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Mixed Method for Compressible Miscible Displacement with Dispersion in Porous Media

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Abstract. Compressible miscible displacement of one fluid by another in porous media is modelled by a nonlinear parabolic system. A finite element procedure is introduced to approximate the concentration of one fluid and the pressure of the mixture. The concentration is treated by a Galerkin method while the pressure is treated by a parabolic mixed finite element method. The effect of dispersion, which is neglected in [1], is considered. Optimal order estimates in L^2 are derived for the errors in the approximate solutions.

Key words: Compressible miscible displacement; mixed finite element method; dispersion; error estimate.

AMS subject classifications: 65N30

1 Introduction

Miscible displacement of one compressible fluid by another in a porous medium is modeled by a nonlinear parabolic system [1, 4]

$$d(c)\frac{\partial p}{\partial t} + \nabla \cdot u = d(c)\frac{\partial p}{\partial t} - \nabla \cdot (a(c)\nabla p) = q, \tag{1a}$$

$$\phi(x)\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + u \cdot \nabla c - \nabla \cdot (D(u)\nabla c) = (\hat{c} - c)q, \tag{1b}$$

where c denotes the volumetric concentration of one of the two components of the fluid ($c = c_1 = 1 - c_2$), and p denotes its pressure. The coefficients $a(c)$, $b(c)$, $d(c)$, $\phi(x)$ (porosity of the rock) are assumed bounded below positively and $a(c)$, $b(c)$, $d(c) \in C^1$; q is the external volumetric flow rate; $\hat{c}$ is the concentration of the external flow, which is specified at points where injection ($q > 0$) takes place, or assumed to be equal to c at production points; u is the Darcy velocity of the fluid satisfying

$$u = -a(c)\nabla p, \tag{2}$$

$D(u)$ combines the effects of molecular diffusion and dispersion [5], defined by

$$D = \phi\{d_m I + |u|(d_l E(u) + d_t E^{\perp}(u))\} \tag{3}$$

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where $E(u) = [u_k u_l/|u|^2]$ is an 2×2 matrix representing orthogonal projection along the velocity vector and $E^{\perp}(u) = I - E(u)$ is its orthogonal complement. $D(u)$ is a positive definite matrix since the effect of molecular diffusion is much greater than that of dispersion. In addition, the reservoir Ω will be taken to be of unit thickness and be identified with a bounded domain in R^2.

We shall also assume that no flow occurs across the boundary:

$$u \cdot \nu = 0 \qquad on\ \partial\Omega, \tag{4a}$$
$$(D\nabla c - cu) \cdot \nu = 0 \quad on\ \partial\Omega, \tag{4b}$$

where ν is the outer normal to $\partial\Omega$. The initial conditions are

$$p(x,0) = p_0(x) \quad x \in \Omega, \tag{5a}$$
$$c(x,0) = c_0(x) \quad x \in \Omega. \tag{5b}$$

Douglas [1] introduced a mixed finite element procedure for the same problem while dispersion was neglected such that $D = \phi(x)d_m I$. Cheng [8] introduced a Galerkin procedure with dispersion on rectangle element and derived optimal error estimates. Wang and Cheng [9] considered the Galerkin method with dispersion to another similar problem on quasi-regular element, and derived nearly optimal error estimates. In this paper, a mixed finite element procedure on quasi-regular element is introduced with dispersion so that $D = D(u)$ (see (3)). The analysis of the procedure is based on [1] while different test functions are selected and two projections are introduced to derive the optimal error estimates in L^2.

2 Formulation of the mixed finite element procedure

It is well known that physical transport dominates the diffusive effects in realistic examples of compressible miscible displacement. Thus it is more important to obtain good approximate velocities than to achieve high accuracy in pressure. This motivates the use of mixed method in the calculation of the pressure and velocity.

Firstly, the weak form for the parabolic system (1a), (1b) and (2) is given by

$$\left(\phi\frac{\partial c}{\partial t}, z\right) + \left(b(c)\frac{\partial p}{\partial t}, z\right) + (u \cdot \nabla c, z) + (D(u)\nabla c, \nabla z) = ((\hat{c} - c)q, z) \quad z \in H^1(\Omega), t \in J, \tag{6a}$$
$$\left(d(c)\frac{\partial p}{\partial t}, w\right) + (\nabla \cdot u, w) = (q, w) \qquad w \in L^2(\Omega), t \in J, \tag{6b}$$
$$(\alpha(c)u, v) - (\nabla \cdot v, p) = 0 \qquad v \in V, t \in J, \tag{6c}$$

where $V = \{v \in H(div;\Omega) : v \cdot \nu = 0\ on\ \partial\Omega\}$, $\alpha(c) = a(c)^{-1}$ and $J = (0,T]$.

Let $h = (h_c, h_p)$, where h_c and h_p are positive. Let M_h denote a standard finite element space whose elements diameters are bounded by h_c. Assume that M_h is associated with a quasi-regular polygonalization of Ω and piecewise-polynomial functions of some fixed degree greater or equal to l. As a result, all standard inverse relations hold on M_h, which will be used frequently in our analysis. Then let $V_h \times W_h$ be a Raviart-Thomas [6] space of index at least k associated with quasi-regular triangulation or quadrilateralization (or a mixture of the two) of Ω such that the elements diameters are bounded by h_p. If the approximations for the pressure, concentration and velocity are denoted by $c_h \in M_h$, $p_h \in W_h$ and $u_h \in V_h$ respectively, then they are defined to be the solutions of the equations

$$\left(\phi\frac{\partial c_h}{\partial t}, z\right) + \left(b(c_h)\frac{\partial p_h}{\partial t}, z\right) + (u_h \cdot \nabla c_h, z) + (D(u_h)\nabla c_h, \nabla z) = ((\hat{c}_h - c_h)q, z)$$
$$z \in M_h, t \in J, \tag{7a}$$

$$\left(d(c_h)\frac{\partial p_h}{\partial t}, w\right) + (\nabla \cdot u_h, w) = (q, w) \quad w \in W_h, t \in J, \tag{7b}$$

$$(\alpha(c_h)u_h, v) - (\nabla \cdot v, p_h) = 0 \qquad v \in V_h, t \in J, \tag{7c}$$

where function $\hat{c}_h = \hat{c}$ if $q > 0$ and $\hat{c}_h = c_h$ if $q < 0$. Initial values must be specified for $c_h(0)$ and $p_h(0)$; and consistent initial value $u_h(0)$ can then be computed from (7c).

3 Analysis of the mixed method procedure

Because the primary concern in the evaluation of a miscible displacement process will lie in obtaining accurate information about the behavior in the interior of the domain, we shall emphasize the interior behavior by considering either the no-flow boundary conditions (4) or by assuming Ω to be rectangle and by replacing the no-flow boundary conditions with the assumption that the problem is periodic with Ω as periodic. To make the analysis convenient [1], we project the solution of the differential problem (1) into the finite element spaces by means of coercive elliptic forms associated with the differential system. Let $\tilde{c} = \tilde{c}_h : J \to M_h$, $\tilde{p} = \tilde{p}_h : J \to W_h$ and $\tilde{u} = \tilde{u}_h : J \to V_h$ be determined by the relations

$$(D(u)\nabla(c - \tilde{c}), \nabla z) + (u \cdot \nabla(c - \tilde{c}), z) + \lambda(c - \tilde{c}, z) = 0 \quad z \in M_h, t \in J, \tag{8a}$$

$$\left(d(c)\frac{\partial p}{\partial t}, w\right) + (\nabla \cdot \tilde{u}, w) = (q, w) \quad w \in W_h, t \in J, \tag{8b}$$

$$(\alpha(c)\tilde{u}, v) - (\nabla \cdot v, \tilde{p}) = 0 \qquad v \in V_h, t \in J, \tag{8c}$$

$$(\tilde{p}, 1) = (p, 1), \tag{8d}$$

where the constant λ is chosen large enough to insure the coercivity of the bilinear form. Let

$$\begin{aligned} \zeta &= c - \tilde{c}, \quad & \xi &= \tilde{c} - c_h, \quad & \eta &= p - \tilde{p}, \\ \pi &= \tilde{p} - p_h, \quad & \rho &= u - \tilde{u}, \quad & \sigma &= \tilde{u} - u_h. \end{aligned} \tag{9}$$

The initial values are

$$\xi(0) = \pi(0) = 0$$

and it is easy to know that $\sigma(0) = 0$. The effect of adding (8d) is to drop the necessity that the "inf sup" condition H2 of Brezzi [7] hold for constant functions in W_h. Thus, the following estimate is given by [1(4.5)]

$$\|\rho\|_{H(div;\Omega)} + \|\eta\|_0 \le M\|p\|_{k+3}h_p^{k+1}, \tag{10}$$

with the constant M dependent only on the bound of the coefficient $\alpha(c)$. It is a standard result [2] in the theory of Galerkin methods for elliptic problems that

$$\|\zeta\|_0 + h_c\|\zeta\|_1 \le M\|c\|_{l+1}h_c^{l+1}, \quad \|\eta\|_0 + h_p\|\eta\|_1 \le M\|p\|_{k+1}h_p^{k+1}, \tag{11}$$

for $t \in J$ and M depending on the bounds of lower order derivatives of c and p.

The following estimates are shown in [1, 3]:

$$\left\|\frac{\partial \zeta}{\partial t}\right\|_0 + h_c \left\|\frac{\partial \zeta}{\partial t}\right\|_1 \le M \left\{ \|c\|_{l+1} + \left\|\frac{\partial c}{\partial t}\right\|_{l+1} \right\} h_c^{l+1}, \tag{12a}$$

$$\left\|\frac{\partial \eta}{\partial t}\right\|_0 + h_p \left\|\frac{\partial \eta}{\partial t}\right\|_1 \le M \left\{ \|p\|_{k+1} + \left\|\frac{\partial p}{\partial t}\right\|_{k+1} \right\} h_p^{k+1}, \tag{12b}$$

$$\left\|\frac{\partial \rho}{\partial t}\right\|_{H(div;\Omega)} + \left\|\frac{\partial \eta}{\partial t}\right\|_0 \le M \left(\|p\|_{k+3} + \left\|\frac{\partial p}{\partial t}\right\|_{k+3} \right) h_p^{k+1}, \tag{12c}$$

where M depends on lower order derivatives of c, p and their first derivatives with respect to time.

In the analysis below, the constant K always depends on lower order derivatives of c and p without further notation. In order to derive an evolution inequality for π, we subtract (8b) from (7b) to obtain

$$\left(d(c_h)\frac{\partial \pi}{\partial t}, w\right) + (\nabla \cdot \sigma, w) = \left((d(c_h) - d(c))\frac{\partial \tilde{p}}{\partial t}, w\right) - \left(d(c)\frac{\partial \eta}{\partial t}, w\right). \tag{13}$$

Using (8c) and (7c), and by letting $v = \sigma$, we have

$$(\alpha(c)\tilde{u} - \alpha(c_h)u_h, \sigma) - (\nabla \cdot \sigma, \pi) = 0. \tag{14}$$

In (13), choose the test function $w = \pi$. Adding the resulting equation to (14) gives

$$\left(d(c_h)\frac{\partial \pi}{\partial t}, \pi\right) = -((\alpha(c) - \alpha(c_h))\tilde{u}, \sigma) - (\alpha(c_h)\sigma, \sigma) + \left((d(c_h) - d(c))\frac{\partial \tilde{p}}{\partial t}, \pi\right) - \left(d(c)\frac{\partial \eta}{\partial t}, \pi\right). \tag{15}$$

Observe that

$$\begin{aligned}
\left(d(c_h)\frac{\partial \pi}{\partial t}, \pi\right) &= \frac{1}{2}\frac{d}{dt}(d(c_h)\pi, \pi) - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial t}\pi, \pi\right) \\
&= \frac{1}{2}\frac{d}{dt}(d(c_h)\pi, \pi) - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial c_h}{\partial t}\pi, \pi\right) \\
&= \frac{1}{2}\frac{d}{dt}(d(c_h)\pi, \pi) - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial \tilde{c}}{\partial t}\pi, \pi\right) + \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial \xi}{\partial t}\pi, \pi\right).
\end{aligned} \tag{16}$$

Combining (15) and (16) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(d(c_h)\pi, \pi) + (\alpha(c_h)\sigma, \sigma) = &- ((\alpha(c) - \alpha(c_h))\tilde{u}, \sigma) + \left((d(c_h) - d(c))\frac{\partial \tilde{p}}{\partial t}, \pi\right) \\
&- \left(d(c)\frac{\partial \eta}{\partial t}, \pi\right) + \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial \tilde{c}}{\partial t}\pi, \pi\right) - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial \xi}{\partial t}\pi, \pi\right).
\end{aligned} \tag{17}$$

Consider the case $\alpha(c) = a(c)^{-1} \ge \alpha_\star > 0$. Then the second term on the left-hand side of (17) gives

$$(\alpha(c_h)\sigma, \sigma) \ge \alpha_\star \|\sigma\|_0^2.$$

The first term on the right-hand side of (17) can be bounded by

$$((\alpha(c) - \alpha(c_h))\tilde{u}, \sigma) \le (\|\zeta\|_0 + \|\xi\|_0)\|\sigma\|_0 \le K(\|\xi\|_0^2 + \|\zeta\|_0^2) + \varepsilon\|\sigma\|_0^2.$$

Let $\varepsilon = \frac{1}{2}\alpha_\star$. We can obtain from the above three estimates that

$$\frac{1}{2}\frac{d}{dt}(d(c_h)\pi,\pi) + \frac{1}{2}\alpha_\star\|\sigma\|_0^2 \le K\left(\|\zeta\|_0 + \|\xi\| + \left\|\frac{\partial\eta}{\partial t}\right\|_0 + \|\pi\|_0\right)\|\pi\|_0 - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi\right)$$
$$\le K(\|\xi\|_0^2 + \|\pi\|_0^2 + h_c^{2l+2} + h_p^{2k+2}) - \frac{1}{2}\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi\right), \quad (18)$$

where the constant K depends on $\tilde{u}$, $\partial d/\partial c$, $\partial\tilde{p}/\partial t$, d, $\partial\tilde{c}/\partial t$, and $\alpha_\star$. To estimate $(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi)$, we need the evolution inequality of ξ.

Eqs. (6a) and (8a) can be differeced to show that

$$\begin{aligned}&\left(\phi\frac{\partial\tilde{c}}{\partial t},z\right) + (u\cdot\nabla\tilde{c},z) + (D(u)\nabla\tilde{c},\nabla z) + \left(b(c)\frac{\partial\tilde{p}}{\partial t},z\right)\\ =\;& ((\hat{c}-c)q,z) + \lambda(\zeta,z) - \left(\phi\frac{\partial\zeta}{\partial t},z\right) - \left(b(c)\frac{\partial\eta}{\partial t},z\right).\end{aligned} \quad (19)$$

Similarly, (19) and (7a) can be differenced to show that

$$\begin{aligned}&\left(\phi\frac{\partial\xi}{\partial t},z\right) + (u\cdot\nabla\xi,z) + \left(b(c)\frac{\partial\pi}{\partial t},z\right) + (D(u)\nabla\tilde{c} - D(u_h)\nabla c_h,\nabla z)\\ =\;& (-(\zeta+\xi)q^+,z) + \left(\phi\frac{\partial\zeta}{\partial t},z\right) + \lambda(\zeta,z) - \left(b(c)\frac{\partial\eta}{\partial t},z\right) + ((u_h-u)\cdot\nabla\tilde{c},z)\\ &+ \left((b(c_h)-b(c))\frac{\partial\tilde{p}}{\partial t},z\right) \qquad z\in M_h.\end{aligned} \quad (20)$$

Noticing that

$$(D(u)\nabla\tilde{c} - D(u_h)\nabla c_h,\nabla z) = ((D(u)-D(u_h))\nabla\tilde{c},\nabla z) + (D(u_h)\nabla\xi,\nabla z),$$

we can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi) + (D(u_h)\nabla\xi,\nabla\xi) \le\;& K\left(\|u_h\|_{0,\infty}\|\nabla\zeta\|_0 + \|\zeta\|_0 + \|\xi\|_0 + \left\|\frac{\partial\zeta}{\partial t}\right\|_0 + \left\|\frac{\partial\eta}{\partial t}\right\|_0\right)\|\xi\|_0\\ &- \left(b(c_h)\frac{\partial\pi}{\partial t},\xi\right) - ((D(u)-D(u_h))\nabla\tilde{c},\nabla\xi) - ((u_h-u)\cdot\nabla\tilde{c},\xi),\end{aligned} \quad (21)$$

where $\|\nabla\tilde{c}\|_{0,\infty}$ is bounded by $M\|c\|_{l+1,\infty}$ for any $l \ge 0$ and sufficiently small h_c due to the $L^\infty(\Omega)$ error estimate ([1](3.23))

$$\|\nabla\zeta\|_{0,\infty} \le M\|c\|_{l+1,\infty}h_c^l$$

with M depending on some lower norms of c and p. Thus

$$\begin{aligned}&((D(u)-D(u_h))\nabla\tilde{c},\nabla\xi) + ((u_h-u)\cdot\nabla\tilde{c},\xi)\\ \le\;& K(\|\sigma\|_0 + \|\rho\|_0)(\|\nabla\xi\|_0 + \|\xi\|_0) \le K'(\|\sigma\|_0^2 + h_p^{2k+2} + \|\xi\|_0^2) + \varepsilon\|\nabla\xi\|_0^2.\end{aligned}$$

In order to show the boundedness of $\|u_h\|_{0,\infty}$, we need the induction hypothesis

$$\|\sigma\|_{0,\infty} \le K^\star. \quad (22)$$

Because $D(u)$ satisfies $D(u) \geq D_\star > 0$, similar to (18), we can make the coefficient of $\|\nabla \xi\|_0^2$ less than $\frac{1}{2}D_\star$ to show that

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)+\frac{1}{2}D_\star\|\nabla\xi\|_0^2 \leq K(K^\star)(\|\xi\|_0^2+h_c^{2l+2}+h_p^{2k+2})+K_1\|\sigma\|_0^2-\left(b(c_h)\frac{\partial\pi}{\partial t},\xi\right) \tag{23}$$

with K_1 dependent on $D_\star$.

The term $(b(c_h)\frac{\partial\pi}{\partial t},\xi)$ can be estimated by using (13) for such equation as

$$\left(b(c_h)\frac{\partial\pi}{\partial t},\xi\right)=\left(d(c_h)\frac{\partial\pi}{\partial t},\frac{b(c_h)}{d(c_h)}\xi\right).$$

Since the test function $w \in W_h$ in (13), we need to introduce a projection operator $P_0 : H_1(\Omega) \to W_h$ satisfying

$$(d(c_h)v_h, v) = (d(c_h)v_h, P_0 v) \quad v_h \in W_h, \quad v \in H_1(\Omega). \tag{24}$$

Lemma 3.1. *For any projection operator defined in* (24)*, we have*

$$\|P_0 v\|_0 \leq K\|v\|_0, \tag{25}$$

$$\|\nabla P_0 v\|_0 \leq K\|v\|_1. \tag{26}$$

Proof By letting $v_h = P_0 v$ in (24), we can show that (25) is right. Let I_h be the interpolation operator satisfying $I_h = I_h : H^1(\Omega) \to N_h$. It is a standard result for I_h that

$$\|v - I_h v\|_0 \leq K h_p^{k+1}\|v\|_{k+1}, \quad \|\nabla(I_h v)\|_0 \leq K\|\nabla v\|_0.$$

Therefore, we have

$$\begin{aligned}\|\nabla P_0 v\|_0 &\leq \|\nabla P_0(v-I_h v)\|_0+\|\nabla P_0(I_h v)\|_0\\ &\leq h_p^{-1}\|P_0(v-I_h v)\|_0+\|\nabla(I_h v)\|_0 \leq K h_p^{-1}\|v-I_h v\|_0+\|\nabla(I_h v)\|_0\\ &\leq K h_h^{-1}h_p\|v\|_1+K\|\nabla v\|_0 \leq K\|v\|_1.\end{aligned}$$

This completes the proof of Lemma (3.1). ■

By letting $w = \gamma_0 \equiv P_0(\frac{b(c_h)}{d(c_h)}\xi)$ in (13), we can obtain that

$$\begin{aligned}(b(c_h)\frac{\partial\pi}{\partial t},\xi) &= (d(c_h),\gamma_0)=\left((d(c_h)-d(c))\frac{\partial\tilde{p}}{\partial t},\gamma_0\right)-\left(d(c)\frac{\partial\eta}{\partial t},\gamma_0\right)-(\nabla\cdot\sigma,\gamma_0)\\ &\leq K\left(\|\xi\|_0+\|\zeta\|_0+\left\|\frac{\partial\eta}{\partial t}\right\|_0\right)\|\xi\|_0-(\nabla\cdot\sigma,\gamma_0).\end{aligned} \tag{27}$$

Applying Green's formula and the periodic assumption on the last term of (27) gives

$$\begin{aligned}-(\nabla\cdot\sigma,\gamma_0) &= -\langle\sigma,\gamma_0\rangle+(\sigma,\nabla\gamma_0)\\ &= (\sigma,\nabla\gamma_0) \leq K\|\sigma\|_0\|\nabla\gamma_0\|_0 \leq K\|\sigma\|_0\|\xi\|_1,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denote the inner product in $L^2(\partial\Omega)$. Therefore,

$$\left(b(c_h)\frac{\partial\pi}{\partial t},\xi\right) \leq K_2(\|\xi\|_0^2+\|\sigma\|_0^2+h_c^{2l+2}+h_p^{2k+2})+\varepsilon_0\|\xi\|_1^2, \tag{28}$$

where K_2 depends on ε_0 and $K^\star$. Then we have, by combining (23) and (28), that

$$\frac{d}{dt}(\phi\xi,\xi)+D_\star\|\xi\|_1^2\le K(\|\xi\|_0^2+\|\sigma\|_0^2+h_c^{2l+2}+h_p^{2k+2})+\varepsilon_0\|\xi\|_1^2.$$

In addition, we can make ε_0 sufficiently small to get

$$\frac{d}{dt}(\phi\xi,\xi)+\|\xi\|_1^2\le K_3(\|\xi\|_0^2+\|\sigma\|_0^2+h_c^{2l+2}+h_p^{2k+2}) \tag{29}$$

with K_3 dependent on $K^\star$ and $D_\star$.

Now the only term in (18) which we have not estimated is $(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi)$. To estimate it, we shall use (23). In general, the process is similar to what we have done from (23) to (28). Firstly, we shall introduce a projection operator $P_1: H^1(\Omega)\to M_h$ satisfying

$$(\phi(x)v,v_h)=(\phi(x)P_1v,v_h)\quad v\in H^1(\Omega), v_h\in M_h.$$

Similar to Lemma 3.1 we have

Lemma 3.2. *For a projection operator defined above, we have such inequalities as*

$$\|P_1v\|_0\le K\|v\|_0, \tag{30}$$
$$\|\nabla P_1v\|_0\le K\|v\|_1. \tag{31}$$

Observe

$$\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi\right)=\left(\phi\frac{\partial\xi}{\partial t},\frac{1}{\phi}\frac{\partial d(c_h)}{\partial c}\pi^2\right)=\left(\phi\frac{\partial\xi}{\partial t},P_1\left(\frac{1}{\phi}\frac{\partial d(c_h)}{\partial c}\pi^2\right)\right).$$

By letting $z=\gamma_1\equiv P_1(\frac{1}{\phi}\frac{\partial d(c_h)}{\partial c}\pi^2)$ in (20), we have

$$\begin{aligned}
&\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial\xi}{\partial t}\pi,\pi\right)=\left(\phi\frac{\partial\xi}{\partial t},\gamma_1\right)\\
=\;&(-(\zeta+\xi)q^+,\gamma_1)+\left(\phi\frac{\partial\zeta}{\partial t},\gamma_1\right)+\lambda(\zeta,\gamma_1)-\left(b(c)\frac{\partial\eta}{\partial t},\gamma_1\right)+((u_h-u)\cdot\nabla\tilde{c},\gamma_1)\\
&+\left((b(c_h)-b(c))\frac{\partial\tilde{p}}{\partial t},\gamma_1\right)-(u_h\cdot\nabla\xi,\gamma_1)-(D(u)\nabla\tilde{c}-D(u_h)\nabla c_h,\nabla\gamma_1)-\left(b(c_h)\frac{\partial\pi}{\partial t},\gamma_1\right)\\
\le\;&K\left(\|\xi\|_0+\|\zeta\|_0+\left\|\frac{\partial\zeta}{\partial t}\right\|_0+\left\|\frac{\partial\eta}{\partial t}\right\|_0+\|\rho\|_0+\|\sigma\|_0+\|u_h\|_\infty\|\nabla\xi\|\right)\|\gamma_1\|_0\\
&+\|\nabla\xi\|_0\|\nabla\gamma_1\|_0-\left(b(c_h)\frac{\partial\pi}{\partial t},\gamma_1\right).
\end{aligned} \tag{32}$$

In order to estimate $\|\nabla\gamma_1\|_0$, we need another induction hypothesis

$$\|\pi\|_{1,\infty}\le K^{\star\star}. \tag{33}$$

So $\|\nabla\pi\|_\infty$ can be bounded and

$$\begin{aligned}
\|\nabla\gamma_1\|_0\;&=\;\left\|\nabla P_1\left(\frac{1}{\phi}\frac{\partial d(c_h)}{\partial c}\pi^2\right)\right\|_0\le K\left\|\frac{1}{\phi}\frac{\partial d(c_h)}{\partial c}\pi^2\right\|_1\\
&\le\;K\|\pi\|_{1,\infty}\|\pi\|_0\le K(K^\star,K^{\star\star})\|\pi\|_0.
\end{aligned} \tag{34}$$

Applying (34) to (32) gives

$$\left(\frac{\partial d(c_h)}{\partial c}\frac{\partial \xi}{\partial t}\pi,\pi\right) \le K\left(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2}\right)+\varepsilon_1\|\nabla\xi\|_0^2+\varepsilon_2\|\sigma\|_0^2-\left(b(c_h)\frac{\partial \pi}{\partial t},\gamma_1\right), \quad (35)$$

where K depends on ε_1, ε_2, $K^\star$, and $K^{\star\star}$.

To estimate the last term of (35), we need to repeat the process from (23) to (28) to get

$$\frac{d}{dt}(d(c_h)\pi,\pi)+\alpha_\star\|\sigma\|_0^2 \le K_4(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2})+\varepsilon_1\|\nabla\xi\|_0^2+(\varepsilon_2+\varepsilon_3)\|\sigma\|_0^2, \quad (36)$$

where $K_4=K_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)$. Letting ε_2, ε_3 sufficiently small yields

$$\frac{d}{dt}(d(c_h)\pi,\pi)+\|\sigma\|_0^2 \le K_5(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2})+\varepsilon_1\|\nabla\xi\|_0^2. \quad (37)$$

By integrating (29) and (37) we can obtain

$$\|\xi\|_0^2+\int_0^t\|\xi\|_1^2d\tau \le K_3\int_0^t(\|\xi\|_0^2+\|\sigma\|_0^2+h_c^{2l+2}+h_p^{2k+2})d\tau, \quad (38)$$

$$\|\pi\|_0^2+\int_0^t\|\sigma\|_0^2d\tau \le K_5\int_0^t(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2})d\tau+\varepsilon_1\int_0^t\|\nabla\xi\|^2d\tau. \quad (39)$$

It follows from (38) and(39) that

$$\begin{aligned}&(K_3+1)\|\pi\|_0^2+\int_0^t\|\sigma\|_0^2d\tau+\|\xi\|_0^2+\int_0^t\|\xi\|_1^2d\tau\\ \le\ & K_6\left(\int_0^t(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2})d\tau\right)+\varepsilon_1(K_3+1)\int_0^t\|\xi\|_1^2d\tau.\end{aligned}$$

In the above inequality, let $\varepsilon_1=1/(2K_3+2)$ (noticing that this will have no influence on K_3). Consequently,

$$\|\pi\|_0^2+\int_0^t\|\sigma\|_0^2d\tau+\|\xi\|_0^2+\int_0^t\|\xi\|_1^2d\tau \le K\int_0^t(\|\xi\|_0^2+\|\pi\|_0^2+h_c^{2l+2}+h_p^{2k+2})d\tau.$$

Thus, it follows from Gronwall lemma that

$$\|\pi\|_{L^\infty(J;L^2(\Omega))}+\|\sigma\|_{L^2(J;L^2(\Omega)^2)}+\|\xi\|_{L^\infty(J;L^2(\Omega))}+\|\xi\|_{L^2(J;H^1(\Omega))} \le K(h_p^{k+1}+h_c^{l+1}). \quad (40)$$

The justification of the induction hypothesis (22) can be given at this point. We can see from (14) that

$$([\alpha(c)-\alpha(c_h)]\tilde{u},\sigma)+(\alpha(c_h)\sigma,\sigma)+(\sigma,\nabla\pi)=0.$$

So we have

$$\begin{aligned}\alpha_*\|\sigma\|_0^2 &\le (\alpha(c_h)\sigma,\sigma)=-([\alpha(c)-\alpha(c_h)]\tilde{u},\sigma)-(\sigma,\nabla\pi)\\ &\le K[\|\nabla\pi\|_0+(\|\xi\|_0+\|\zeta\|_0)\|\tilde{u}\|_{0,\infty}]\|\sigma\|_0.\end{aligned}$$

Thus,

$$\|\sigma\|_0 \le K(\|\nabla\pi\|_0+\|\xi\|_0+\|\zeta\|_0) \quad (41)$$

with K dependent on α_*, $\alpha'(c)$, $\|\tilde{u}\|_{0,\infty}$. The quasi-regularity of the polygonalization for the space V_h implies that

$$\|\sigma\|_{0,\infty} \leq K h_p^{-1} \|\sigma\|_0, \quad \|\nabla \pi\|_0 \leq K h_p^{-1} \|\pi\|_0.$$

Thus, we can know from (40) and (41) that

$$\|\sigma\|_{0,\infty} \leq K h_p^{-1} (h_p^{-1} \|\pi\|_0 + \|\xi\|_0 + \|\zeta\|_0) \leq K(h_p^{k-1} + h_c^{l+1} h_p^{-2}). \tag{42}$$

It is similar to justify (33). Observe

$$\|\pi\|_{1,\infty} \leq K h_p^{-2} \|\pi\|_0 \leq K(h_p^{k-1} + h_c^{l+1} h_p^{-2}). \tag{43}$$

Therefore, if we choose

$$k \geq 2 \quad \text{and} \quad h_c^{l+1} h_p^{-2} \to 0 \tag{44}$$

when $\max(h_p, h_c) \to 0$, then it follows from (42) and (43) that (22) and (33) hold.

We see that, under the constraint (44),

$$\begin{aligned} &\|p - p_h\|_{L^\infty(J;L^2(\Omega))} + \|u - u_h\|_{L^2(J;L^2(\Omega)^2)} + \|c - c_h\|_{L^\infty(J;L^2(\Omega))} \\ &+\|c - c_h\|_{L^2(J;L^2(\Omega))} + h_c \|c - c_h\|_{L^2(J;H^1(\Omega))} \leq K(h_p^{k+1} + h_c^{l+1}), \end{aligned}$$

where the constant K depends on the spatial derivatives of order not greater than $l+1$ of c and $\partial c / \partial t$, and of order not greater than $k+1$ of p and $\partial p / \partial t$.

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