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On The Maximal-Like Solution of Matrix Equation $X + A^*X^{-2}A = I$* †

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Abstract. In this paper, we study several iterative methods for finding the maximal-like solution of the matrix equation $X + A^*X^{-2}A = I$, and deduce some properties of the maximal-like solution with these methods.

Key words: Matrix equation; maximal-like solution.

AMS subject classifications: 65F10, 65F30

1 Introduction

In this paper we consider the matrix equation

$$X + A^*X^{-2}A = I \tag{1}$$

where I is the $n \times n$ identity matrix and A is an $n \times n$ complex matrix.

Throughout this paper we denote $\|\cdot\|$ the Euclidean vector norm, or corresponding subordinate matrix norm (simply 2-norm). $\lambda(M)$, $\rho(M)$ are respectively the spectrum and spectral radius of a square matrix M, A^* is conjugate transpose of a matrix A. For two positive definite (Hermitian) matrices P, Q of the same dimension, $P > Q$ $(P \geq Q)$ means that $P - Q$ is positive definite (semi-definite). For any positive definite solution X of Eq. (1), we have $X_S \leq X \leq X_L$, where X_L and X_S are respectively the maximal solution and minimal solution, and X_l is the maximal-like solution whose inverse has the minimal 2-norm.

In the literature, matrix equations of type like Eq. (1) have been extensively studied. Articles [3, 4, 11] discuss the matrix equation $X + A^*X^{-1}A = I$ and obtained some properties of the equation, including the existence of maximal and minimal solutions. [1, 2, 10] generalize the results, and [7, 8] directly discuss nonlinear matrix equation of type in Eq. (1). [7, 8] mainly study the following algorithms:

$$\begin{cases} X_0 = \alpha I \\ X_k = I - A^*X_{k-1}^{-2}A \end{cases}, \quad \begin{cases} X_0 = \alpha I \\ X_{k+1} = \sqrt{A(I - X_k)^{-1}A^*} \end{cases}, \tag{2}$$

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and provide some convergence properties under different conditions. However, they do not show the existence of the maximal and minimal solutions and the properties of solutions. [5] proves the existence of the minimal solutions. [9] studies more general matrix equations of the type $X^s \pm A^T X^{-t} A = I$.

In this paper, we discuss the maximal-like solution X_l, which is the maximal solution X_L when X_L exists. In Sections 2 and 3 we propose two algorithms for finding X_l, and study properties of these algorithms; in Section 4 we provide some numerical experiments.

2 An algorithm for computing X_l

In this section, we propose an iterative algorithm for computing X_l. We will prove that the algorithm is linearly convergent, and derive some properties of X_l. Unlike the commonly used algorithms given in Eq. (2) which involve computing the inverse, our algorithm only requires matrix multiplications.

We first give a necessary condition for existence of a solution of Eq. (1)

Theorem 2.1 ([5])**.** *If Eq. (1) has a positive definite solution X, then*

$$\rho(A) \leq \frac{2\sqrt{3}}{9}.$$

Corollary 2.1. *Suppose that A is normal. If Eq. (1) has a positive definite solution, then*

$$\|A\| \leq \frac{2\sqrt{3}}{9}.$$

Lemma 2.1. *Define*

$$f(\eta) = \frac{\eta}{(1+\eta)^3}, \ \ \eta \geq 0.$$

Then f is increasing for $0 \leq \eta \leq \frac{1}{2}$, decreasing for $\frac{1}{2} \leq \eta \leq +\infty$, and

$$f_{\max} = f(\frac{1}{2}) = \frac{4}{27}.$$

Proof: From

$$f'(\eta) = \frac{1}{(1+\eta)^4}(1-2\eta),$$

we know that $f(\eta)$ is increasing in $[0, \frac{1}{2}]$, and decreasing in $[\frac{1}{2}, +\infty]$. When $\eta = \frac{1}{2}$, $f\max = f(\frac{1}{2}) = \frac{4}{27}$. ■

We now present the main result of this section.

Theorem 2.2. *If $\|A\| < \frac{2\sqrt{3}}{9}$, then there exists a* unique *solution X_l of Eq. (1) satisfying*

$$\|X_l^{-1}\| < \frac{3}{2}.$$

Moreover, for any other positive definite solution X we have

$$\|X^{-1}\| \geq \frac{3}{2}.$$

Proof It is easy to verify that X is a solution of Eq. (1), if and only if $Y = X^{-1}$ satisfies

$$Y = A^* Y^2 AY + I. \tag{3}$$

Now define matrix sequence $\{Y_k\}$:

$$Y_0 = 0,\ Y_k = A^* Y_{k-1}^2 AY_{k-1} + I,\ k = 1, 2, \cdots. \tag{4}$$

Obviously, for $k = 0, 1, \cdots$,

$$\|Y_k\| \le 1 + \eta_k, \tag{5}$$

where

$$\eta_1 = 0, \eta_k = \|A\|^2 (1 + \eta_{k-1})^3, k = 1, 2, \cdots. \tag{6}$$

We now prove by induction that,

$$\frac{1}{2} > \eta_{k+1} > \eta_k \ge 0, k = 1, 2 \cdots. \tag{7}$$

In fact, for $k = 1,\ 2$ we have

$$\eta_1 = 0 < \frac{1}{2} \quad \text{and} \quad \eta_2 = \|A\|^2 < \frac{4}{27} < \frac{1}{2}.$$

Suppose that $\eta_k < \frac{1}{2}$ for $2 \le k \le p$. Then

$$\eta_{p+1} = \|A\|^2 (1 + \eta_p)^3 < \|A\|^2 \times \frac{27}{8} < \frac{4}{27} \times \frac{27}{8} = \frac{1}{2}.$$

Hence there exists a positive number η with $0 < \eta < \frac{1}{2}$, such that $\eta = \lim\limits_{k\to\infty} \eta_k$, and it follows from Eq. (6) that

$$\eta = \|A\|^2 (1 + \eta)^3,$$

and from Lemma 2.1,

$$0 < \eta < \frac{1}{2}. \tag{8}$$

Consequently, we have

$$\|Y_k\| \le 1 + \eta_k \le 1 + \eta,\ k = 1, 2, \cdots$$

Therefore, the iterative sequence $\{Y_k\}$ satisfies

$$\begin{aligned}
\|Y_{k+1} - Y_k\| &= \|A^* Y_k^2 AY_k - A^* Y_{k-1}^2 AY_{k-1}\| \\
&= \|A^* \Big[Y_k (Y_k - Y_{k-1}) AY_k + (Y_k - Y_{k-1}) Y_{k-1} AY_k + Y_{k-1}^2 A (Y_k - Y_{k-1}) \Big]\| \\
&\le \|A\|^2 (\|Y_k\|^2 + \|Y_{k-1}\| \|Y_k\| + \|Y_{k-1}^2\|) \|Y_k - Y_{k-1}\| \\
&\le 3\|A\|^2 (1 + \eta)^2 \|Y_k - Y_{k-1}\| \\
&\le \rho \|Y_k - Y_{k-1}\| \\
&\le \rho^k \|Y_1 - Y_0\| = \rho^k,
\end{aligned}$$

where $\rho = 3\|A\|^2 (1 + \eta)^2 < 3 \times \frac{4}{27} \times \frac{9}{4} = 1$. It follows that $\{Y_k\}$ is a Cauchy sequence, therefore is convergent. Let $\bar{Y} = \lim\limits_{k\to\infty} Y_k$, then $\bar{Y}$ is a solution of Eq. (3) satisfying $\|\bar{Y}\| \le 1 + \eta$.

In order to prove that $\bar{Y}$ is a positive definite matrix, we consider the following matrix equation

$$Y = I + \frac{1}{2} (A^* Y^2 AY + YA^* Y^2 A). \tag{9}$$

It turns out that for any solution Y of Eq. (3), Y must satisfy Eq. (9). In fact, if Y is a solution of Eq. (3), then

$$(I - A^*Y^2A)Y = I,$$

which implies that $I - A^*Y^2A$ is the inverse of Y, and so

$$Y(I - A^*Y^2A) = I.$$

That means

$$Y = YA^*Y^2A + I$$

and Y is a solution of Eq. (9). Now define the sequence $\{Z_k\}$:

$$Z_0 = 0,\ Z_{k+1} = I + \frac{1}{2}(A^*Z_k^2AZ_k + Z_kA^*Z_k^2A),\ k = 1, 2, \cdots.$$

Notice that Z_k are Hermitian for $k = 0, 1, \cdots$. Following the same way as in the proof for $\|Y_k\|$, it can be proved that

$$\|Z_k\| \leq 1 + \eta < \frac{3}{2},\ \|Z_{k+1} - Z_k\| \leq \rho^k.$$

Define $M_k = \frac{1}{2}(A^*Z_k^2AZ_k + Z_kA^*Z_k^2A)$. Then M_k is Hermitian, and because $\|M_k\| \leq \|A\|^2\|Z_k\|^3 < 1/2$ we have $Z_{k+1} = I + M_k$ is positive definite. Therefore, there exists a positive definite matrix Z, such that

$$Z = \lim_{k\to\infty} Z_k, \text{ and } \|Z\| \leq 1 + \eta < \frac{3}{2}.$$

The next task is to prove that Eq. (9) has a unique solution Y satisfying $\|Y\| < \frac{3}{2}$. In fact, for any solution Y of Eq. (9) satisfying $\|Y\| < 3/2$,

$$\begin{aligned}
\|Z_{k+1} - Y\| &= \tfrac{1}{2}\|A^*Z_k^2AZ_k + Z_kA^*Z_k^2A - A^*Y_k^2AY_k - Y_kA^*Y_k^2A\| \\
&\leq \|A^*Z_k^2AZ_k - A^*Y_k^2AY_k\| \\
&\leq \|A\|^2(\|Z_k\|^2 + \|Z_k\|\|Y\| + \|Y\|^2)\|Z_k - Y\| \\
&\leq \|A\|^2((1+\eta)^2 + (1+\eta)\|Y\| + \|Y\|^2)\|Z_k - Y\| \\
&= \rho_1^{k+1}\|Y\|,
\end{aligned}$$

where $\rho_1 = \|A\|^2((1+\eta)^2 + (1+\eta)\|Y\| + \|Y\|^2) < 1$. Hence

$$Y = \lim_{k\to\infty} Z_k = Z.$$

Because $\bar{Y}$ is a solution of Eqs. (3) and (9) satisfying $\|\bar{Y}\| \leq 1 + \eta < \frac{3}{2}$, therefore $\bar{Y} = Z = Y$. It follows that for any other positive definite solution Y of Eq. (3), we have $\|Y\| \geq \frac{3}{2}$, which completes the proof of the theorem. ■

As an immediate consequence of Theorem 2.2, we have

Corollary 2.2. *If* $\|A\| < \frac{2\sqrt{3}}{9}$, *then we have*

$$\kappa(X_l) = \|X_l\|\|X_l^{-1}\| < \frac{3}{2},$$

therefore X_l *is well-conditioned.*

Remark 2.1. Corollary 2.1 shows that we cannot improve the condition of Theorem 2.2 if only a spectral norm bound is used.

Remark 2.2. $X \geq Y > 0$ implies $\|Y^{-1}\| \geq \|X^{-1}\|$, therefore, when Eq. (1) has the maximal solution X_L, then

$$\bar{Y} = X_L^{-1}$$

and X_l in Theorem 2.2 and Corollary 2.2 can be replaced by X_L.

Remark 2.3. From the proof of Theorem 2.2 we see that, when $\|A\| < \frac{2\sqrt{3}}{9}$, the optimization problem

$$\min\{\|Y\| : \ Y > 0, \ Y = A^* Y^2 A Y + I\}$$

has a unique solution Y. Moreover, Y is positive definite satisfying $\|Y\| < \frac{3}{2}$.

Remark 2.4. The proof of Theorem 2.2 proposes an iterative method for evaluating X_l^{-1},

$$\text{Method 1:} \left\{ \begin{array}{l} Y_0 = 0, \\ Y_k = A^* Y_{k-1}^2 A Y_{k-1} + I, \ k = 1, 2, \cdots \end{array} \right. \tag{10}$$

This method only involves matrix multiplications, and the algorithm is linearly convergent.

3 Another algorithm for computing X_l

In this section we will prove that the iterative sequence

$$\text{Method 2:} \left\{ \begin{array}{l} X_0 = I \\ X_k = I - A^* X_{k-1}^{-2} A, \ k = 1, 2, \cdots \end{array} \right. \tag{11}$$

converges to X_l under the condition of Theorem 2.2. This algorithm is commonly used, and has been extensively discussed in the literature, while the sequence converges to what solution is not discussed.

Theorem 3.1. *If $\|A\| < \frac{2\sqrt{3}}{9}$, then the iterative sequence $\{X_k\}$ defined by Method 2 converges to X_l.*

Proof Because $\|A^*A\| < \frac{4}{27}$, there exist two constants α and β satisfying $1 \geq \alpha \geq \beta > \frac{2}{3}$, such that

$$\alpha^2(1-\alpha)I \leq A^*A \leq \beta^2(1-\beta)I.$$

By Theorem 5 of [5], the sequence $\{X_k\}$ converges to $\bar{X}$ which satisfies

$$\beta I \leq \bar{X} \leq I, \quad \|\bar{X}^{-1}\| \leq \frac{1}{\beta} < \frac{3}{2}.$$

By applying Theorem 2.2, $X_l = \bar{X}$. ■

Table 1: Computational results for Method 1

n	cpu-time	norm(X)	cond(X)	abs
5	0	9. 999994987e-001	1. 000756149e+000	7. 535543911e-016
10	5e-002	9. 999862301e-001	1. 002397325e+000	2. 447112065e-016
20	0	9. 999978680e-001	1. 006290253e+000	4. 099287099e-015

Table 2: Computational results for Method 2

n	cpu-time	norm(X)	cond(X)	abs
5	0	9. 999994987e-001	1. 000756149e+000	1. 301720233e-017
10	0	9. 999862301e-001	1. 002397325e+000	2. 131704101e-015
20	0	9. 999978680e-001	1. 006290253e+000	4. 557376069e-017

Table 3: Differences of solutions between two methods

n	5	10	20
	5. 6330885e-016	1. 354791659e-015	3. 080567139e-015

4 Numerical experiments

In this section we provide several examples to verify the results obtained in the previous sections. In the following examples, matrices A are obtained by MATLAB function randn and n are the orders of A,

$$\text{abs}= \|X + A^*X^{-2}A - I\|_\infty,\ \text{norm(X)}= \|X\|_2,\ \text{cond(X)}= \|X\|_2\|X^{-1}\|_2.$$

From the tables we see that X_l's satisfy Corollary 2.2, and the both iterative solutions by using Methods 1 and 2 converge to X_l.

Many other examples have been tested. It is observed from all these numerical experiments that X_l's are well-conditioned and $\kappa(X_l) < \frac{3}{2}$.

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