NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **16** No. 1, pp. 83-91
February 2007

The Equivalence of Ishikawa-Mann and Multistep Iterations in Banach Space†

Liping Yang*

Faculty of Applied Mathematics, Guangdong University of Technology, Guangdong 510090, China.

Received June 8, 2005; Accepted (in revised version) February 16, 2006

Abstract. Let E be a real Banach space and T be a continuous $\Phi-$strongly accretive operator. By using a new analytical method, it is proved that the convergence of Mann, Ishikawa and three-step iterations are equivalent to the convergence of multistep iteration. The results of this paper extend the results of Rhoades and Soltuz in some aspects.

Key words: $\Phi-$strongly accretive operator; $\Phi-$strongly pseudocontractive operator; continuous; Mann iteration; Ishikawa iteration; multistep iteration.

AMS subject classifications: 47H09, 47H10

1 Introduction

Let E denote an arbitrary real Banach space and E^* denote the dual space of E. The duality map $J: E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. We first recall and define some concepts as follows:

Definition 1.1. Let K be a nonempty subset of E and let $T: K \to E$ be an operator.

(i). T is said to be accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0. \tag{1}$$

(ii). T is said to be strongly accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2, \tag{2}$$

where $k > 0$ is a constant. Without loss of generality, we assume that $k \in (0,1)$.

*Correspondence to: Liping Yang, Faculty of Applied Mathematics, Guangdong University of Technology, Guangdong 510090, China. Email: `yanglping2003@126.com`

†Supported by the Nature Science Foundation of Guangdong Province (020163).

(iii). T is said to be $\Phi-$strongly accretive if, for $\forall x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \Phi(\|x-y\|)\|x-y\|, \tag{3}$$

where Φ: $[0,\infty) \to [0,\infty)$ is a function for which $\Phi(0)$=0, $\Phi(r)$>0 for all r>0, $\liminf\limits_{r\to\infty} \Phi(r)$>0 and the function $h(r) = r\Phi(r)$ is nondecreasing on $[0,\infty)$.

If I denotes the identity operator, it follows from inequalities (1)-(3) that T is pseudocontractive (respectively, strongly pseudocontractive, $\Phi-$strongly pseudocontractive) if and only if $(I-T)$ is an accretive (respectively, strongly accretive,$\Phi-$strongly accretive). It is shown in [1] that the class of single-valued strongly pseudocontractive operators is a proper subclass of the class of single-valued $\Phi-$strongly pseudocontractive operators. The classes of single-valued operators have been studied by many authors (see, for example [1]- [13]).

Now, we state concepts which will be needed in the sequel.

(a). The iteration (see [9])

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T x_n, \quad n = 0, 1, 2, \cdots \end{cases} \tag{4}$$

is called the Ishikawa iteration sequence, where $\{\alpha_n\}, \{\beta_n^1\}$ are real sequences in $[0,1]$ satisfying some appropriate conditions.

(b). In particular, if $\beta_n^1 = 0$ for $n \geq 0$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, 2, \cdots \tag{5}$$

is called the Mann iteration (see [10]).

(c). In [11], Noor introduced the three-step procedure

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + T y_n^1, \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T y_n^2, \\ y_n^2 = (1-\beta_n^2)x_n + \beta_n^2 T x_n, \quad n = 0, 1, 2, \cdots, \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{\beta_n^1\}, \{\beta_n^2\}$ are real sequences in $[0,1]$ satisfying some appropriate conditions.

(d). In [13], Rhoades and Soltuz introduced the multi-step procedure

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + T y_n^1, \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1, \cdots, p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \quad n = 0, 1, 2, \cdots, \end{cases} \tag{7}$$

where $p \geq 2$ is a fixed order, $\{\alpha_n\}$ is a sequence in $(0,1)$ such that for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{8}$$

Moreover, for all $n \in \mathbb{N}$

$$\{\beta_n^i\} \subset [0,1), \quad 1 \leq i \leq p-1, \quad \lim_{n\to\infty} \beta_n^1 = 0. \tag{9}$$

Taking p=3 in (7), we get the three-step iteration (6). Taking p=2 in (7), we get the Ishikawa iteration (4). Iterative methods for approximating fixed points of strongly ($\Phi-$strongly) accretive operator have been studied by some authors (see, e.g., [1–10]), using the Mann iteration process or the Ishikawa iteration process. Then we have a question: are there any differences on convergence between these sequences? Can we prove the equivalence on convergence between the two sequences? Rhoades and Soltuz in [13] show that the convergence of Mann, Ishikawa iterations are equivalent to the multi-step iteration for strongly pseudocontractive operator and strongly accretive operator in uniformly convex Banach space. Motivated and inspired by the results of [13], we prove in this paper that the convergence of Mann, Ishikawa iterations are equivalent to the multi-step iteration for $\Phi-$strongly pseudocontractive operator and $\Phi-$strongly accretive operator in any Banach space. The results of this paper extend and improve Rhoades and Soltuz's results in some aspects.

We shall use the following lemmas in the sequel.

Lemma 1.1. ([8]) *Let $\{\sigma_n\}, \{\mu_n\}, \{t_n\}$ be nonnegative real sequences. Assume there exists a positive integer n_0 such that*

$$\sigma_{n+1} \leq (1-t_n)\sigma_n + \mu_n, \qquad for\ \forall\ n \geq n_0,$$

where $0 \leq t_n \leq 1$, $\sum_{n=1}^{\infty} t_n = \infty$, $\mu_n = o(t_n)$. Then $\sigma_n \to 0$ $(n \to \infty)$.

Lemma 1.2. ([14]) *Let E be a real Banach space and T: $E \to E$ be a continuous $\Phi-$strongly accretive operator. Then the equation $Tx = f$ has a unique solution for any $f \in E$.*

2 Main results

Theorem 2.1. *Let E be a real Banach space and T: $E \to E$ be a continuous $\Phi-$strongly accretive operator. Assume $\{\alpha_n\} \subset (0,1)$ satisfies (8) and $\{\beta_n^i\} \subset [0,1)$ $(i = 1, \cdots, p-1)$ satisfies (9). If the sequences $\{Tu_n\}_{n=0}^{\infty}, \{Tx_n\}_{n=0}^{\infty}, \{Ty_n^i\}_{n=0}^{\infty}$ $(i = 1, \cdots, p-1)$ or the sequences $\{u_n - Tu_n\}_{n=0}^{\infty}, \{x_n - Tx_n\}_{n=0}^{\infty}, \{y_n^i - Ty_n^i\}_{n=0}^{\infty}$ $(i = 1, \cdots, p-1)$ are bounded and $u_0 = x_0 \in E$, then the following are equivalent:*

(i). *the Mann iterative sequence $\{u_n\}$ defined for $\forall u_0 \in E$ by*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n(f + u_n - Tu_n), \quad n = 0, 1, \cdots \tag{10}$$

converges strongly to the solution of the equation $Tx = f$ for any given $f \in E$.

(ii). *the multi-step iterative sequence $\{x_n\}$ defined for $\forall x_0 \in E$ by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + y_n^1 - Ty_n^1), \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i(f + y_n^{i+1} - Ty_n^{i+1}), \quad i = 1, \cdots, p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1}(f + x_n - Tx_n), \quad n = 0, 1, 2, \cdots \end{cases} \tag{11}$$

converges strongly to the solution of the equation $Tx = f$ for any given $f \in E$.

Proof It follows from Lemma 1.2 that the equation $Tx = f$ has a unique solution $q \in E$. Define S: $E \to E$ by $Sx = f + (I-T)x$ for $\forall x \in E$. Then we know that S is continuous and q is a unique fixed point of S. If the multi-step iterative sequence (11) converges strongly to a point x^*, we can prove that x^* is a fixed point following a procedure in [12]. Setting $\beta_n^i = 0$ $(i = 1, 2, \cdots, p-1)$ in (11), we get the convergence of the Mann iteration. Conversely, we will prove that the convergence of the Mann iteration implies the convergence of the multistep iteration. Since T is $\Phi-$strongly accretive operator, then the operator S is $\Phi-$strongly pseudocontractive and for $\forall x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \Phi(\|x-y\|)\|x-y\|,$$

which implies that

$$\Phi(\|x-y\|) \leq \|Tx - Ty\|.$$

Then, for $\forall x, y \in E$, we have

$$\|Sx - Sy\| \leq \|x-y\| + \|Tx - Ty\| \leq \Phi^{-1}(\|Tx-Ty\|) + \|Tx - Ty\|, \tag{12}$$

and

$$\|Sx - Sy\| \leq \|x - Tx\| + \|y - Ty\|. \tag{13}$$

It follows from the boundedness assumptions on Tx_n, Tu_n, Ty_n^i, as well as (12) and (13) that the sequences $\{Sx_n\}, \{Su_n\}, \{Sy_n^i, i = 1, \cdots, p-1\}$ are bounded. If we denote $M = \max\{\sup_{n\in\mathbb{N}}\{\|Sx_n - q\|\}, \sup_{n\in\mathbb{N}}\{\|Su_n - q\|\}, \sup_{n\in\mathbb{N}}\{\|Sy_n^i - q\| : 1 \leq i \leq p-1\}\} + \|x_0 - q\|$, then $M < \infty$. By induction, we have that

$$\|u_n - q\| \leq M, \quad \|x_n - q\| \leq M, \quad n \in \mathbb{N}. \tag{14}$$

Using (10) and (11) gives

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\|^2 &\leq (1-\alpha_n)\|x_n - u_n\| \cdot \|x_{n+1} - u_{n+1}\| + \alpha_n\langle Sy_n^1 - Su_n, j(x_{n+1} - u_{n+1})\rangle \\
&= (1-\alpha_n)\|x_n - u_n\| \cdot \|x_{n+1} - u_{n+1}\| + \alpha_n\langle Sx_{n+1} - Su_{n+1}, j(x_{n+1} - u_{n+1})\rangle \\
&\quad +\alpha_n\langle Sy_n^1 - Su_n - (Sx_{n+1} - Su_{n+1}), j(x_{n+1} - u_{n+1})\rangle \\
&\leq (1-\alpha_n)\|x_n - u_n\| \cdot \|x_{n+1} - u_{n+1}\| \\
&\quad +\alpha_n[\|x_{n+1} - u_{n+1}\|^2 - \Phi(\|x_{n+1} - u_{n+1}\|)\|x_{n+1} - u_{n+1}\|] \\
&\quad +\alpha_n q_n\|x_{n+1} - u_{n+1}\|,
\end{aligned} \tag{15}$$

where $q_n = \|Sy_n^1 - Su_n - (Sx_{n+1} - Su_{n+1})\|$. From (10) and (8), we have

$$\|u_{n+1} - u_n\| \leq \alpha_n[\|u_n - q\| + \|Su_n - q\|] \leq 2M\alpha_n \to 0, \ (n \to \infty).$$

It follows from (11), (8) and (9) that

$$\begin{aligned}
\|x_{n+1} - y_n^1\| &\leq \alpha_n\|x_n - Ty_n^1\| + \beta_n^1\|x_n - Ty_n^2\| \\
&\leq 2M(\alpha_n + \beta_n^1) \to 0, \quad (n \to \infty).
\end{aligned}$$

Since S is continuous, we have

$$q_n \leq \|Sx_{n+1} - Sy_n^1\| + \|Su_{n+1} - Su_n\| \to 0, \quad (n \to \infty).$$

Note that

$$\begin{aligned}
&(1-\alpha_n)\|x_n - u_n\| \cdot \|x_{n+1} - u_{n+1}\| \leq \frac{1}{2}((1-\alpha_n)^2\|x_n - u_n\|^2 + \|x_{n+1} - u_{n+1}\|^2), \\
&\|x_{n+1} - u_{n+1}\| \leq \frac{1}{2}(1 + \|x_{n+1} - u_{n+1}\|^2).
\end{aligned}$$

If we set $b_n = \|x_n - u_n\|$, then it follows from (15) that

$$[1-\alpha_n(2+q_n)]b_{n+1}^2 \le (1-\alpha_n)^2 b_n^2 - 2\alpha_n \Phi(b_{n+1})b_{n+1} + \alpha_n q_n. \tag{16}$$

Owing to $\lim_{n\to\infty}[1-\alpha_n(2+q_n)] = 1 > 0$, then there exists a positive integer N_0 such that $1-\alpha_n(2+q_n) > 0$ for $n \ge N_0$. Without loss of generality, let $1-\alpha_n(2+q_n) > 0$ for $\forall n > 0$. Thus, for $\forall n > 0$, from (16), we get

$$b_{n+1}^2 \le \frac{(1-\alpha_n)^2}{1-\alpha_n(2+q_n)} b_n^2 - \frac{2\alpha_n}{1-\alpha_n(2+q_n)}\Phi(b_{n+1})b_{n+1} + \frac{\alpha_n q_n}{1-\alpha_n(2+q_n)}. \tag{17}$$

It readily follows from (17) that $b_n \to 0$ as $n \to \infty$. However, for completeness, we present the details. We consider the following cases which cover all the possibilities:

1. Suppose $\inf_{n\ge 0} b_{n+1} > 0$. Then due to the assumption on the function Φ, there exists $r > 0$ such that

$$r < \frac{\Phi(b_{n+1})}{b_{n+1}}, \qquad \forall n > 0. \tag{18}$$

Combining (17) and (18) yields

$$b_{n+1}^2 \le \frac{(1-\alpha_n)^2}{1-\alpha_n(2+q_n)+2r\alpha_n} b_n^2 + \frac{\alpha_n q_n}{1-\alpha_n(2+q_n)+2r\alpha_n}. \tag{19}$$

Since $\alpha_n \to 0, q_n \to 0$ $(n\to\infty)$, there exists n_0 such that

$$\begin{aligned} & \Big(1-\alpha_n\Big)^2 - \Big(1-\frac{r}{2}\alpha_n\Big)\Big[1-\alpha_n(2+q_n)+2r\alpha_n\Big] \\ = \;& \alpha_n\Big[\alpha_n + q_n - \frac{r}{2}\alpha_n(2+q_n) + r^2\alpha_n - \frac{3}{2}r\Big] \\ \le \;& 0, \quad \forall n \ge n_0. \end{aligned}$$

Therefore, we get

$$\frac{(1-\alpha_n)^2}{1-\alpha_n(2+q_n)+2r\alpha_n} \le 1 - \frac{r}{2}\alpha_n, \quad \forall n \ge n_0. \tag{20}$$

It follows from (19) and (20) that

$$b_{n+1}^2 \le \Big(1-\frac{r}{2}\alpha_n\Big)b_n^2 + \frac{\alpha_n q_n}{1-\alpha_n(2+q_n)+2r\alpha_n}, \quad \forall n \ge n_0.$$

Let $\mu_n = \dfrac{\alpha_n q_n}{1-\alpha_n(2+q_n)+2r\alpha_n}$. Using Lemma 1.1, we have $b_n \to 0$ $(n\to\infty)$, which is a contradiction.

2. If $\inf_{n\ge 0} b_{n+1} = 0$, then there exists subsequence $\{b_{n_j+1}\} \subset \{b_{n+1}\}$ such that $b_{n_j+1} \to 0$ $(j\to\infty)$. Since $\alpha_n \to 0, q_n \to 0$ $(n\to\infty)$, $\forall \varepsilon \in (0,1)$, there exists a positive integer n_{j_0} such that

$$\left.\begin{aligned} & b_{n_{j_0}+1} < \varepsilon, \\ & \alpha_n < \frac{1}{4}\Phi(\varepsilon), \quad q_n < \frac{1}{4}\Phi(\varepsilon)\varepsilon \end{aligned}\right\} \tag{21}$$

for $n \geq n_{j_0}$. Our next step is to show that

$$b_{n_{j_0}+i} \leq \varepsilon, \quad \forall i \geq 1. \tag{22}$$

In fact, for i=1, we know the conclusion holds from (21). For i=2, we assume the conclusion does not holds. Then we have

$$b_{n_{j_0}+2} > \varepsilon. \tag{23}$$

Since $h(r)$ is nondecreasing, we have $\Phi(b_{n_{j_0}+2})b_{n_{j_0}+2} \geq \Phi(\varepsilon)\varepsilon$. Let $h_n = \dfrac{1}{1-\alpha_n(2+q_n)}$. Then the first term of the right-hand side of (17) becomes

$$\frac{(1-\alpha_n)^2}{1-\alpha_n(2+q_n)} b_n^2 = b_n^2 + h_n\alpha_n(\alpha_n + q_n)b_n^2.$$

Consequently, it follows from (17) and (21) that

$$\begin{aligned}
b^2_{n_{j_0}+2} &\leq b^2_{n_{j_0}+1} + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\left\{[\alpha_{n_{j_0}+1} + q_{n_{j_0}+1}]b^2_{n_{j_0}+1} - 2\Phi(\varepsilon)\varepsilon + q_{n_{j_0}+1}\right\} \\
&\leq \varepsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\left\{\left[\frac{1}{4}\Phi(\varepsilon) + \frac{1}{4}\Phi(\varepsilon)\varepsilon\right]\varepsilon^2 - 2\Phi(\varepsilon)\varepsilon + \frac{1}{4}\Phi(\varepsilon)\varepsilon\right\} \\
&\leq \varepsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\left\{\frac{3}{4}\Phi(\varepsilon)\varepsilon - 2\Phi(\varepsilon)\varepsilon\right\} < \varepsilon^2,
\end{aligned}$$

which is a contradiction with (23). Hence $b_{n_{j_0}+2} < \varepsilon$ holds and inductively we can show that (22) holds. This implies that $\lim_{n\to\infty} b_n = 0$, i.e.,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{24}$$

Suppose that $\lim\limits_{n\to\infty} u_n = q^*$. The inequality

$$0 \leq \|x_n - q^*\| \leq \|u_n - q^*\| + \|x_n - u_n\|$$

and (24) imply that $\lim_{n\to\infty} x_n = q^*$. This completes the proof. ■

Remark 2.1. Theorem 2.1 extends and improves Theorem 2.1 of [13] in the following aspects:

1. Abolish the condition that E^* is uniformly convex used in [13];
2. The hypotheses conditions that a closed, convex, bounded subset B of E in [13] is replaced by the more general conditions $\{(I-T)u_n\}$, $\{(I-T)x_n\}$, $\{(I-T)y_n^i\}$ or $\{Tu_n\}$, $\{Tx_n\}$, $\{Ty_n^i\}(1,\cdots,p-1)$ are bounded;
3. The strongly pseudocontractive operator in [13] is replaced by the $\Phi-$strongly pseudocontractive operator.

Taking $p = 2, 3$ in (11), respectively, Theorem 2.1 leads to the following result.

Corollary 2.1. *If the assumptions in Theorem 2.1 hold, then the following are equivalent:*

(i). *the Mann iterative sequence* (10) *converges strongly to the solution of the equation* $Tx = f$ *for any given* $f \in E$*;*

(ii). *the Ishikawa iterative sequence* $\{x_n\}$ *defined for any* $x_0 \in E$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + y_n^1 - Ty_n^1), \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1(f + x_n - Tx_n), \quad n = 0, 1, 2, \cdots \end{cases} \tag{25}$$

converges strongly to the solution of the equation $Tx = f$ *for any given* $f \in E$;

(iii). *the three-step iterative sequence* $\{x_n\}$ *defined for any* $x_0 \in E$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + y_n^1 - Ty_n^1), \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1(f + y_n^2 - Ty_n^2), \\ y_n^2 = (1-\beta_n^2)x_n + \beta_n^2(f + x_n - Tx_n), \quad n = 0, 1, 2, \cdots \end{cases} \tag{26}$$

converges strongly to the solution of the equation $Tx = f$ *for any given* $f \in E$;

(iv). *the multi-step iterative sequence* (11) *converges strongly to the solution of the equation* $Tx = f$ *for any given* $f \in E$.

If we put $S = I + T$ and T: $E \to E$ be a continuous $\Phi-$strongly accretive operator. It is easy to prove that S is a continuous $\Phi-$strongly accretive operator. For all $x \in E$, we have $f - Tx = f - (S - I)x = f + x - Sx$. Thus, the Mann iterative sequence (10) becomes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n(f - Tu_n), \quad n = 0, 1, \cdots. \tag{27}$$

The Ishikawa iterative sequence (25) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n^1), \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1(f - Tx_n), \quad n = 0, 1, 2, \cdots. \end{cases} \tag{28}$$

The three-step iterative sequence (26) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n^1), \\ y_n^1 = (1-\beta_n^1)x_n + \beta_n^1(f - Ty_n^2), \\ y_n^2 = (1-\beta_n^2)x_n + \beta_n^2(f - Tx_n), \ n = 0, 1, 2, \cdots. \end{cases} \tag{29}$$

The multi-step iterative sequence (11) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n^1), \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i(f - Ty_n^{i+1}), \quad i = 1, \cdots, p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1}(f - Tx_n), \quad n = 0, 1, 2, \cdots. \end{cases} \tag{30}$$

The following result follows from Corollary 2.1.

Corollary 2.2. *Let* E *be a real Banach space and* T: $E \to E$ *be a continuous* $\Phi-$*strongly accretive operator. Assume that* $\{\alpha_n\} \subset (0,1)$ *satisfies* (8) *and* $\{\beta_n^i\} \subset [0,1)(i = 1, \cdots, p-1)$ *satisfy* (9). *If the sequences* $\{u_n + Tu_n\}_{n=0}^\infty, \{x_n + Tx_n\}_{n=0}^\infty, \{y_n^i + Ty_n^i\}_{n=0}^\infty$ $(i = 1, \cdots, p-1)$ *or the sequences* $\{Tu_n\}_{n=0}^\infty, \{Tx_n\}_{n=0}^\infty, \{Ty_n^i\}_{n=0}^\infty$ $(i = 1, \cdots, p-1)$ *are bounded and* $u_0 = x_0 \in E$, *then the following are equivalent:*

(i). *the Mann iterative sequence* (27) *converges strongly to the solution of the equation* $x + Tx = f$ *for any given* $f \in E$;

(ii). *the Ishikawa iterative sequence* (28) *converges strongly to the solution of the equation* $x + Tx = f$ *for any given* $f \in E$;

(iii). *the three-step iterative sequence* (29) *converges strongly to the solution of the equation* $x + Tx = f$ *for any given* $f \in E$;

(iv). *the multi-step iterative sequence* (30) *converges strongly to the solution of the equation* $x + Tx = f$ *for any given* $f \in E$.

Let $S = I - T$ and $f = 0$. If T is a continuous $\Phi-$strongly pseudocontractive operator, then S is continuous $\Phi-$strongly accretive operator. It follows from Lemma 1.2 that $Sx = 0$ has a unique solution $p \in E$ if and only if the operator T has a unique fixed point $p \in E$. On the other hand, $\forall x \in E$, we have $Tx = f + (I - S)x = (I - S)x$. Thus, the Mann iterative sequence (10) becomes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, n = 0, 1, \cdots, \tag{31}$$

and the Ishikawa iterative sequence (25) becomes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^1 \quad = (1-\beta_n^1)x_n + \beta_n^1 T x_n, \quad n = 0, 1, 2, \cdots. \end{cases} \tag{32}$$

Moreover, the multi-step iterative sequence (11) becomes

$$\begin{cases} |Dx_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^i \quad = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1, \cdots, p-2, \\ y_n^{p-1} \quad = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \quad n = 0, 1, 2, \cdots. \end{cases} \tag{33}$$

It follows from Theorem 2.1 that we get the following result.

Corollary 2.3. *Let* E *be a real Banach space and* T: $E \to E$ *be a continuous* $\Phi-$*strongly pseudocontractive operator. If* $\{\alpha_n\}$, $\{\beta_n^i\}$, T *and* u_0 *satisfy the assumptions of Theorem* 2.1, *then the following are equivalent:*

(i). *the Mann iterative sequence* (31) *converges strongly to the fixed point of* T;

(ii). *the Ishikawa iterative sequence* (32) *converges strongly to the fixed point of* T;

(iii). *the multi-step iterative sequence* (33) *converges strongly to the fixed point of* T.

Remark 2.2. The iteration parameters $\{\alpha_n\}$, $\{\beta_n^i\}$ $(i = 1, \cdots, p-1)$ in Theorem 2.1 and Corollaries 2.1-2.3 do not depend on any geometric structure of the Banach space E or on any property of the operator T.

Following the above results, we know that if the simple Mann iterative sequence converges, then the convergence properties of the Ishikawa, three-step, and multi-step iterative processes are obtained.

References

[1] Osilike M O, Iterative solution of nonlinear equations of the $\phi-$strongly accretive type. J. Math. Anal. Appl., 1996, 200(2): 259-271.

[2] Chidume C E, Iterative solution of nonlinear equations in smooth Banach spaces. Nonlinear Anal., 1996, 26: 1823-1834.

[3] Chidume C E, Approximation of fixed point of strictly pseudocontractive mappings. Proc. Am. Math. Soc., 1994, 120: 545-551.

[4] Chidume C E, Iterative solution of nonlinear equations with strongly accretive operators. J. Math. Anal. Appl. 1995, 192: 501-518.

[5] Browder F E, Nonlinear mappings of nonexpansive and accretive type in Banach spaces. Bull. Am. Math. Soc., 1967, 73: 875-882.
[6] Kato T, Nonlinear semigroups and evolution equations. J. Math. Soc. Japan., 1967, 19: 508-520.
[7] Morales C, Jung J S, Convergence of paths for pseudocontractive mappings in Banach space. Proc. Am. Math. Soc., 2000, 128: 3411-3419.
[8] Weng X, Fixed point iteration for local strictly pseudocontractive mapping. Proc. Am. Math. Soc., 1991, 113: 727-731.
[9] Ishikawa S, Fixed point by a new iteration method. Proc. Am. Math. Soc., 1974, 44: 147-150.
[10] Mann W R, Mean value in iteration. Proc. Am. Math. Soc., 1953, 4: 506-510.
[11] Noor M A, New approximation schemes for general variational inequalities. J. Math. Anal. Appl., 2000, 251: 217-229.
[12] Rhoades B E, Soltuz, S M, The equivalence of Mann iteration and Ishikawa iteration for non-Lipschitzian operators. Int. J. Math. Math., 2003, 2003: 2645-2652.
[13] Rhoades B E, Soltuz S M, The equivalence between Mann-Ishikawa iterations and multistep iteration. Nonlinear Anal., 2004, 58:219-228.
[14] Liu Z Q, Kang S M, Convergence theorems for $\phi-$strongly accretive and $\phi-$hemicontractive operators. J. Math. Anal. Appl., 2001, 253: 35-49.