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Least-Squares Solutions of the Equation $AX = B$ Over Anti-Hermitian Generalized Hamiltonian Matrices†

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Abstract. Upon using the denotative theorem of anti-Hermitian generalized Hamiltonian matrices, we solve effectively the least-squares problem $\min \|AX - B\|$ over anti-Hermitian generalized Hamiltonian matrices. We derive some necessary and sufficient conditions for solvability of the problem and an expression for general solution of the matrix equation $AX = B$. In addition, we also obtain the expression for the solution of a relevant optimal approximate problem.

Key words: Least-squares problem; anti-Hermitian generalized Hamiltonian matrices; optimal approximation.

AMS subject classifications: 65F15, 65F20, 65D99

1 Introduction

A typical least-squares problem is: Given a set S of matrices and given matrices X and B, find all matrices $A \in S$ for which $\|AX - B\| = \min_{G\in S} \|GX - B\|$.

We get different least-squares problems according to different sets S. The least-squares problems and relevant constrained matrix equation problems have been widely used in particle physics and geology[1], inverse problems of vibration theory[2,3], inverse Sturm-Liouville problem[4], control theory and multidimensional approximation[5,6]. In recent years a series of good results have been made for this problem[2−14]. For example, J. G. Sun considered the problem for the case of real symmetric matrices in [10]. K. G. Woodgate studied the problem for the case of symmetric positive semidefinite matrices in [3]. D. X. Xie studied the problem for the case of anti-symmetric matrices, nonnegative definite matrices (may be nonsymmetric), as well as bisymmetric matrices in [11-13]. In this paper, we discuss the problem for a set S which is defined in the following way.

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Definition 1.1. Assume that $J \in R^{n\times n}$ is a given orthogonal anti-symmetric matrix. $A \in C^{n\times n}$ is said to be an anti-Hermitian generalized Hamiltonian matrix if

$$A^H = -A \quad \text{and} \quad JAJ = A^H$$

where A^H stands for the conjugate transformation of matrix A. The set of all n-by-n anti-Hermitian generalized Hamiltonian matrices is denoted by $\mathrm{AHHC}^{n\times n}$, i.e.,

$$\mathrm{AHHC}^{n\times n} = \{A \in C^{n\times n} | A^H = -A \ \text{ and } \quad JAJ = A^H\}.$$

It is clear that the set $\mathrm{AHHC}^{n\times n}$ is a linear subspace of $C^{n\times n}$ and depends on matrix J. Throughout the paper, we always assume that the matrix J is fixed. In addition, by the properties of the matrix J, we have $J^2 = -I_n$. Consequently, n must be an even integer.

In this paper, we study the following two problems.

Problem I Given $X, B \in C^{n\times m}$, find a matrix $A \in \mathrm{AHHC}^{n\times n}$ such that

$$\min f(A) = \min \|AX - B\|.$$

Problem II Given $A^* \in C^{n\times n}$, find a matrix $\hat{A} \in S_{X,B}$ such that

$$\|A^* - \hat{A}\| = \min_{\forall A \in S_{X,B}} \|A^* - A\|,$$

where $S_{X,B}$ is the set of solutions of Problem I and $\|A\|$ stands for the Frobenius norm of matrix A.

In this paper, we derive an expression of the solution for Problems I and II. We prove the necessary and sufficient conditions of the solvability for the matrix equation $AX = B$ in $\mathrm{AHHC}^{n\times n}$.

Let us introduce some notations that will be used in this paper. Let $\mathrm{HC}^{n\times n}(\mathrm{AHC}^{n\times n})$ be the set of all $n \times n$ Hermitian matrices (anti-Hermitian matrices). The notation $UC^{n\times n}$ stands for the set of all $n\times n$ unitary matrices. We denote the Moore-Penrose generalized inverse of a matrix A by A^+, the identity matrix of order n by I_n. For $A, B \in C^{n\times m}$, we use $< A, B >= \mathrm{tr}(B^H A)$ to define the inner product of matrices A and B. The induced matrix norm is the so called Frobenius norm, i.e.,

$$\|A\| = \sqrt{< A, A >} = [\mathrm{tr}(A^H A)]^{\frac{1}{2}}.$$

It is clear that $C^{n\times m}$ is a complete inner product space. For $A, B \in C^{n\times m}$, $A * B$ stands for the Hadamard product of A and B.

This paper is organized as follows. In Section 2, we discuss the properties of the $\mathrm{AHHC}^{n\times n}$. In Section 3, we derive the expression of the general solution for Problem I, and then establish the necessary and sufficient conditions of the solvability for $AX = B$ in $\mathrm{AHHC}^{n\times n}$. In Section 4, we prove the existence and uniqueness of the solution and derive the expression of the solution for Problem II.

2 Characterization of anti-Hermitian generalized Hamiltonian matrices

In this section, we prove the denotative theorem of anti-Hermitian generalized Hamiltonian matrices. Let

$$P_1 = \frac{1}{2}(I + iJ), \quad P_2 = \frac{1}{2}(I - iJ). \tag{1}$$

where $i = \sqrt{-1}$. It is not difficult to prove that P_1 and P_2 are orthogonal projection matrices. Moreover, we have $P_1 + P_2 = I$ and $P_1 P_2 = 0$. Hence, there exist unit column-orthogonal matrices $Q_1 \in C^{n\times k}$ and $Q_2 \in C^{n\times k}(n = 2k)$ such that

$$P_1 = Q_1 Q_1^H, \qquad P_2 = Q_2 Q_2^H, \tag{2}$$

where $Q_1^H Q_2 = 0$. Let $Q = (Q_1, Q_2)$. By properties of P_1 and P_2, it is easy to prove that Q is an n-by-n unitary matrix.

Theorem 2.1. *A matrix A belongs to $AHHC^{n\times n}$ if and only if it is of the form*

$$A = Q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H \tag{3}$$

for some $E_1, E_2 \in \mathrm{AHC}^{k\times k}$.

Proof. First we note that for $A \in \mathrm{AHHC}^{n\times n}$, it holds that

$$\begin{aligned} P_1AP_1 + P_2AP_2 &= \frac{1}{4}\left[(I+iJ)A(I+iJ) + (I-iJ)A(I-iJ)\right] \\ &= \frac{1}{4}[2A - 2JAJ] = \frac{1}{4}\cdot 4A = A. \end{aligned}$$

Then it follows from (2) that

$$\begin{aligned} A &= Q_1Q_1^HAQ_1Q_1^H + Q_2Q_2^HAQ_2Q_2^H \\ &= Q\begin{pmatrix} Q_1^HAQ_1 & 0 \\ 0 & Q_2^HAQ_2 \end{pmatrix} Q^H \\ &\triangleq Q\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H, \end{aligned}$$

where $E_1 = Q_1^HAQ_1$ and $E_2 = Q_2^HAQ_2$. Since $A \in AHC^{n\times n}$, we have $E_1, E_2 \in \mathrm{AHC}^{k\times k}$. Conversely, the matrix A of the form (3) clearly belongs to $\mathrm{AHC}^{n\times n}$. Furthermore, it follows from $J = -i(P_1 - P_2) = -i(Q_1Q_1^H - Q_2Q_2^H)$ that

$$JQ = iQ\begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix} \quad \text{and} \quad Q^HJ = i\begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix} Q^H.$$

Consequently,

$$\begin{aligned} JAJ &= JQ\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^HJ \\ &= -Q\begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix} Q^H \\ &= -Q\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H = -A, \end{aligned}$$

which completes the proof. ■

3 General Solution of Problem I

In this section, we derive the expression for the general solution for Problem I, and then establish necessary and sufficient conditions about the solvability of the matrix equation $AX = B$ in $\mathrm{AHHC}^{n\times n}$.

The following Lemma comes from [11].

Lemma 3.1. *Let $X \in C^{n\times m}$, $B \in C^{n\times m}$. Suppose that the singular value decomposition of matrix X is as follows.*

$$X = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^H$$

where $U = (U_1, U_2) \in UC^{n\times n}$, $V = (V_1, V_2) \in UC^{m\times m}$, $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, $\sigma_j > 0, j = 1, \cdots, r$. Let $\phi = (\phi_{ij}), \phi_{ij} = \dfrac{1}{\sigma_i^2 + \sigma_j^2}, 1 \le i, j \le r$. Then

- *(1) the problem $\min\limits_{A\in \mathrm{AHC}^{n\times n}} \|AX - B\|$ has a solution. Moreover, the solution can be expressed as*

$$A = U \begin{pmatrix} \phi * (U_1^H B V_1 \Sigma - \Sigma V_1^H B^H U_1) & -\Sigma^{-1} V_1^H B^H U_2 \\ U_2^H B V_1 \Sigma^{-1} & G \end{pmatrix} U^H, \quad G \in AHC^{(n-r)\times(n-r)}. \tag{4}$$

- *(2) the matrix equation $AX = B$ has a solution in $AHC^{n\times n}$ if and only if*

$$B = BX^+X \quad \textit{and} \quad X^H B = -B^H X. \tag{5}$$

In addition, the solution can be expressed as

$$A = U \begin{pmatrix} U_1^H B V_1 \Sigma^{-1} & -\Sigma^{-1} V_1^H B^H U_2 \\ U_2^H B V_1 \Sigma^{-1} & G \end{pmatrix} U^H, \quad G \in AHC^{(n-r)\times(n-r)}. \tag{6}$$

Theorem 3.1. *Let $X, B \in C^{n\times m}$ and*

$$Q^H X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, Q^H B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \textit{where} \quad X_1, B_1 \in C^{k\times m}, n = 2k. \tag{7}$$

Suppose that the singular value decomposition of matrices X_1 and X_2 are as follows, respectively,

$$X_1 = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^H, \qquad X_2 = M \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} N^H,$$

where $U = (U_1, U_2) \in UC^{k\times k}$, $V = (V_1, V_2) \in UC^{m\times m}$, $M = (M_1, M_2) \in UC^{k\times k}$, $N = (N_1, N_2) \in UC^{m\times m}$, $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, $\sigma_j > 0, j = 1, \cdots, r$, $\Gamma = \mathrm{diag}(\delta_1, \cdots, \delta_s)$, $\delta_j > 0, j = 1, \cdots, s$, $\mathrm{rank}(X_1) = r$, and $\mathrm{rank}(X_2) = s$. Let $\phi = (\phi_{ij}), \psi = (\psi_{ij})$ with

$$\phi_{ij} = \frac{1}{\sigma_i^2 + \sigma_j^2}, 1 \le i, j \le r; \quad \psi_{ij} = \frac{1}{\delta_i^2 + \delta_j^2}, \ 1 \le i, j \le s.$$

Then the solution of Problem I can be expressed as

$$A = Q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H \tag{8}$$

where

$$E_1 = U \begin{pmatrix} \phi * (U_1^H B_1 V_1 \Sigma - \Sigma V_1^H B_1^H U_1) & -\Sigma^{-1} V_1^H B_1^H U_2 \\ U_2^H B_1 V_1 \Sigma^{-1} & G_1 \end{pmatrix} U^H, \quad G \in AHC^{(k-r)\times(k-r)}. \quad (9)$$

$$E_2 = M \begin{pmatrix} \psi * (M_1^H B_2 N_1 \Gamma - \Gamma N_1^H B_2^H M_1) & -\Gamma^{-1} N_1^H B_2^H M_2 \\ M_2^H B_2 N_1 \Gamma^{-1} & G_2 \end{pmatrix} M^H, \quad G_2 \in AHC^{(k-s)\times(k-s)}. \quad (10)$$

Proof. For any $A \in \mathrm{AHHC}^{n\times n}$, by Theorem 2.1, there exist $E_1, E_2 \in AHC^{k\times k}$ such that

$$A = Q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H. \quad (11)$$

It follows from (11) that

$$\begin{aligned} \|AX - B\|^2 &= \left\| Q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H X - B \right\|^2 \\ &= \left\| \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right\|^2 \\ &= \|E_1 X_1 - B_1\|^2 + \|E_2 X_2 - B_2\|^2. \end{aligned}$$

Hence, the problem $\min_{A \in \mathrm{AHHC}^{n\times n}} \|AX - B\|$ is equivalent to the following problems

$$\min_{E_1 \in \mathrm{AHC}^{k\times k}} \|E_1 X_1 - B_1\| \quad (12)$$

and

$$\min_{E_2 \in \mathrm{AHC}^{k\times k}} \|E_2 X_2 - B_2\|. \quad (13)$$

By Lemma 3.1, the solutions E_1 and E_2 of the problems (12) and (13) are given by (9), (10). Substituting (9) and (10) into (11), we get the desired result. ■

By Lemma 3.1 and Theorem 3.1, the following result can be established.

Theorem 3.2. *Let $X, B \in C^{n\times m}$ be the same as those in Theorem 3.1. Then the matrix equations $AX = B$ has a solution in $AHHC^{n\times n}$ if and only if*

$$X_i^H B_i = -B_i^H X_i \quad \text{and} \quad B_i = B_i X_i^+ X_i, i = 1, 2. \quad (14)$$

Moreover, the solution can be expressed as

$$A = Q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H, \quad (15)$$

where

$$E_1 = U \begin{pmatrix} U_1^H B_1 V_1 \Sigma^{-1} & -\Sigma^{-1} V_1^H B_1^H U_2 \\ U_2^H B_1 V_1 \Sigma^{-1} & G_1 \end{pmatrix} U^H, \quad G_1 \in AHC^{(k-r)\times(k-r)},$$

$$E_2 = M \begin{pmatrix} M_1^H B_2 N_1 \Gamma^{-1} & -\Gamma^{-1} N_1^H B_2^H M_2 \\ M_2^H B_2 N_1 \Gamma^{-1} & G_2 \end{pmatrix} M^H, \quad G_2 \in AHC^{(k-s)\times(k-s)}.$$

4 The expression of the solution for problem II

In this section, we prove the existence and uniqueness of the solution and derive an expression of the solution for Problem II.

Theorem 4.1. *Given $A^* \in C^{n\times n}$. Then the problem II has a unique solution $\hat{A}$. Moreover, we have*

$$\hat{A} = Q\begin{pmatrix} \hat{E}_1 & 0 \\ 0 & \hat{E}_2 \end{pmatrix} Q^H, \tag{16}$$

where

$$\hat{E}_1 = U\begin{pmatrix} \phi * (U_1^H B_1 V_1 \Sigma - \Sigma V_1^H B_1^H U_1) & -\Sigma^{-1} V_1^H B_1^H U_2 \\ U_2^H B_1 V_1 \Sigma^{-1} & \frac{1}{2}(Q_1U_2)^H (A^* - A^{*H}) Q_1 U_2 \end{pmatrix} U^H,$$

$$\hat{E}_2 = M\begin{pmatrix} \psi * (M_1^H B_2 N_1 \Gamma - \Gamma N_1^H B_2^H M_1) & -\Gamma^{-1} N_1^H B_2^H M_2 \\ M_2^H B_2 N_1 \Gamma^{-1} & \frac{1}{2}(Q_2M_2)^H (A^* - A^{*H}) Q_2 M_2 \end{pmatrix} M^H.$$

Proof. It is not difficult to prove from (8) that the solution set $S_{X,B}$ of the Problem I is a closed convex set. So, we get from [15] that for $A^* \in C^{n\times n}$, it has a unique optimal approximation. For $A^* \in C^{n\times n}$, let

$$\overline{A} = Q^H A^* Q \triangleq \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{22} & \overline{A}_{22} \end{pmatrix}, \tag{17}$$

where

$$\overline{A}_{11} = Q_1^H A^* Q_1,\ \overline{A}_{12} = Q_1^H A^* Q_2,\ \overline{A}_{21} = Q_2^H A^* Q_1,\ \overline{A}_{22} = Q_2^H A^* Q_2.$$

For any $A \in S_{X,B}$, by (8) and (17), we have

$$\begin{aligned} \|A^* - A\|^2 &= \left\| A^* - Q\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} Q^H \right\|^2 \\ &= \left\| \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{pmatrix} - \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \right\|^2 \\ &= \|\overline{A}_{12}\|^2 + \|\overline{A}_{21}\|^2 + \|\overline{A}_{11} - E_1\|^2 + \|\overline{A}_{22} - E_2\|^2. \end{aligned} \tag{18}$$

Since

$$\begin{aligned} &\|\overline{A}_{11} - E_1\|^2 = \|U^H \overline{A}_{11} U - U^H E_1 U\|^2 \\ = &\left\| \begin{pmatrix} U_1^H \overline{A}_{11} U_1 & U_1^H \overline{A}_{11} U_2 \\ U_2^H \overline{A}_{11} U_1 & U_2^H \overline{A}_{11} U_2 \end{pmatrix} - \begin{pmatrix} \phi * (U_1^H B_1 V_1 \Sigma - \Sigma V_1^H B_1^H U_1) & -\Sigma^{-1} V_1^H B_1^H U_2 \\ U_2^H B_1 V_1 \Sigma^{-1} & G_1 \end{pmatrix} \right\|^2 \\ = &\|U_1^H \overline{A}_{11} U_1 - \phi * (U_1^H B_1 V_1 \Sigma - \Sigma V_1^H B_1^H U_1)\|^2 + \|U_1^H \overline{A}_{11} U_2 + \Sigma^{-1} V_1^H B_1^H U_2\|^2 \\ &+ \|U_2^H \overline{A}_{11} U_1 - U_2^H B_1 V_1 \Sigma^{-1}\|^2 + \|U_2^H \overline{A}_{11} U_2 - G_1\|^2 \\ = &\|U_1^H \overline{A}_{11} U_1 - \phi * (U_1^H B_1 V_1 \Sigma - \Sigma V_1^H B_1^H U_1)\|^2 + \|U_1^H \overline{A}_{11} U_2 + \Sigma^{-1} V_1^H B_1^H U_2\|^2 \\ &+ \|U_2^H \overline{A}_{11} U_1 - U_2^H B_1 V_1 \Sigma^{-1}\|^2 + \|\frac{1}{2} U_2^H (\overline{A}_{11} + \overline{A}_{11}^H) U_2\|^2 \\ &+ \|\frac{1}{2} U_2^H (\overline{A}_{11} - \overline{A}_{11}^H) U_2 - G_1\|^2. \end{aligned} \tag{19}$$

Similarly, we have

$$\begin{aligned} \|\overline{A}_{22} - E_2\|^2 = &\|M_1^H \overline{A}_{22} M_1 - \psi * (M_1^H B_2 N_1 \Gamma - \Gamma N_1^H B_2^H M_1)\|^2 + \\ &\|M_1^H \overline{A}_{22} M_2 + \Gamma^{-1} N_1^H B_2^H M_2\|^2 + \|M_2^H \overline{A}_{22} M_1 - M_2^H B_2 N_1 \Gamma^{-1}\|^2 + \\ &\|\tfrac{1}{2} M_2^H (\overline{A}_{22} + \overline{A}_{22}^H) M_2\|^2 + \|\tfrac{1}{2} M_2^H (\overline{A}_{22} - \overline{A}_{22}^H) M_2 - G_2\|^2. \end{aligned} \tag{20}$$

Hence, the equalities (18), (19) and (20) imply that the problem $\min_{A\in S_{X,B}} \|A^* - A\|$ is equivalent to the following problems

$$\min_{G_1\in \mathrm{AHC}^{(k-r)\times(k-r)}} \|\frac{1}{2}U_2^H(\overline{A}_{11} - \overline{A}_{11}^H)U_2 - G_1\| \tag{21}$$

and

$$\min_{G_2\in \mathrm{AHC}^{(k-s)\times(k-s)}} \|\frac{1}{2}M_2^H(\overline{A}_{22} - \overline{A}_{22}^H)M_2 - G_2\|^2. \tag{22}$$

It is obvious that the solutions of problems (21) and (22) are given by

$$G_1 = \frac{1}{2}U_2^H(\overline{A}_{11} - \overline{A}_{11}^H)U_2 = \frac{1}{2}U_2^H Q_1^H(A^* - A^{*H})Q_1U_2 \tag{23}$$

and

$$G_2 = \frac{1}{2}M_2^H(\overline{A}_{22} - \overline{A}_{22}^H)M_2 = \frac{1}{2}M_2^H Q_2^H(A^* - A^{*H})Q_2U_2 \tag{24}$$

respectively. Substituting (23) and (24) to (9) and (10), respectively, we then obtain the solution (16) of Problem II. The proof is completed. ■

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