|x+ty\|}\right)+I_\Phi\left(\frac{k_{x,y,t}ty}{\|x+ty\|}\right),$$

so $c_{x,y,t,k_{x,y,t}}=\|x+ty\|$. Therefore, $c_{x,y,t}\leq\|x+ty\|$, which together with the opposite inequality proved already gives the equality $c_{x,y,t}=\|x+ty\|$ for any couple $x,y\in h_\Phi$.

For any $x\in S(h_\Phi)$ with $\max(supp(x))<\infty$, we define for any $n\in\mathbb{N}$ the number $c_{x,n}(t)$ as follows

$$c_{x,n}(t)=\sup\Big\{c_{x,y,t}>0:\ y\in S(h_\Phi)\text{ with }n\leq\min(supp(y))<\infty\Big\}.$$

Since h_Φ is a symmetric space, it is clear that $c_{x,n}(t)=c_{x,1}(t)$ for any $n\in\mathbb{N}$. Put $c_x(t)=c_{x,1}(t)$.

Theorem 4.2. *If Φ is an Orlicz function satisfying the δ_2-condition and its complementary function Ψ satisfies the δ_2-condition, then for the Orlicz sequence space ℓ_Φ equipped with the Amemiya-Orlicz norm there holds the equality*

$$\Gamma_{\ell_\Phi}(t)=\sup\{c_x(t):x\in S(\ell_\Phi)\ with\ \max(supp(x))<\infty\}-1.$$

Proof. Let

$$d_\Phi:=\sup\{c_x(t):x\in S(\ell_\Phi)\text{ with }\max(supp(x))<\infty\}.$$

The number d_Φ is well-defined because the numbers $c_x(t)$ exist by the fact that if $\max(supp(x))<\infty$, then $x\in h_\Phi$. Then for any $\varepsilon\in(0,d_\Phi)$, there exists $\|x\|_\Phi=1$ with $\max(supp(x))<\infty$ such that

$$c_x(t)>d_\Phi-\varepsilon.$$

By the definition of $c_x(t)$ there exists $n_1\in N$ such that

$$c_{x,n_1}(t)>d_\Phi-\varepsilon.$$

By the definition of $c_{x,n_1}(t)$, there exists $y_1\in S(\ell_\Phi)$ with $\min(supp(y_1))\geq n_1$ such that $c_{x,y_1,t}>d_\Phi-\varepsilon$. Hence, for all $k>1$ we have

$$c_{x,y_1,t,k}>d_\Phi-\varepsilon.$$

Put

$$y_n=\left(\overbrace{0,\cdots,0}^{l_n\ times},y_1(\min(supp(y_1))),\cdots,y_1(\max(supp(y_1))),0,\cdots\right)$$

for $n=2,3,\cdots$, where $l_n=\min(supp(y_1))-1+(n-1)\cdot\text{card}(supp(y_1))$. By

$$\lim_{\lambda\to 0}\sup_n\frac{I_\Phi(\lambda y_n)}{\lambda}=\lim_{\lambda\to 0}\frac{I_\Phi(\lambda y_1)}{\lambda}=0,$$

we have $y_n\overset{w}{\to}0$ (see [21]). Since $\|y_n\|=\|y_1\|=1$ for any $n\in\mathbb{N}$, there exists a subsequence (y_{n_i}) of (y_n) which is a basic sequence. We may assume, without loss of generality, that this subsequence is just the sequence (y_n).

For any $n\in N$, we have

$$k-1=I_\Phi\left(\frac{kx}{c_{x,y_1,t,k}}\right)+I_\Phi\left(\frac{kty_n}{c_{x,y_1,t,k}}\right),$$

whence

$$\begin{aligned}1&=\frac{1}{k}\left(1+I_\Phi\left(\frac{kx}{c_{x,y_1,t,k}}\right)+I_\Phi\left(\frac{kty_n}{c_{x,y_1,t,k}}\right)\right)\\&\leq\frac{1}{k}\left(1+I_\Phi\left(\frac{kx}{d_\Phi-\varepsilon}\right)+I_\Phi\left(\frac{kty_n}{d_\Phi-\varepsilon}\right)\right)\end{aligned}$$

for all $k>1$. Therefore, for all $k>0$ the inequality

$$\frac{1}{k}\left(1+I_\Phi\left(\frac{kx}{d_\Phi-\varepsilon}\right)+I_\Phi\left(\frac{kty_n}{d_\Phi-\varepsilon}\right)\right)\geq 1$$

holds. By the definition of the Amemiya-Orlicz norm, we have $\left\|\frac{x+ty_n}{d_\Phi-\varepsilon}\right\|\geq 1$, that is, $\|x+ty_k\|>d_\Phi-\varepsilon$. Therefore,

$$\Gamma_{\ell_\Phi}(t)\geq d_\Phi-\varepsilon-1,$$

and, by the arbitrariness of $\varepsilon>0$, we have that $\Gamma_{\ell_\Phi}(t)\geq d_\Phi-1$.

Now, we will prove that $\Gamma_{\ell_\Phi}(t)\leq d_\Phi-1$. By the definition of d_Φ, we always have $c_x(t)\leq d_\Phi$ for any $x\in S(\ell_\Phi)$ with finite $supp(x)$ and any $t>0$.

Taking any basic sequence (x_n) in $S(\ell_\Phi)$ and any $\varepsilon>0$, by $\Phi\in\delta_2$, there exists $i_0>0$ such that

$$\left\|\sum_{i=i_0+1}^{\infty}x_1(i)e_i\right\|<\varepsilon.$$

Since in reflexive Banach space every basic sequence is weakly null (see [25]), we have that $x_n\to 0$ weakly and we conclude that the sequence (x_n) converges to 0 coordinatewise. There is $n_1\in N$ such that

$$\left\|\sum_{i=1}^{i_0}x_n(i)e_i\right\|<\varepsilon$$

for all $n\geq n_1$. In virtue of $\Phi\in\delta_2$ again, there exists $i_1>i_0$ such that

$$\left\|\sum_{i=i_1+1}^{\infty}x_{n_1}(i)e_i\right\|<\varepsilon.$$

In such a way we obtain that there are two sequences $i_0<i_1<\cdots$ and $n_1<n_2<\cdots$ such that

$$\left\|\sum_{i=1}^{i_0}x_{n_m}(i)e_i\right\|<\varepsilon,\quad \left\|\sum_{i=i_m+1}^{\infty}x_{n_m}(i)e_i\right\|<\varepsilon.$$

Hence

$$\begin{aligned}&\|x_1+tx_{n_m}\|\\ \leq&\left\|\sum_{i=1}^{i_0}(x_1(i)+tx_{n_m}(i))e_i+\sum_{i=i_0+1}^{\infty}x_1(i)e_i+\sum_{i=i_0+1}^{i_m}tx_{n_m}(i)e_i+\sum_{i=i_m+1}^{\infty}tx_{n_m}(i)e_i\right\|\\ \leq&\left\|\sum_{i=1}^{i_0}x_1(i)e_i+\sum_{i=i_0+1}^{i_m}tx_{n_m}(i)e_i\right\|+(1+2t)\varepsilon.\end{aligned}$$

Next, we will estimate the second last term in the above estimate for $n>n_1$. Put

$$z_1=\sum_{i=1}^{i_0}x_1(i)e_i \Big/ \left\|\sum_{i=1}^{i_0}x_1(i)e_i\right\|, \quad z_m=\sum_{i=i_0+1}^{i_m}x_{n_m}(i)e_i \Big/ \left\|\sum_{i=i_0+1}^{i_m}x_{n_m}(i)e_i\right\|$$

for $n>n_0$. There exists $k>1$ such that

$$\frac{1}{k}\left(1+I_\Phi\left(\frac{kz_1}{c_{z_1,z_n,t}}\right)+I_\Phi\left(\frac{ktz_m}{c_{z_1,z_n,t}}\right)\right)=1.$$

So

$$\begin{aligned}\left\|\frac{z_1+tz_m}{d_\Phi}\right\| &\le \frac{1}{k}\left(1+I_\Phi\left(\frac{kz_1+ktz_m}{d_\Phi}\right)\right)\\ &=\frac{1}{k}\left(1+I_\Phi\left(\frac{kz_1}{d_\Phi}\right)+I_\Phi\left(\frac{ktz_m}{d_\Phi}\right)\right)\\ &\le\frac{1}{k}\left(1+I_\Phi\left(\frac{kz_1}{c_{z_1}(t)}\right)+I_\Phi\left(\frac{ktz_m}{c_{z_1}(t)}\right)\right)\\ &\le\frac{1}{k}\left(1+I_\Phi\left(\frac{kz_1}{c_{z_1,z_n,t}}\right)+I_\Phi\left(\frac{ktz_m}{c_{z_1,z_n,t}}\right)\right)=1\end{aligned}$$

for all $m\ge n_0$, whence

$$\begin{aligned}&\left\|\sum_{i=1}^{i_0}x_1(i)e_i+\sum_{i=i_0+1}^{i_m}tx_{n_m}(i)e_i\right\|\\ \le&\left\|\left\|\sum_{i=1}^{i_0}x_1(i)e_i\right\|z_1+t\left\|\sum_{i=i_0+1}^{i_m}x_{n_m}(i)e_i\right\|z_m\right\|\le\ \|z_1+tz_m\|\le\ d_\Phi.\end{aligned}$$

By the arbitrariness of $\varepsilon>0$, we get the inequality

$$\Gamma_{\ell_\Phi}(t)\le d_\Phi-1.$$

Since the opposite inequality was already proved, we obtain the equality $\Gamma_{\ell_\Phi}(t)=d_\Phi-1$. □

Corollary 4.1. *The Orlicz sequence space ℓ_Φ is nearly uniformly smooth if and only if $\Phi\in\delta_2$ and $\Phi\in\overline{\delta}_2$.*

Proof. Since nearly uniform smoothness implies reflexivity and the space ℓ_Φ is reflexive if and only if $\Phi\in\delta_2$ and $\Phi\in\overline{\delta}_2$, so we only need to prove that if $\Phi\in\delta_2$ and $\Phi\in\overline{\delta}_2$, then ℓ_Φ is nearly uniformly smooth. For any $x,y\in S(\ell_\Phi)$ with with $supp(x)\cap supp(y)=\emptyset$, any $\varepsilon>0$ and any $t_0>0$, by $\Phi\in\overline{\delta}_2$ we have that the number

$$K:=\sup\left\{k>1:\frac{1}{k}(1+I_\Phi(k(x+ty)))=\|x+ty\|\right\}$$

is finite for any $t\in[0,t_0]$ (see Theorem 1.35 in [9]). Take arbitrary $k>1$ such that

$$\|x\|=\frac{1}{k}(1+I_\Phi(kx)).$$

Then

$$\begin{aligned}\|x+ty\|&\le\frac{1}{k}\Big(1+I_\Phi(kx+kty)\Big)\\&=\frac{1}{k}\Big(1+I_\Phi(kx)+I_\Phi(kty)\Big)\\&=1+\frac{I_\Phi(kty)}{k}.\end{aligned}$$

By $\Phi\in\overline{\delta}_2$, there exists $\theta\in(0,1)$ such that

$$\Phi(\frac{u}{2})\le\frac{\theta}{2}\Phi(u)$$

whenever $|u|\le\Phi^{-1}(1)$ (see [9]). For any $\varepsilon>0$, there is $n\in N$ such that $\theta^n<\varepsilon$. Put $\delta=\frac{1}{2^n}$. Then

$$\Phi(\delta u)=\Phi(\frac{u}{2^n})\le\frac{\theta^n}{2^n}\Phi(u)\le\delta\varepsilon\Phi(u)$$

whenever $|u|\le\Phi^{-1}(1)$. Hence

$$\Phi(tu)=\Phi\left(\frac{t}{\delta}\delta u\right)\le\frac{t}{\delta}\Phi(\delta u)\le\frac{t}{\delta}\delta\varepsilon\Phi(u)=t\varepsilon\Phi(u)$$

whenever $t\in(0,\delta)$ and $|u|\le\Phi^{-1}(1)$.

Take $t_0>0$ small enough such that $t_0K<\delta$. We have $I_\Phi(x)\le1$ thanks to $x\in S(\ell_\Phi)$. Therefore,

$$\|x+ty\|\le1+\frac{I_\Phi(kty)}{k}\le1+\frac{tk\varepsilon}{k}I_\Phi(y)=1+t\varepsilon$$

for any $t\in(0,\delta)$. This shows that $c_{x,y,t}=\|x+ty\|\le1+t\varepsilon$, that is, $\Gamma_{\ell_\Phi}(t)\le t\varepsilon$ whenever $t\in(0,\delta)$. Therefore, $\Gamma'_{\ell_\Phi}(0)=0$. □

Corollary 4.2. *Let ℓ_p $(1<p<\infty)$ be the Lebesgue sequence space. Then $\Gamma_{\ell_p}(t)=(1+t^p)^{\frac{1}{p}}$.*

Proof. It is well known that the Lebesgue sequence space ℓ_p $(1<p<\infty)$ is the Orlicz sequence space generated by the Orlicz function $\Phi(u)=|u|^p$. For any $x,y\in S(\ell_p)$ with $supp(x)\cap supp(y)=\emptyset$ and $t>0$, we can easy calculate the Amemiya-Orlicz norm of $\|x\|$ and $\|y\|$, which are the following:

$$\|x\|=C\left(\sum_{i=1}^{\infty}|x(i)|^p\right)^{\frac{1}{p}},\quad\|y\|=C\left(\sum_{i=1}^{\infty}|y(i)|^p\right)^{\frac{1}{p}},$$

where $C=p(p-1)^{\frac{1}{p}-1}$. We have

$$\begin{aligned} c_{x,y,t}^p &= \|x+ty\|^p = C^p\left(\sum_{i=1}^{\infty}|x(i)|^p + t^p\sum_{i=1}^{\infty}|y(i)|^p\right) \\ &= C^p\left(\frac{1}{C^p}+\frac{t^p}{C^p}\right)=1+t^p. \end{aligned}$$

Therefore, $\Gamma_{\ell_p}(t)=(1+t^p)^{\frac{1}{p}}$. □

Remark 4.1. As far as we know, the result from Corollary 4.2 is new.

Using the definition of $R(a,\ell_\Phi)$ and $D(x_n)$, we can easily get the following lemma.

Lemma 4.1. *In any Orlicz sequence space l_Φ, we have that*

$$R(a,\ell_\Phi)=\sup\left\{\liminf_{n\to\infty}\|x_n+x\|\right\},$$

where the supremum is taken over all $\|x\|\le a$ and all weakly null sequences (x_n) in $B(X)$ with $\|x_n-x_m\|\le 1$ for all $n,m\in\mathbb{N}$.

Let us define

$$D(\ell_\Phi)=\left\{x\in B(\ell_\Phi): supp(x) \text{ is finite and there exists } k>1 \text{ such that } \frac{k-1}{2}\ge I_\Phi(kx)\right\}.$$

Take $a>0$ such that $(a,a,0\cdots)\in B(\ell_\Phi)$. Then there exists $k>1$ such that

$$1\ge\|(a,a,0\cdots)\|=\frac{1}{k}\Big(1+2\Phi(ka)\Big)=\frac{1}{k}(1+2I_\Phi(kx)),$$

where $x=(a,0,0,\cdots)$. This shows that the element $x\in D(\ell_\Phi)$ and so that the set $D(l_\Phi)$ is nonempty.

Theorem 4.3. *For any Orlicz sequence space we have the equality*

$$R(a,\ell_\Phi)=\sup\Big\{\|x+y\|: x\in D(\ell_\Phi),\ \|y\|=a, supp(y) \text{ is finite and } supp(x)\cap supp(y)=\emptyset\Big\}.$$

Proof. Let

$$c_\Phi=\sup\Big\{\|x+y\|: x\in D(\ell_\Phi),\ \|y\|=a, supp(y) \text{ is finite and } supp(x)\cap supp(y)=\emptyset\Big\}.$$

Then for any $\varepsilon>0$ there are $x\in D(\ell_\Phi)$ and $y\in\ell_\Phi$ with $\|y\|=a$ such that $supp(y)$ is finite, $supp(x)\cap supp(y)=\emptyset$, and

$$c_\Phi-\varepsilon\le\|x+y\|.$$

Let $s=\max(supp(y))$, $l=\max(supp(x))$ and

$$x_n=\left(\overbrace{0,\cdots,0}^{s+(n-1)l\,times},x(1),\cdots,x(l),0,\cdots\right)$$

for each $n\in N$. Then $x_n\overset{w}{\to}0$ as $n\to\infty$. Since $x\in D(\ell_\Phi)$, there exists $k>1$ such that

$$\frac{k-1}{2}\ge I_\Phi(kx).$$

Hence

$$\|x_n-x_m\|\le\frac{1}{k}(1+I_\Phi(kx_n-kx_m))=\frac{1}{k}(1+2I_\Phi(kx))\le\frac{1}{k}(1+k-1)=1,$$

and so

$$R(a,\ell_\Phi)\ge\liminf_{n\to\infty}\|x_n+y\|=\|x+y\|\ge c_\Phi-\varepsilon.$$

By the arbitrariness of $\varepsilon>0$, we get the inequality $R(a,\ell_\Phi)\ge c_\Phi$.

We need to prove the inverse inequality. For any weakly null sequence $\{x_n\}$ in $B(\ell_\Phi)$ with $\|x_n-x_m\|\le1$ for any $m,n\in\mathbb{N}$ and $\|y\|=a$, by the same argumentation as in the proof of Theorem 4.2, we may assume that $supp(y)\cap supp(x_n)=\emptyset$, $supp(x_n)\cap supp(x_m)=\emptyset$ for $n,m\in N,n\ne m$, and both $supp(y)$ and $supp(x_n)$ are finite.

Taking $k_{n,m}>1$ such that

$$\|x_n-x_m\|=\frac{1}{k_{n,m}}\Big(1+I_\Phi(k_{n,m}x_n)+I_\Phi(k_{n,m}x_m)\Big),$$

we have

$$\frac{k_{n,m}-1}{2}\ge I_\Phi(k_{n,m}x_n)\quad\text{(equivalently } x_n\in D(\ell_\Phi))$$

and

$$\frac{k_{n,m}-1}{2}\ge I_\Phi(k_{n,m}x_m)\quad\text{(equivalently } x_m\in D(\ell_\Phi)).$$

Therefore, we have that $x_n\in D(\ell_\Phi)$ for any $n\in N$. So $\|x_n+y\|\le c_\Phi$, whence $R(a,\ell_\Phi)\le c_\Phi$. □

Corollary 4.3. Let ℓ_p $(1<p<\infty)$ be the Lebesgue sequence space. Then $R(a,\ell_p)=(1+a^p)^{\frac{1}{p}}$.

Proof. It is easy to prove that

$$R\left(a,\ell_p\right)=\left\{\|x+y\|:\|y\|=a,\ \|x\|=\frac{1}{2^{\frac{1}{p}}},supp(y)\cap supp(x)=\emptyset\right\}.$$

Then

$$\begin{aligned}\|x+y\|^p&=C^p\left(\sum_{i=1}^{\infty}|x(i)|^p+\sum_{i=1}^{\infty}|y(i)|^p\right)\\&=C^p\left(\frac{1}{2C^p}+\frac{a^p}{C^p}\right)=\frac{1}{2}+a^p,\end{aligned}$$

where C was defined in the proof of Corollary 4.2. So, $R\left(a,\ell_p\right)=\left(\frac{1}{2}+a^p\right)^{\frac{1}{p}}$. □

Corollary 4.4. For an Orlicz sequence space ℓ_Φ, the inequality $R(a,\ell_\Phi)<1+a$ holds if and only if $\Phi\in\delta_2$.

Proof. If $\Phi\notin\delta_2$, for any $\varepsilon>0$ there exists $x\in B(\ell_\Phi)$ such that

$$\lim_{n\to\infty}\|(0,\cdots,0,x(n+1),x(n+2),\cdots)\|>1-\varepsilon.$$

There are $n_1<n_2<\cdots$ such that

$$\left\|\sum_{i=1}^{n_{j+1}-n_j}x(n_j+i)e_i\right\|\geq 1-\varepsilon$$

for each $j\in N$. Put $x_j=\sum_{i=1}^{n_{j+1}-n_j}x(n_j+i)e_i$. Then $\|x_j-x_s\|\leq\|x\|\leq 1$ for all $j,s\in\mathbb{N}$ and so, if $f\in l_\Psi$, then we have

$$f(x_j)=\sum_{i=1}^{n_{j+1}-n_j}x(n_j+i)f(i)\to 0$$

thanks to $f(x)=\sum_{i=1}^{\infty}x(i)f(i)<\infty$. This fact and the fact that any singular functional vanishes over h_Φ show that the sequence $\{x_j\}$ is weakly null. Moreover,

$$\|x_j+ax\|\geq\|x_j+ax_j\|=(1+a)\|x_j\|\geq(1+a)(1-\varepsilon)$$

for any $j\in\mathbb{N}$. So, $R(a,\ell_\Phi)=1+a$.

On the other hand, if $\Phi\in\delta_2$ there is a number $\delta>0$ such that

$$\Phi((1+\delta))\leq 2\Phi(u)$$

whenever $|u|\leq\Phi^{-1}(3)$. For any $x\in D(\ell_\Phi)$ there is $k>1$ such that $\frac{k-1}{2}\geq I_\Phi(kx)$, whence

$$1\geq\frac{1}{k}(1+2I_\Phi(kx))\geq\frac{1}{k}(1+I_\Phi((1+\delta)kx))\geq(1+\delta)\|x\|.$$

Therefore, $\|x\| \leq \frac{1}{1+\delta}$ and by $\|y\| = a$ and the triangle inequality for the norm, we get

$$\|x+y\| \leq \frac{1}{1+\delta} + a < 1+a.$$

Summing up, we obtain that $R(a,\ell_\Phi) < 1+a$. □

Corollary 4.5. *If $\Phi \in \delta_2$, then Orlicz sequence spaces equipped with the Amemiya-Orlicz norm have the weak fixed point property.*

Acknowledgments

The first author gratefully acknowledges the support of the Natural Science Foundation of Heilongjiang Province of China (A2015018). The authors are grateful to Professor Grzegorz Lewicki for his information concerning the proofs of Lemma 2.3 and Theorem 2.4. The fourth author Haifeng Ma gratefully acknowledges the support the National Natural Science Foundation of China (Grant No. 11401143) and Overseas Returning Foundation of Hei Long Jiang Province (Grant No. LC201402). The fifth author Yuwen Wang gratefully acknowledges the support of the National Natural Science Foundation of China (Grant No. 11471091).

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