
TOEPLITZ AND POSITIVE SEMIDEFINITE COMPLETION PROBLEM FOR CYCLE GRAPH*

He Ming(何明) Michael K. Ng(吴国宝)

Abstract *We present a sufficient and necessary condition for a so-called C_n^k pattern to have positive semidefinite (PSD) completion. Since the graph of the C_n^k pattern is composed by some simple cycles, our results extend those given in [1] for a simple cycle. We also derive some results for a partial Toeplitz PSD matrix specifying the C_n^k pattern to have PSD completion and Toeplitz PSD completion.*

Key words *Partial matrix, Toeplitz matrix, completion problem, C_n^k pattern.*

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1 Introduction

A partial matrix is a matrix in which some entries are specified, while the remains entries are free to be chosen(from a certain set). A completion of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. For most of matrix completion problems there are some obvious conditions that must be satisfied for a completion of a certain class to exist. For example, for real symmetric positive semidefinite (PSD) completions, all completely specified submatrices must be symmetric PSD. A partial matrix which satisfies such a condition is called a partial PSD matrix. In this paper another class of matrices which we concern is Toeplitz matrix. It is known that an $n \times n$ matrix $A = (a_{ij})$ is called a symmetric Toeplitz matrix if $a_{i,j} = r_{|i-j|}$ for all $i, j = 1, 2, \dots, n$. So a partial symmetric Toeplitz matrix is a partial symmetric matrix and if an entry in position (i, j) is specified then all entries in positions $(i + l, j + l) \pmod n$ are also specified and these (specified) entries are equal. Other

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types of partial matrices are defined similarly, see [4].

A pattern for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$. A partial matrix specifies the pattern if its specified entries are exactly those listed in the pattern. A pattern Q is called symmetric if $(i, j) \in Q$ implies $(j, i) \in Q$. We assume throughout this paper that all the patterns we discuss are symmetric and include all diagonal positions, and when we define a pattern or calculate the number of the specified entries of a pattern, we shall ignore diagonal and symmetric positions of the pattern (e.g., Definition 1 below) and the entries in the positions. For a certain class of matrices, one important area of research is to decide for which patterns of specified entries all partial matrices in the class are completable to the class of matrices.

Among the types of matrix completion problems that have been studied are completion to positive definite (semidefinite) matrices [2],[7], to M-matrices and inverse M-matrices [5], to P-matrices and P_0 matrices[6],[4] and to contractions [8], etc.. For the positive definite completion problem, a solution was given by Grone, Johnson, Sa, and Wolkowicz in [2]. For the partial positive definite Toeplitz completion problem, an interesting open problem was presented in [7].

Now we give definition of the C_n^k pattern and describe the completion problems we shall consider.

Definition 1 Let $n/2 > k \geq 1$, the symmetric pattern

$$Q = \{(1, k+1), (2, k+2), \dots, (n-k, n), (n-k+1, 1), \dots, (n, k)\} \quad (1)$$

is called a C_n^k pattern. A Toeplitz C_n^k pattern is a C_n^k pattern with a restriction that the partial matrix specifying the pattern is a partial Toeplitz matrix.

Remark 1 In the definition, replacing k by $n-k$ gives the same pattern. So this is why we call it C_n^k pattern. It is trivial to show that when n is even and $k = n-k$, the pattern has PSD completion. So in the following, we always assume that $k < n-k$.

Remark 2 If we number the diagonals for an $n \times n$ symmetric Toeplitz matrix in the following way: The main diagonal is given the number 0 and then the diagonals are numbered in increasing order to the last, which becomes the $(n-1)$ th, one. Then the (Toeplitz) C_n^k pattern is, in fact, the k th and $(n-k)$ th diagonal.

In terms of the definition, our problems are as follows.

Problem 1 Whether does the C_n^k pattern have a PSD completion?

Without loss of generality, associated with a normalized of the data (see [1]), we assume

that the partial PSD matrix specifying the C_n^k pattern is of the form:

$$C = \begin{pmatrix} 1 & \cos \theta_1 & \cos \theta_{n-k+1} & & ? \\ & 1 & \cos \theta_2 & ? & \ddots \\ \cos \theta_1 & & 1 & ? & \ddots & \cos \theta_n \\ & \cos \theta_2 & ? & 1 & & \ddots \\ \cos \theta_{n-k+1} & ? & \ddots & & \ddots & \cos \theta_{n-k} \\ ? & \ddots & \cos \theta_n & \ddots & \cos \theta_{n-k} & \ddots \\ & & & & \cos \theta_{n-k} & \ddots & 1 \end{pmatrix}, \quad (2)$$

where $0 \leq \theta_i \leq \pi$, the ?s indicate unspecified entries. Thus Problem 1 can be restated as:

Problem 1' When does the partial PSD matrix (2) admit a PSD completion?

Next problem is about Toeplitz C_n^k pattern and so more special than the previous one.

Problem 2 Whether does a Toeplitz C_n^k pattern have a Toeplitz PSD completion?

Like Problem 1, Problem 2 can be restated as:

Problem 2' When does the partial Toeplitz PSD matrix

$$T = \begin{pmatrix} 1 & \cos \theta & \cos \phi & & ? \\ & 1 & \cos \theta & ? & \ddots \\ \cos \theta & & 1 & ? & \ddots & \cos \phi \\ & \cos \theta & ? & 1 & & \ddots \\ \cos \phi & ? & \ddots & & \ddots & \cos \theta \\ ? & \ddots & \cos \phi & \ddots & \cos \theta & \ddots \\ & & & & \cos \theta & \ddots & 1 \end{pmatrix} \quad (3)$$

admit a Toeplitz PSD completion? Here we assume that in the partial matrix T, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$ and $\cos \theta, \cos \phi$ lies in the k th, the $(n - k)$ th diagonal, respectively.

In the case $k = 1$, Problem 1 or Problem 1' has been resolved by W. Barrett, C. R. Johnson and P. Tarazaga in [1] (see Theorem 2). A partial answer to Problem 2 or 2' in the case $k = 1$ was also given by them (see Theorem 9 below). Our main contribution in this note is to extend their results to arbitrary k ($n/2 > k \geq 1$) for Problem 1 (see next section) and give some results for Problem 2 (see section 3).

2 Completion of C_n^k pattern

Graphs and digraphs are very effective to study matrix completion problems [3]. Since the C_n^k pattern is a symmetric pattern, we shall use undirected graph to study its completion problem.

Let Q be a symmetric pattern for $n \times n$ matrices, the graph $G=(V,E)$ of Q is defined to have nodes set $V = N \equiv \{1, 2, \dots, n\}$ and edge $(i, j) \in E$ if and only if $(i, j) \in Q$.

A path in a graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \dots, v_k, v_{k+1}$ in V such that for $i = 1, 2, \dots, k$, $(v_i, v_{i+1}) \in E$ and all nodes are distinct except possibly $v_1 = v_{k+1}$. If $v_1 = v_{k+1}$ in the path $v_1, v_2, \dots, v_k, v_{k+1}$, then the path forms a simple circle, which will be denoted by $\{v_1, v_2, \dots, v_{k+1}(=v_1)\}$, its length is k . According to the definition, the graph of the C_n^1 pattern is a simple circle of length n , for whose PSD completion problem the following result was given in [1].

Theorem 2 Let $n \geq 4$ and $N = \{1, 2, \dots, n\}$, $0 \leq \theta_1, \theta_2, \dots, \theta_n \leq \pi$, then the matrix

$$C = \begin{pmatrix} 1 & \cos \theta_1 & & & \cos \theta_n \\ \cos \theta_1 & 1 & \cos \theta_2 & ? & \\ & \cos \theta_2 & 1 & \ddots & \\ & ? & \ddots & \ddots & \cos \theta_{n-1} \\ \cos \theta_n & & & \cos \theta_{n-1} & 1 \end{pmatrix} \quad (4)$$

has a PSD completion if and only if for each $S \subseteq N$ with $|S|$ odd,

$$\sum_{i \in S} \theta_i \leq (|S| - 1)\pi + \sum_{i \in N \setminus S} \theta_i. \quad (5)$$

A graph is chordal if it has no minimal simple cycle of length four or more. The following result is well-known[2].

Theorem 3 Every partial PSD matrix specifying a pattern Q has a PSD completion if and only if the graph of Q is chordal.

A graph is connected if there is a path from any node to any other node; otherwise it is disconnected. A subgraph of the graph $G = (V_G, E_G)$ is a graph $H = (V_H, E_H)$, where $V_H \subseteq V_G$ and $E_H \subseteq E_G$ and that $(u, v) \in E_H$ requires $u, v \in V_H$ since H is a graph. A component of a graph is maximal connected subgraph.

The graph $G = (V_G, E_G)$ is isomorphic to the graph $H = (V_H, E_H)$ by isomorphism ϕ if ϕ is a one-to-one map from V_G onto V_H and $(u, v) \in E_G$ if and only if $(\phi(u), \phi(v)) \in E_H$. Relabeling the nodes of a graph diagram corresponds to perform a graph isomorphism. Since the class of PSD matrices is closed under permutation similarity, we are free to relabel graphs as desired and we have

Lemma 4 Let Q be a pattern and G its graph. If every component of G is isomorphic to a graph of a pattern and the pattern has PSD completion, then Q has PSD completion.

Proof The proof of the lemma is similar to that of Lemma 3.4 of [4] and is omitted here.

In the following, we let $\gcd(n, k)$ denote the greatest common divisor of the integers n and k . Let $\bar{x} = x \pmod{n}$, $d = \gcd(n, k)$ and $t = \frac{n}{d}$. Now we describe the graph of the C_n^k pattern.

Lemam 5 Every component of the graph of C_n^k pattern is isomorphic to a simple circle

of length t .

Proof Let Q be the C_n^k pattern defined in (1) and $G = (N, E)$ its graph. Using the notation of \overline{x} , the edge set can be written as

$$E = \{(\overline{i + (j - 1)k}, \overline{i + jk}) | 1 \leq i \leq d; 1 \leq j \leq t\}.$$

By noting that $\overline{i + tk} = i$, it is not difficult to prove that the graph G is composed by the following d independent simple cycles of length t :

$$\begin{aligned} & \{1, \overline{1 + k}, \overline{1 + 2k}, \dots, \overline{1 + (t - 1)k}, \overline{1 + tk}\} \\ & \{2, \overline{2 + k}, \overline{2 + 2k}, \dots, \overline{2 + (t - 1)k}, \overline{2 + tk}\} \\ & \{3, \overline{3 + k}, \overline{3 + 2k}, \dots, \overline{3 + (t - 1)k}, \overline{3 + tk}\} \\ & \dots, \dots, \dots, \dots, \dots \\ & \{d, \overline{d + k}, \overline{d + 2k}, \dots, \overline{d + (t - 1)k}, \overline{d + tk}\}. \end{aligned}$$

The proof is completed.

Remark 3 The lemma shows that the graph of the C_n^k pattern is a cycle graph[9], that is, it is composed by some simple cycles. But only one simple cycle also is possible even for $k > 1$ if n and k have no common divisor larger than and equal to 2. In this case, $t = n$ and (2) is permutation similar to (4). For example, let $n = 5$, $k = 2$, then it is easy to verify that the following two partial PSD matrices are permutation similar:

$$\begin{pmatrix} 1 & ? & c_1 & c_4 & ? \\ ? & 1 & ? & c_2 & c_5 \\ c_1 & ? & 1 & ? & c_3 \\ c_4 & c_2 & ? & 1 & ? \\ ? & c_5 & c_3 & ? & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & c_1 & ? & ? & c_4 \\ c_1 & 1 & c_3 & ? & ? \\ ? & c_3 & 1 & c_5 & ? \\ ? & ? & c_5 & 1 & c_2 \\ c_4 & ? & ? & c_2 & 1 \end{pmatrix},$$

here we denote $\cos\theta_i$ by c_i .

By using Lemma 4 and 5, we easily derive the following results for Problem 1 and Problem 1'.

Theorem 6 The C_n^k pattern has a PSD completion if and only if the C_n^1 pattern has a PSD completion.

Theorem 7 The partial PSD matrix (2) is permutation similar to the partial PSD matrix:

$C' = C_1 \oplus C_2 \oplus \cdots \oplus C_d$, where

$$C_i = \begin{pmatrix} 1 & \cos \theta_i & & & \cos \theta_{\overline{i+(t-1)k}} \\ \cos \theta_i & 1 & \cos \theta_{\overline{i+k}} & ? & \\ & \cos \theta_{\overline{i+k}} & 1 & \ddots & \\ & ? & \ddots & \ddots & \cos \theta_{\overline{i+(t-2)k}} \\ \cos \theta_{\overline{i+(t-1)k}} & & & \cos \theta_{\overline{i+(t-2)k}} & 1 \end{pmatrix} \quad (6)$$

and all entries outside the diagonal blocks of C' are unspecified.

Thus the partial PSD matrix C of (2) can be completed to a PSD matrix if and only if the partial PSD matrix C' can. It is easy to complete C' (then complete C of (2) to a PSD matrix: complete every C_i to a PSD matrix by using the method given in [1], and then set all (unspecified) entries outside the diagonal blocks of C' to zero. On the other hand, if the partial PSD matrix C of (2) has PSD completion, then C' and so every C_i of (6) must have a PSD completion too. So for Problem 1', the following solution is given by applying Theorem 7 and Theorem 2.

Theorem 8 Let $n \geq 4$, $1 \leq k < n/2$, $d = \gcd(n, k)$ and $t = n/d$. Then we have

a) If $t = 3$, then the partial PSD matrix (2) has a PSD completion.

b) If $t \geq 4$, then the partial PSD matrix (2) has a PSD completion if and only if for each $S_i \subseteq N_i \equiv \{i, \overline{i+k}, \overline{i+2k}, \dots, \overline{i+(t-1)k}\}$ with $|S_i|$ odd,

$$\sum_{j \in S_i} \theta_j \leq (|S_i| - 1)\pi + \sum_{j \in N_i \setminus S_i} \theta_j, \quad (7)$$

$i = 1, 2, \dots, d$.

Proof From the assumption of $k < n/2$, we know that t is an integer larger than or equal to 3.

When $t = 3$, Lemma 5 shows that the graph of the C_n^k pattern is chordal (d simple cycles of length 3). So C has a PSD completion by Theorem 3, a) is proved.

When $t \geq 4$, applying Theorem 2 to each C_i of (6), b) is obtained.

3 Completion of Toeplitz C_n^k Pattern

Now we consider Problem 2 or 2': Toeplitz PSD completion problem for the Toeplitz C_n^k pattern, which is more difficult than non-Toeplitz completion problem. We shall see that Lemma 5 is not directly applicable to the problem. [1] has not solved Problem 2' for the case $k = 1$ and only presented a result for non-Toeplitz PSD completion as follows.

Theorem 9^[1] Let C be an $n \times n$ matrix, if $n \geq 4$ and $\theta, \phi \in [0, \pi]$, then

$$C = \begin{pmatrix} 1 & \cos \theta & & & \cos \phi \\ \cos \theta & 1 & \cos \theta & ? & \\ & \cos \theta & 1 & \ddots & \\ \cos \phi & ? & \ddots & \ddots & \cos \theta \\ & & & \cos \theta & 1 \end{pmatrix} \quad (8)$$

has PSD completion if and only if

$$\phi \leq (n - 1)\theta \leq (n - 2)\pi + \phi, \quad n \text{ is even}; \quad (9)$$

$$\phi \leq (n - 1)\theta \leq (n - 1)\pi - \phi, \quad n \text{ is odd}. \quad (10)$$

The following result generalizes the theorem.

Theorem 10 Let $n \geq 4$, $1 \leq k < n/2$, $d = \gcd(n, k)$ and $t = n/d$. Then we have

a) If $t = 3$, then the partial Toeplitz PSD matrix (3) has a PSD completion.

b) If $t \geq 4$, then the partial Toeplitz PSD matrix (3) has a PSD completion if and only if :

1) when $\frac{n-k}{d}$ is odd and $\frac{k}{d}$ is even,

$$(n - k)\theta \leq (n - k - d)\pi + k\phi \leq (2n - k - 2d)\pi - (n - k)\theta. \quad (11)$$

2) when $\frac{n-k}{d}$ is even and $\frac{k}{d}$ is odd,

$$k\phi \leq (k - d)\pi + (n - k)\theta \leq (n + k - 2d)\pi - k\phi. \quad (12)$$

3) when both $\frac{n-k}{d}$ and $\frac{k}{d}$ are odd,

$$(n - k)\theta \leq (n - k - d)\pi + k\phi \leq (n - 2d)\pi + (n - k)\theta. \quad (13)$$

Proof a) is obvious from the proof of Theorem 8. We prove b) by applications of Theorem 7 and Theorem 8.

First, note that it is impossible that both $\frac{n-k}{d}$ and $\frac{k}{d}$ are even, since $d = \gcd(n, k)$.

Secondly, we know from Theorem 7 that T is permutation similar to $T' = C_1 \oplus C_2 \oplus \dots \oplus C_d$ where C_i is defined in (6), but θ_j is θ or ϕ and each C_i contains $\frac{n-k}{d} \cos \theta$ and $\frac{k}{d} \cos \phi$. Thus C_i is generally not a partial Toeplitz matrix, so it is Theorem 8 other than Theorem 9 that is applied in the following proof.

T has a PSD completion if and only if every C_i has. In fact, all C_i ($i = 1, 2, \dots, d$) have the same form. So we only need to consider the PSD completion of one C_i .

It is obvious that the inequality (7) holds automatically if θ or ϕ occurs on both sides of (7), since $\theta \leq \pi$ ($\phi \leq \pi$). So we only need to consider the following two cases: one case is that t is odd and all θ and ϕ appear on the left side of the inequality (7). The another case is that all $\frac{n-k}{d}$ θ just appears on one side of (7), all $\frac{k}{d}$ ϕ appears in the other side of (7). Thus we have

1) When $\frac{n-k}{d}$ is odd, $\frac{k}{d}$ is even, we only need to consider S_i , in Theorem 8, such that $|S_i| = \frac{n-k}{d}$ and $|S_i| = t$. The required inequalities are

$$\frac{(n-k)}{d}\theta \leq \left(\frac{(n-k)}{d} - 1\right)\pi + \frac{k}{d}\phi \quad \text{and} \quad \frac{(n-k)}{d}\theta + \frac{k}{d}\phi \leq \left(\frac{n}{d} - 1\right)\pi. \quad (14)$$

2) When $\frac{n-k}{d}$ is even, $\frac{k}{d}$ is odd, we only need to consider S_i such that $|S_i| = \frac{k}{d}$ and $|S_i| = t$. The required inequalities are

$$\left(\frac{n}{d}\right)\phi \leq \left(\frac{n}{d} - 1\right)\pi + \frac{n-k}{d}\theta \quad \text{and} \quad \frac{(n-k)}{d}\theta + \frac{k}{d}\phi \leq \left(\frac{n}{d} - 1\right)\pi. \quad (15)$$

3) When both $\frac{n-k}{d}$ and $\frac{k}{d}$ are odd, we have to consider S_i such that $|S_i| = \frac{n-k}{d}$ and $|S_i| = \frac{k}{d}$. The required inequalities are

$$\frac{(n-k)}{d}\theta \leq \left(\frac{(n-k)}{d} - 1\right)\pi + \frac{k}{d}\phi \quad \text{and} \quad \left(\frac{n}{d}\right)\phi \leq \left(\frac{n}{d} - 1\right)\pi + \frac{n-k}{d}\theta. \quad (16)$$

We get (11), (12) and (13) from (14), (15) and (16), respectively. The theorem is proved.

Corollary 11 When $\theta = \phi$, if t is three or even, then the partial Toeplitz PSD matrix (3) has a PSD completion; if t is odd larger than three, then the partial Toeplitz PSD matrix (3) has a PSD completion if and only if

$$\theta \leq \frac{n-d}{n}\pi \quad \text{or} \quad \cos \theta \geq -\cos \frac{d}{n}\pi. \quad (17)$$

Proof We do not need to prove the case $t \leq 3$. When t is even, both $\frac{n-k}{d}$ and $\frac{k}{n}$ are odd. By applying 3) of Theorem 10 and noting that $d \leq k$, the inequalities (13) hold automatically.

When t is odd larger than three, applying 1) and 2) of Theorem 10, we find the required inequality only is (17).

By using a similar proof, we get another interesting result.

Corollary 12 If $t \geq 4$, $\theta = \pi - \phi$ or $\cos \theta = -\cos \phi$, then the partial Toeplitz PSD matrix (3) has a PSD completion if and only if

1) when $\frac{n-k}{d}$ is odd and $\frac{k}{d}$ is even, $\theta \leq \frac{n-d}{n}\pi$ or $\cos \theta \geq -\cos \frac{d}{n}\pi$.

2) when $\frac{n-k}{d}$ is even and $\frac{k}{d}$ is odd, $\theta \geq \frac{d}{n}\pi$ or $\cos \theta \leq \cos \frac{d}{n}\pi$.

3) when both $\frac{n-k}{d}$ and $\frac{k}{d}$ are odd, $\frac{d}{n}\pi \leq \theta \leq \frac{n-d}{n}\pi$ or $\cos \frac{d}{n}\pi \leq \cos \theta \leq \cos \frac{d}{n}\pi$.

In terms of the two corollaries, the two partial Toeplitz PSD matrices

$$\begin{pmatrix} 1 & ? & 1 & 1 & ? \\ ? & 1 & ? & 1 & 1 \\ 1 & ? & 1 & ? & 1 \\ 1 & 1 & ? & 1 & ? \\ ? & 1 & 1 & ? & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & ? & 1 & -1 & ? \\ ? & 1 & ? & 1 & -1 \\ 1 & ? & 1 & ? & 1 \\ -1 & 1 & ? & 1 & ? \\ ? & -1 & 1 & ? & 1 \end{pmatrix}$$

have PSD completions, but the two partial Toeplitz PSD matrices

$$\begin{pmatrix} 1 & ? & -1 & -1 & ? \\ ? & 1 & ? & -1 & -1 \\ -1 & ? & 1 & ? & -1 \\ -1 & -1 & ? & 1 & ? \\ ? & -1 & -1 & ? & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & ? & -1 & 1 & ? \\ ? & 1 & ? & -1 & 1 \\ -1 & ? & 1 & ? & -1 \\ 1 & -1 & ? & 1 & ? \\ ? & 1 & -1 & ? & 1 \end{pmatrix}$$

have no PSD completions.

Now we turn to Problem 2' and discuss the case $k = 1$ first. For notation convenience and obvious reason, in the following by $To(a_1, a_2, \dots, a_n)$ we shall denote a symmetric Toeplitz matrix whose first row is (a_1, a_2, \dots, a_n) . The partial Toeplitz PSD matrix (8) can be written as $To(1, \cos \theta, ?, \dots, ?, \cos \phi)$.

Theorem 13 The partial Toeplitz PSD matrix (8) has a Toeplitz PSD completion if

$$-\frac{1}{n-1} \leq \cos \theta \quad \text{and} \quad 0 \leq \frac{\cos \phi - \cos(n-1)\theta}{\cos \theta - \cos(n-1)\theta} \leq 1. \quad (18)$$

or if

$$\cos \theta \leq \frac{1}{n-1} \quad \text{and} \quad 0 \leq \frac{\cos \phi - \cos(n-1)\theta}{(-1)^n \cos \theta - \cos(n-1)\theta} \leq 1. \quad (19)$$

or if

$$-\frac{1}{n-1} \leq \cos \theta \leq \frac{1}{n-1} \quad \text{and} \quad \begin{cases} 0 \leq \frac{\cos \theta - \cos \phi}{2 \cos \theta} \leq 1, & \text{if } n \text{ is odd,} \\ \cos \phi = \cos \theta, & \text{if } n \text{ is even.} \end{cases} \quad (20)$$

Proof Define three symmetric Toeplitz matrices:

$$B = To(1, \cos \theta, \cos 2\theta, \dots, \cos(n-1)\theta), \quad (21)$$

$$E = To(1, \cos \theta, \cos \theta, \dots, \cos \theta), \quad (22)$$

and

$$F = To(1, \cos \theta, -\cos \theta, \dots, (-1)^n \cos \theta). \quad (23)$$

We first prove that B is a PSD matrix, E is positive semidefinite if $\cos \theta \geq -\frac{1}{n-1}$, and F is positive semidefinite if $\cos \theta \leq \frac{1}{n-1}$.

In fact, B is positive semidefinite, since

$$B = \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \\ \cos 2\theta & \sin 2\theta \\ \dots & \dots \\ \cos(n-1)\theta & \sin(n-1)\theta \end{pmatrix} \begin{pmatrix} 1 & \cos \theta & \cos 2\theta & \dots & \cos(n-1)\theta \\ 0 & \sin \theta & \sin 2\theta & \dots & \sin(n-1)\theta \end{pmatrix}.$$

E is positive semidefinite if $\cos \theta \geq -\frac{1}{n-1}$, since it can be written as $E = (1 - \cos \theta)I_n + (\cos \theta)ee^T$, where I_n is $n \times n$ identity matrix and $e = (1, 1, \dots, 1)^T$. Thus it is easy to verify that E is symmetric and has $n-1$ eigenvalues $1 - \cos \theta$ and one eigenvalue $1 + (n-1)\cos \theta$. all eigenvalues of E are nonnegative under the assumption. Similarly, F is positive semidefinite if $\cos \theta \leq \frac{1}{n-1}$, since it has $n-1$ eigenvalues $1 + \cos \theta$ and one eigenvalue $1 - (n-1)\cos \theta$.

Now we prove that (8) has a Toeplitz PSD completion if (18) holds. Define $T_t = (1-t)B + tE$, then T_t is a Toeplitz PSD matrix for any $t \in [0, 1]$ if $\cos \theta \geq -\frac{1}{n-1}$, since it is the sum of two Toeplitz PSD matrices: $(1-t)B$ and tE .

Let $t_0 = \frac{\cos \phi - \cos(n-1)\theta}{\cos \theta - \cos(n-1)\theta}$, then $0 \leq t_0 \leq 1$ by the assumption (18) and $\cos \phi = (1 - t_0)\cos(n-1)\theta + t_0\cos \theta$. Thus

$$T_{t_0} = To(1, \cos \theta, (1-t_0)\cos 2\theta + t_0\cos \theta, \dots, (1-t_0)\cos(n-2)\theta + t_0\cos \theta, \cos \phi)$$

is a Toeplitz PSD completion of (8).

Similarly, if we define $T_t = (1-t)B + tF$, we can prove that (8) has a Toeplitz PSD completion if (19) holds.

Finally, we prove that (8) has a Toeplitz PSD completion if (20) holds. Now we define $T_t = (1-t)E + tF$, then T_t is a Toeplitz PSD matrix for any $t \in [0, 1]$ if $-\frac{1}{n-1} \leq \cos \theta \leq \frac{1}{n-1}$.

Note now that

$$T_t = To(1, \cos \theta, (1-2t)\cos \theta, \dots, [1 - (1 - (-1)^n)t]\cos \theta).$$

When n is even, it is necessary that $\cos \phi = \cos \theta$ in order that T_t is a Toeplitz PSD completion of (8). When n is odd, let $t_0 = \frac{\cos \theta - \cos \phi}{2\cos \theta}$, then $0 \leq t_0 \leq 1$ by the assumption (20) and $\cos \phi = (1 - 2t_0)\cos \theta$. Therefore T_{t_0} is a Toeplitz PSD completion of (8).

Now we give a result for Problem 2':

Theorem 14 The partial Toeplitz PSD matrix (3) has a Toeplitz PSD completion if

$$-\frac{1}{n-1} \leq \cos \theta \quad \text{and} \quad 0 \leq \frac{\cos \phi - \cos \frac{n-k}{k}\theta}{\cos \theta - \cos \frac{n-k}{k}\theta} \leq 1. \quad (24)$$

or if when k is odd,

$$\cos \theta \leq \frac{1}{n-1} \quad \text{and} \quad 0 \leq \frac{\cos \phi - \cos \frac{n-k}{k}\theta}{(-1)^{n-k+1}\cos \theta - \cos \frac{n-k}{k}\theta} \leq 1. \quad (25)$$

when k is even,

$$-\frac{1}{n-1} \leq \cos \theta \quad \text{and} \quad 0 \leq \frac{\cos \phi - \cos \frac{n-k}{k}\theta}{(-1)^{n-k} \cos \theta - \cos \frac{n-k}{k}\theta} \leq 1. \quad (26)$$

or if when k is odd,

$$-\frac{1}{n-1} \leq \cos \theta \leq \frac{1}{n-1} \quad \text{and} \quad \begin{cases} 0 \leq \frac{\cos \theta - \cos \phi}{2 \cos \theta} \leq 1, & \text{if } n-k \text{ is even,} \\ \cos \phi = \cos \theta, & \text{if } n-k \text{ is odd.} \end{cases} \quad (27)$$

when k is even,

$$-\frac{1}{n-1} \leq \cos \theta \quad \text{and} \quad \begin{cases} 0 \leq \frac{\cos \theta - \cos \phi}{2 \cos \theta} \leq 1, & \text{if } n-k \text{ is odd,} \\ \cos \phi = \cos \theta, & \text{if } n-k \text{ is even.} \end{cases} \quad (28)$$

Proof To prove the theorem, we need to define two more symmetric Toeplitz matrices except E, F in (22),(23)

$$\tilde{B} = To(1, \cos \alpha, \cos 2\alpha, \dots, \cos(n-1)\alpha), \quad (29)$$

here $\alpha = \frac{\theta}{k}$ and

$$\tilde{F} = To(1, -\cos \theta, \cos \theta, \dots, (-1)^{n-1} \cos \theta). \quad (30)$$

\tilde{B} is clearly positive semidefinite. \tilde{F} is positive semidefinite if $\cos \theta \geq -\frac{1}{n-1}$, since it has $n-1$ eigenvalues $1 - \cos \theta$ and one eigenvalue $1 + (n-1) \cos \theta$.

Similar to the proof of Theorem 13, define T_t to be one of the following matrices:

- 1) $(1-t)\tilde{B} + tE$,
- 2) $(1-t)\tilde{B} + tF$ when k is odd, $(1-t)\tilde{B} + t\tilde{F}$ when k is even,
- 3) $(1-t)E + tF$ when k is odd, $(1-t)E + t\tilde{F}$ when k is even.

We can complete the proof of the theorem.

From Theorem 14, we easily get

Corollary 15 When $\theta = \phi$, the partial Toeplitz PSD matrix (3) has a Toeplitz PSD completion if $\cos \theta \geq -\frac{1}{n-1}$ or if n is even, k is odd and $\cos \theta \leq \frac{1}{n-1}$.

When $\theta = \pi - \phi$ and n is odd, the partial Toeplitz PSD matrix (3) has a Toeplitz PSD completion if k is odd and $\cos \theta \leq \frac{1}{n-1}$, or if k is even and $\cos \theta \geq -\frac{1}{n-1}$.

From the proof of Theorem 14, we can see that it is not necessary to connect $(n-k)th$ diagonal to kth diagonal, so the result of Theorem 14 can easily be extended to the following more general pattern:

$$Q1 = \{(1, k+1), (2, k+2), \dots, (n-k, n), (l+1, 1), \dots, (n, n-l)\},$$

where $1 \leq k < l \leq n - 1$. In fact, the pattern just is two different diagonals: the k th and the l th diagonal.

A final remark. We have tried many numerical examples but failed to find a partial Toeplitz PSD matrix specifying the C_n^k pattern which has a PSD completion but no Toeplitz PSD completion. This make us to believe the following is true:

Conjecture The partial Toeplitz PSD matrix (3) that has a PSD completion has a Toeplitz PSD completion.

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He Ming Department of Information and Computational Mathematics, Xiamen University, Xiamen 361005, PRC.

Michael K. Ng Department of Mathematics, University of Hong Kong, Pokfulam Road, Hongkong.