J. Math. Study
doi: 10.4208/jms.v49n4.16.01

Vol. **49**, No. 4, pp. 319-324
December 2016

Hermite Expansion of the Riemann Zeta Function

Bang-He Li*

Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijnig 100190, P. R. China

Received March 15, 2016; Accepted May 2, 2016

Abstract. Let $\zeta(s)$ be the Riemann zeta function, $s=\sigma+it$. For $0<\sigma<1$, we expand $\zeta(s)$ as the following series convergent in the space of slowly increasing distributions with variable t :

$$\zeta(\sigma+it)=\sum_{n=0}^{\infty}a_n(\sigma)\psi_n(t),$$

where

$$\psi_n(t)=(2^n n!\sqrt{\pi})^{-1/2}e^{\frac{-t^2}{2}}H_n(t),$$

$H_n(t)$ is the Hermite polynomial, and

$$a_n(\sigma)=2\pi(-1)^{n+1}\psi_n(i(1-\sigma))+(-i)^n\sqrt{2\pi}\sum_{m=1}^{\infty}\frac{1}{m^{\sigma}}\psi_n(\ln m).$$

This paper is concerned with the convergence of the above series for $\sigma>0$. In the deduction, it is crucial to regard the zeta function as Fourier transfomations of Schwartz' distributions.

AMS subject classifications: 11M06, 33C45, 46F05
Key words: Riemann zeta function, Hermite expansion, Schwartz distributions.

1 Results

Let $\zeta(s)$ be the famous Riemann Zeta function which is holomorphic for $s=\sigma+it\in\mathcal{C}-\{1\}$. It is well known that if $0<\sigma<1$ then

$$\zeta(s)=s\int_0^{\infty}\frac{[x]-x}{x^{s+1}}dx$$

(see [1] or [2]). By the substitute of variables $x=e^y$, we get

$$\zeta(s)=s\int_{-\infty}^{\infty}([e^y]-e^y)e^{-sy}dy.$$

*Corresponding author. *Email address:* `libh@amss.ac.cn` (B.-H. Li)

Set

$$f(y)=[e^y]-e^y. \tag{1.1}$$

Then for $0<\sigma<1$, $e^{-\sigma y}f(y)$ is a slowly increasing function , so can be regarded as an element of $\mathcal{S}'$, the dual space of the space $\mathcal{S}$ of rapidly decreasing functions on R. The Laplace transformation $\mathcal{L}(f)(s)$ of f is then defined on the trip $0<\sigma<1$ both in the ordinary and distributional sense, that is

$$\mathcal{L}(f)(s)=\zeta(s)/s.$$

Let $f'\in\mathcal{S}'$ be the derivative of f in the distributional sense. then $e^{-\sigma y}f'(y)\in\mathcal{S}'$ for $0<\sigma<1$, and Laplace transformation $\mathcal{L}(f')(s)$ is defined on the strip $0<\sigma<1$ such that

$$\mathcal{L}(f')(s)=\zeta(s)$$

([3] Chapter 8). So by the relation of Fourier and Laplace transformation of distribution (see also [3]), we see that $\zeta(\sigma+it)$ as function of t is the Fourier transformation of $e^{-\sigma y}f'(y)$ in the distributional sense, where the Fourier transformation is defined by $g(x)\to\int_{-\infty}^{\infty}g(x)e^{-ixy}dx$ for $g\in L^1(R)$.

Recall that the Hermite polynomials are defined as

$$H_n(x)=e^{x^2}(\frac{d}{dx})^n e^{-x^2},\ n=0,1,\dots$$
$$\psi_n(t)=(2^n n!\sqrt{\pi})^{-1/2}e^{\frac{-t^2}{2}}H_n(t),\ n=0,1,\dots$$

form a complete normalized orthogonal system in $L^2(R)$.

Xiaqi Ding and his collaborators introduced and developed the theory of Hermite expansions of generalized functions [4]. The aim of this paper is to give the Hermite expansion of $\zeta(\sigma+it)$ as function of t for $0<\sigma<1$. For this, we give first the Hermite expansion of $e^{-\sigma y}f'(y)\in\mathcal{S}'$ for fixed σ. Now

$$f'(y)=-e^y+\sum_{m=1}^{\infty}\delta(y-\ln m),$$

where δ is the Dirac δ-function. So

$$e^{-\sigma y}f'(y)=-e^{(1-\sigma)y}+\sum_{m=1}^{\infty}\frac{1}{m^\sigma}\delta(y-\ln m). \tag{1.2}$$

The following lemma gives the Hermite expansion of $-e^{(1-\sigma)y}$.

Lemma 1.1. *For any complex number a,*

$$\int_{-\infty}^{\infty}e^{ax}\psi_n(x)dx=(-i)^n\sqrt{2\pi}\psi_n(ia).$$

Proof. Since $e^{ax}\psi_n(x)\in L^1(R)$, its ordinary Fourier transformation exists. Let $c_n=(2^n n!\sqrt{\pi})^{\frac{1}{2}}$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{ax}\psi_n(x)e^{-ixy}dx=\frac{(-1)^n}{c_n}\int_{-\infty}^{\infty} e^{\frac{x^2}{2}-i(y+ia)x}\left(\frac{d}{dx}\right)^n e^{-x^2}dx\\
=&\frac{1}{c_n}\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d}{dx}\right)^n e^{\frac{x^2}{2}-i(y+ia)x}dx\\
=&\frac{e^{\frac{1}{2}(y+ia)^2}}{c_n}\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{d}{dx}\right)^n e^{\frac{1}{2}(x-i(y+ia))^2}dx\\
=&\frac{i^n}{c_n}e^{\frac{1}{2}(y+ia)^2}\int_{-\infty}^{\infty}\left(\frac{d}{dy}\right)^n e^{-x^2+\frac{1}{2}(x-i(y+ia))^2}dx\\
=&\frac{i^n}{c_n}e^{\frac{1}{2}(y+ia)^2}\left(\frac{d}{dy}\right)^n (e^{\frac{-1}{2}(y+ia)^2}\int_{-\infty}^{\infty} e^{-x^2+\frac{1}{2}(x-i(y+ia))x}dx)\\
=&\frac{i^n}{c_n}e^{\frac{1}{2}(y+ia)^2}\left(\frac{d}{dy}\right)^n (e^{\frac{-1}{2}(y+ia)^2}\sqrt{2\pi}e^{-\frac{1}{2}(y+ia)^2})\\
=&\frac{(-i)^n\sqrt{2\pi}}{c_n}e^{-\frac{1}{2}(y+ia)^2}H_n(y+ia)\\
=&(-i)^n\sqrt{2\pi}\psi_n(y+ia).
\end{aligned}$$

Taking $y=0$ in the above formula, we get

$$\int_{-\infty}^{\infty} e^{ax}\psi_n(x)dx=(-i)^n\sqrt{2\pi}\psi_n(ia),$$

hence the lemma is proved. □

Notice that although $e^{-\sigma y}f'(y)\in\mathcal{S}'$, both the first and second terms in (1.2) are not in $\mathcal{S}'$. In fact, they are only in $\mathcal{D}'$, the dual space of the compactly supported smooth functions. Especially, the series of the second term is only convergent in $\mathcal{D}'$. Since ψ_n is in $\mathcal{S}'$ but not in $\mathcal{D}'$, although it is easy to get

$$<\frac{1}{m^\sigma}\delta(y-\ln m),\psi_n(y)>=\frac{1}{m^\sigma}\psi_n(\ln m),$$

it is not obvious if the series

$$\sum_{m=1}^{\infty}\frac{1}{m^\sigma}\psi_n(\ln m) \tag{1.3}$$

converges. The following theorem addresses this issue.

Theorem 1.1. *For $0<\sigma<1$ and $n=0,1,\cdots$, the series* (1.3) *is convergent.*

Proof. Set

$$\chi_1(y)=\begin{cases} 1, & y\in(-\infty,\ln\frac{3}{2}] \\ 0, & \text{otherwise} \end{cases}$$

and for $m=2,3,\cdots$,

$$\chi_m(y)=\begin{cases} 1, & y\in(\ln(m-\frac{1}{2}),\ln(m+\frac{1}{2})] \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $g_m(y)$ the element of $\mathcal{S}'$ given by the integrable function $\chi_m(y)e^{-\sigma y}f(y)$. Then for any $\varphi\in\mathcal{S}$, we have

$$<e^{-\sigma y}f(y),\varphi(y)>=\int_{-\infty}^{\infty}e^{-\sigma y}f(y)\varphi(y)dy$$

$$=\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\chi_m(y)e^{-\sigma y}f(y)\varphi(y)dy=\sum_{m=1}^{\infty}<g_m(y),\varphi(y)>.$$

This means that

$$e^{-\sigma y}f(y)=\sum_{m=1}^{\infty}g_m(y)$$

is convergent in $\mathcal{S}'$. Thus by a fundamental result in distributional theory, its derivative

$$(e^{-\sigma y}f(y))'=\sum_{m=1}^{\infty}g_m'(y)$$

is also convergent in $\mathcal{S}'$. Now

$$\begin{aligned} g_m'(y)&=-\sigma g_m(y)+e^{-\sigma y}\chi_m'(y)f(y)+e^{-\sigma y}\chi_m(y)f'(y), \qquad (1.4)\\ e^{-\sigma y}\chi_1'(y)f(y)&=-e^{-\sigma y}f(y)\delta\left(y-\ln\frac{3}{2}\right)\\ &=-(\frac{3}{2})^{-\sigma}f\left(\ln\frac{3}{2}\right)\delta\left(y-\ln\frac{3}{2}\right),\\ e^{-\sigma y}\chi_m'(y)f(y)&=\left(m-\frac{1}{2}\right)^{-\sigma}f\left(\ln(m-\frac{1}{2})\right)\delta\left(y-\ln(m-\frac{1}{2})\right)\\ &\quad-\left(m+\frac{1}{2}\right)^{-\sigma}f\left(\ln(m+\frac{1}{2})\right)\delta\left(y-\ln(m+\frac{1}{2})\right). \end{aligned}$$

Since for any $\varphi\in\mathcal{S},\varphi(y)\to 0$ as $y\to\pm\infty$, and $|f(y)|\le 1$, we have

$$\begin{aligned} &<\sum_{m=1}^{M}e^{-\sigma y}\chi_m'(y)f(y),\varphi(y)>\\ =&-\left(M+\frac{1}{2}\right)^{-\sigma}f\left(\ln(M+\frac{1}{2})\right)\varphi\left(\ln(M+\frac{1}{2})\right)\to 0,\ \text{as}\ M\to\infty. \end{aligned}$$

Thus by (1.4), the series

$$\sum_{m=1}^{\infty} e^{-\sigma y}\chi_m(y)f'(y)=e^{-\sigma y}f'(y) \tag{1.5}$$

converges in $\mathcal{S}'$. Now

$$e^{-\sigma y}\chi_m(y)f'(y)=-e^{(1-\sigma)y}\chi_m(y)+\frac{1}{m^{\sigma}}\delta(y-\ln m) \tag{1.6}$$

and $e^{(1-\sigma)y}\psi_n(y)$ is integrable on R, so

$$\int_{-\infty}^{\infty} e^{(1-\sigma)y}\psi_n(y)dy=\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} e^{-\sigma y}\chi_m(y)\psi_n(y)dy$$

is a convergent series. Thus by (1.6),

$$<\sum_{m=1}^{\infty}\frac{1}{m^{\sigma}}\delta(y-\ln m),\psi_n(y)>=\sum_{m=1}^{\infty}\frac{1}{m^{\sigma}}\psi_n(\ln m)$$

is a convergent series. The proof is complete. ☐

By the general theory on Dirichlet series, we have

Corollary 1.1. The Dirichlet series

$$\sum_{m=1}^{\infty}\frac{1}{m^{s}}\psi_n(\ln m)$$

is convergent for $Re(s)>0$, and $n=0,1\cdots$.

Corollary 1.2. For $0<\sigma<1$, the following Hermite expansion

$$e^{-\sigma y}f'(y)=\sum_{n=0}^{\infty} b_n(\sigma)\psi_n(y),$$

is convergent in $\mathcal{S}'$, where

$$b_n(\sigma)=-(-i)^n\sqrt{2\pi}\psi_n(i(1-\sigma))+\sum_{m=1}^{\infty}\frac{1}{m^{\sigma}}\psi_n(\ln m).$$

Proof. Since $e^{-\sigma y}f'(y)\in\mathcal{S}'$, it is proved in [4] that

$$e^{-\sigma y}f'(y)=\sum_{n=0}^{\infty} <e^{-\sigma y}f'(x),\psi_n(x)>\psi_n(y)$$

holds in $\mathcal{S}'$. By (1.5) and (1.6), we have

$$< e^{-\sigma x} f'(x), \psi_n(x) > = \sum_{m=1}^{\infty} < e^{-\sigma x} \chi_m(x) f'(x), \psi_n(x) >$$

$$= \sum_{m=1}^{\infty} \left(\int_{-\infty}^{\infty} e^{(1-\sigma)x} \chi_m(x) \psi_n(x) dx + \frac{1}{m^{\sigma}} \psi_n(\ln m) \right).$$

Thus Lemma 1.1 and Theorem 1.1 imply

$$b_n(\sigma) = < e^{-\sigma y} f'(x), \psi_n(x) >,$$

hence the corollary is proved. □

Since

$$\int_{-\infty}^{\infty} \psi_n(x) e^{-ixy} dx = \sqrt{2\pi}(-i)^n \psi_n(y),$$

Corollary 1.2 implies that

Theorem 1.2. *For* $0 < \sigma < 1$,

$$\zeta(\sigma + it) = \sum_{n=0}^{\infty} a_n(\sigma) \psi_n(t)$$

in $\mathcal{S}'$, *where*

$$a_n(\sigma) = 2\pi(-1)^{n+1} \psi_n(i(1-\sigma)) + (-i)^n \sum_{m=1}^{\infty} \frac{\sqrt{2\pi}}{m^{\sigma}} \psi_n(\ln m).$$

Acknowledgments

The author would like to thank Prof. Xiaqi Ding for his high interest in this work, both of us have been interested in studying Riemann Zeta Function from the point view of distributions. Thanks also go to Prof. Shaoji Feng for reading the preprint of this work.

References

[1] E.C. Titchmarsh, D.R. Heath-Brown, The Theory of the Riemann Zeta Function, Second Edition, Clarendon Press, Oxford, 1986

[2] C.-D. Pan, C.-B. Pan, Foundation of Analytic Number Theory (in Chinese), Scientific Press, Beijing, 1999.

[3] A. H. Zemanian, Distribution Theory and Transform Analysis, McGraw-Hill, 1965.

[4] X. Ding, Y. Ding, Hermite Expansions and Generalized Functions (in Chinese), Huazhong Nomal Univ. Press, 2005.