NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **16** No. 1, pp. 63-73
February 2007

Block Based Bivariate Blending Rational Interpolation via Symmetric Branched Continued Fractions†

Qianjin Zhao[1] and Jieqing Tan[2,*]

[1] *School of Computer & Information, Hefei University of Technology, Hefei 230009, China.*

[2] *Institute of Applied Mathematics, Hefei University of Technology, Hefei 230009, China.*

Received March 22, 2005; Accepted (in revised version) February 27, 2006

Abstract. This paper constructs a new kind of block based bivariate blending rational interpolation via symmetric branched continued fractions. The construction process may be outlined as follows. The first step is to divide the original set of support points into some subsets (blocks). Then construct each block by using symmetric branched continued fraction. Finally assemble these blocks by Newton's method to shape the whole interpolation scheme. Our new method offers many flexible bivariate blending rational interpolation schemes which include the classical bivariate Newton's polynomial interpolation and symmetric branched continued fraction interpolation as its special cases. The block based bivariate blending rational interpolation is in fact a kind of tradeoff between the purely linear interpolation and the purely nonlinear interpolation. Finally, numerical examples are given to show the effectiveness of the proposed method.

Key words: Interpolation; block based bivariate partial divided differences; symmetric branched continued fractions; blending method.

AMS subject classifications: 41A20, 65D05

1 Introduction

Bivariate Newton's polynomial interpolation may be the most commonly used bivariate interpolation. It uses the bivariate partial divided differences which can be calculated recursively and produce useful intermediate results. On the other hand, the most powerful bivariate interpolation is the one using bivariate rational functions. The main advantage of the rational functions over polynomials is their ability to model functions with nonlinear characters (such as

*Correspondence to: Jieqing Tan, Institute of Applied Mathematics, Hefei University of Technology, Hefei 230009, China. Email: jqtan@mail.hf.ah.cn

†Project supported by the National Natural Science Foundation of China (No. 10171026, No. 60473114), the Anhui Provincial Natural Science Foundation, China (No. 03046102), and the Research Funds for Young Innovation Group, Education Department of Anhui Province (No. 2005TD03).

poles or other singularity) and their fast convergence properties. Given a set of two dimensional points $\Pi_{mn} = \{(x_i, y_j) \mid i = 0, 1, \dots, m; j = 0, 1, \dots, n\}$, and suppose that $f(x, y)$ is defined on $D \supset \Pi_{mn}$. Then one has two basic approaches for interpolating $f(x, y)$ on Π_{mn}. One is the bivariate Newton's interpolating polynomial ([5])

$$P(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f[x_0,\dots,x_i;y_0,\dots,y_j]\prod_{h=0}^{i-1}(x-x_h)\prod_{k=0}^{j-1}(y-y_k),$$

where the empty products are defined to take the value 1, and

$$\begin{aligned}
&f[x_0;y_0] = f(x_0,y_0),\\
&f[x_0,\dots,x_i;y_0] = \frac{f[x_1,\dots,x_i;y_0]-f[x_0,\dots,x_{i-1};y_0]}{x_i-x_0},\\
&f[x_0,\dots,x_i;y_0,\dots,y_j] = \frac{f[x_0,\dots,x_i;y_1,\dots,y_j]-f[x_0,\dots,x_i;y_0,\dots,y_{j-1}]}{y_j-y_0}.
\end{aligned}$$

The other one is the interpolating symmetric branched continued fraction ([2–4, 7])

$$\begin{aligned}
R(x,y) \;&=\; \varphi_{00} + \sum_{k=1}^{m}\frac{x-x_{k-1}\rfloor}{\lfloor\varphi_{k0}} + \sum_{k=1}^{n}\frac{y-y_{k-1}\rfloor}{\lfloor\varphi_{0k}}\\
&\quad + \sum_{l=1}^{m}\frac{(x-x_{l-1})(y-y_{l-1})\rfloor}{\lfloor\varphi_{ll} + \sum_{k=l+1}^{m}\frac{x-x_{k-1}\rfloor}{\lfloor\varphi_{kl}} + \sum_{k=l+1}^{n}\frac{y-y_{k-1}\rfloor}{\lfloor\varphi_{lk}}}, \quad (m \le n)
\end{aligned}$$

or

$$\begin{aligned}
R(x,y) \;&=\; \varphi_{00} + \sum_{k=1}^{m}\frac{x-x_{k-1}\rfloor}{\lfloor\varphi_{k0}} + \sum_{k=1}^{n}\frac{y-y_{k-1}\rfloor}{\lfloor\varphi_{0k}}\\
&\quad + \sum_{l=1}^{n}\frac{(x-x_{l-1})(y-y_{l-1})\rfloor}{\lfloor\varphi_{ll} + \sum_{k=l+1}^{m}\frac{x-x_{k-1}\rfloor}{\lfloor\varphi_{kl}} + \sum_{k=l+1}^{n}\frac{y-y_{k-1}\rfloor}{\lfloor\varphi_{lk}}}, \quad (n \le m)
\end{aligned}$$

where $\varphi_{ij} = \varphi[x_0,\dots,x_i;y_0,\dots,y_j]$, and

$$\begin{aligned}
&\varphi[x_0;y_0] = f(x_0,y_0),\\
&\varphi[x_0,\dots,x_k;y_0] = \frac{x_k-x_{k-1}}{\varphi[x_0,\dots,x_{k-2},x_k;y_0]-\varphi[x_0,\dots,x_{k-1};y_0]},\\
&\varphi[x_0;y_0,\dots,y_k] = \frac{y_k-y_{k-1}}{\varphi[x_0;y_0,\dots,y_{k-2},y_k]-\varphi[x_0;y_0,\dots,y_{k-1}]},\\
&\varphi[x_0,\dots,x_j;y_0,\dots,y_j] = (x_j-x_{j-1})(y_j-y_{j-1})(\varphi[x_0,\dots,x_{j-2},x_j;y_0,\dots,y_{j-2},y_j]\\
&\qquad -\varphi[x_0,\dots,x_{j-1};y_0,\dots,y_{j-2},y_j] - \varphi[x_0,\dots,x_{j-2},x_j;y_0,\dots,y_{j-1}]\\
&\qquad +\varphi[x_0,\dots,x_{j-1};y_0,\dots,y_{j-1}])^{-1},
\end{aligned}$$

and for $k > j$

$$\begin{aligned}
&\varphi[x_0,\dots,x_k;y_0,\dots,y_j] = \frac{x_k-x_{k-1}}{\varphi[x_0,\dots,x_{k-2},x_k;y_0,\dots,y_j]-\varphi[x_0,\dots,x_{k-1};y_0,\dots,y_j]},\\
&\varphi[x_0,\dots,x_j;y_0,\dots,y_k] = \frac{y_k-y_{k-1}}{\varphi[x_0,\dots,x_j;y_0,\dots,y_{k-2},y_k]-\varphi[x_0,\dots,x_j;y_0,\dots,y_{k-1}]}.
\end{aligned}$$

The above two interpolants are purely linear and nonlinear interpolations, respectively. Obviously, some applications need interpolation by the bivariate functions between the purely linear and purely nonlinear (rational) interpolants. In such cases, the bivariate blending rational interpolants may be better than the purely linear or nonlinear one. The case of bivariate blending rational functions are studied by one of the authors ([9, 11, 12, 14]). The bivariate blending rational interpolants are constructed by the tensor-like product of univariate linear and nonlinear interpolants ([9, 11]). By applying the Neville's algorithm to continued fractions, a Neville-like method is proposed ([12]). By adopting composite interpolation over triangular sub-grids, the composite scheme for multivariate blending rational interpolation is discussed ([14]).

In this paper, the emphasis is put on the study of block based bivariate blending rational interpolation in Newton-like form via symmetric branched continued fractions. The bivariate blending rational interpolants are constructed by taking the policy of "tradeoff" between the purely linear and the purely nonlinear method in the block blending manner. The construction process may be outlined as follows. First of all, divide the original set of support points into some subsets (blocks). Then construct each block by using symmetric branched continued fraction. Finally assemble these blocks by Newton's method to shape the whole interpolation scheme. By introducing so-called block based bivariate partial divided differences which look like divided differences, we give a recursive algorithm for interpolation. Our method offers many flexible bivariate blending rational interpolation schemes which include the classical bivariate Newton's polynomial interpolation and the classical branched continued fraction interpolation as its special cases. Moreover, we investigate the error estimation. Numerical examples are also given to show the effectiveness of our method.

2 Block based bivariate blending rational interpolants

Recently, a kind of block based interpolation method in Thiele-Werner-type form has been proposed for univariate vector-valued osculatory rational interpolation ([15]). In order to obtain flexible bivariate blending rational interpolation, we elaborate on that technique in terms of Newton-like form for the bivariate blending rational interpolation. Some details of the algorithm will be given below.

2.1 Basic idea

Given a set of two dimensional points $\Pi_{mn} = \{(x_i, y_j) \mid i = 0, 1, \dots, m; j = 0, 1, \dots, n\}$. Suppose $\Pi_{mn} \subset D \subset R^2$, and let $f(x, y)$ be a real function defined on D such that

$$f(x_i, y_j) = f_{ij}, \quad i = 0, 1, \dots, m; \quad j = 0, 1, \dots, n.$$

We divide Π_{mn} into the following $(u+1) \times (v+1)$ subsets:

$$\Pi_{mn}^{st} = \{(x_i, y_j) \mid c_s \le i \le d_s; h_t \le j \le r_t\}, \quad (s = 0, 1, \dots, u; \quad t = 0, 1, \dots, v).$$

The subsets may be achieved by reordering the interpolation points if necessary. It is easy to check that

$$\sum_{s=0}^{u}(d_s - c_s + 1) = m + 1, \quad \sum_{t=0}^{v}(r_t - h_t + 1) = n + 1.$$

Let us consider the following function with block based bivariate Newton-like formation:

$$T(x, y) = Z_0(x, y) + Z_1(x, y)\omega_0(x) + \cdots + Z_u(x, y)\omega_0(x)\cdots\omega_{u-1}(x), \tag{1}$$

for $s=0,1,\ldots,u$

$$Z_s(x,y) = I_{s0}(x,y) + I_{s1}(x,y)\omega_0(y) + \cdots + I_{sv}(x,y)\omega_0(y)\cdots\omega_{v-1}(y), \tag{2}$$

where

$$\omega_s(x) = \prod_{i=c_s}^{d_s}(x-x_i), \quad s=0,1,\ldots,u-1, \tag{3}$$

$$\omega_t^*(y) = \prod_{i=h_t}^{r_t}(y-y_i), \quad t=0,1,\ldots,v-1, \tag{4}$$

and $I_{st}(x,y)$ $(s=0,1,\ldots,u; t=0,1,\ldots,v)$ are the interpolating symmetric branched continued fractions on the subsets Π_{mn}^{st} $(s=0,1,\ldots,u; t=0,1,\ldots,v)$. If the above $I_{st}(x,y)$ $(s=0,1,\ldots,u; t=0,1,\ldots,v)$ are chosen so that

$$T(x_i,y_j) = f_{ij}, (x_i,y_j) \in \Pi_{mn}, \tag{5}$$

then $T(x,y)$ defined by (1)-(4) is called block based bivariate blending rational interpolant to $f(x,y)$. To obtain a block based bivariate blending rational interpolant on the whole set X_n, the above $I_{st}(x,y)$ must be computed so that (5) holds.

2.2 Block based bivariate partial divided differences

This subsection is concerned with choosing $I_{st}(x,y)$ $(s=0,1,\ldots,u; t=0,1,\ldots,v)$ such that (5) holds. For convenience, we introduce the following notations:

$$f_{ij}^{00} = f_{ij}, \quad i=0,1,\ldots,m; \quad j=0,1,\ldots,n; \tag{6}$$

and for $t=1,2,\ldots,v$

$$f_{ij}^{0t} = \frac{f_{ij}^{0,t-1} - I_{0,t-1}(x_i,y_j)}{\omega_{t-1}^*(y_j)}, \quad (i=0,1,\ldots,m; \quad j=h_t,h_t+1,\ldots,n), \tag{7}$$

where $I_{0t}(x,y)$ are the symmetric branched continued fraction interpolants on the subsets Π_{mn}^{0t} such that

$$I_{0t}(x_i,y_j) = f_{ij}^{0t}, \quad (c_0 \le i \le d_0, \quad h_t \le j \le r_t, \quad t=0,1,\ldots,v). \tag{8}$$

For $s=1,2,\ldots,u$,

$$f_{ij}^{s0} = \frac{f_{ij}^{s-1,0} - Z_{s-1}(x_i,y_j)}{\omega_{s-1}(x_i)}, \quad (i=c_s,c_s+1,\ldots,m; \quad j=0,1,\ldots,n), \tag{9}$$

and for $t=1,2,\ldots,v$

$$f_{ij}^{st} = \frac{f_{ij}^{s,t-1} - I_{s,t-1}(x_i,y_j)}{\omega_{t-1}^*(y_j)}, \quad (j=h_t,h_t+1,\ldots,n; \quad i=c_s,c_s+1,\ldots,m), \tag{10}$$

where $I_{st}(x,y)$ are the symmetric branched continued fraction interpolants on subsets Π_{mn}^{st} such that

$$I_{st}(x_i,y_j) = f_{ij}^{st}, \quad (c_s \le i \le d_s, h_t \le j \le r_t; \quad s=1,2,\ldots,u; \quad t=0,1,\ldots,v). \tag{11}$$

If all the f_{ij}^{st} exist, then f_{ij}^{st} are called the (s,t)th block based bivariate partial divided differences for the function $f(x,y)$.

Theorem 2.1. *If all the above interpolants $I_{st}(x,y)$ exist and satisfy* (8) *and* (11)*, then*

$$T(x_i,y_j)=f_{ij},\quad i=0,1,\ldots,m;\quad j=0,1,\ldots,n.$$

Proof Let $c_s\le i\le d_s$, and $h_t\le j\le r_t$. By (1), (2) and (6)-(11), we have

$$T(x_i,y_j)=Z_0(x_i,y_j)+Z_1(x_i,y_j)\omega_0(x_i)+\cdots+Z_s(x_i,y_j)\omega_0(x_i)\cdots\omega_{s-1}(x_i)$$

and

$$\begin{aligned} &Z_s(x_i,y_j)=I_{s0}(x_i,y_j)+I_{s1}(x_i,y_j)\omega_0^*(y_j)+\cdots+I_{st}(x_i,y_j)\omega_0^*(y_j)\cdots\omega_{t-1}^*(y_j)\\ =\;&I_{s0}(x_i,y_j)+I_{s1}(x_i,y_j)\omega_0^*(y_j)+\cdots+f_{ij}^{st}\omega_0^*(y_j)\cdots\omega_{t-1}^*(y_j)\\ =\;&I_{s0}(x_i,y_j)+I_{s1}(x_i,y_j)\omega_0^*(y_j)+\cdots+(f_{ij}^{s,t-1}-I_{s,t-1}(x_i,y_j))\omega_0^*(y_j)\cdots\omega_{t-2}^*(y_j)\\ =\;&I_{s0}(x_i,y_j)+I_{s1}(x_i,y_j)\omega_0^*(y_j)+\cdots+f_{ij}^{s,t-1}\omega_0^*(y_j)\cdots\omega_{t-2}^*(y_j)\\ =\;&\cdots=f_{ij}^{s0}. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} &T(x_i,y_j)=Z_0(x_i,y_j)+Z_1(x_i,y_j)\omega_0(x_i)+\cdots+Z_s(x_i,y_j)\omega_0(x_i)\cdots\omega_{s-1}(x_i)\\ =\;&Z_0(x_i,y_j)+Z_1(x_i,y_j)\omega_0(x_i)+\cdots+f_{ij}^{s0}\omega_0(x_i)\cdots\omega_{s-1}(x_i)\\ =\;&Z_0(x_i,y_j)+Z_1(x_i,y_j)\omega_0(x_i)+\cdots+(f_{ij}^{s-1,0}-Z_{s-1}(x_i,y_j))\omega_0(x_i)\cdots\omega_{s-2}(x_i)\\ =\;&Z_0(x_i,y_j)+Z_1(x_i,y_j)\omega_0(x_i)+\cdots+f_{ij}^{s-1,0}\omega_0(x_i)\cdots\omega_{s-2}(x_i)\\ =\;&\cdots=f_{ij}^{00}=f_{ij}. \end{aligned}$$

The proof is thus completed. ■

2.3 Special cases

Block based bivariate blending rational interpolation via symmetric branched continued fractions can be obtained explicitly by means of the above recursive algorithm and has a number of interesting special cases. From the special cases, it is not difficult to find that the above block based interpolation is a kind of tradeoff between the purely linear bivariate Newton interpolation and the purely nonlinear symmetric branched continued fraction interpolation in block blending manner.

Case 1: $u=v=0$, i.e., the whole set Π_{mn} is the unique subset. In this case, if $I_{00}(x,y)$ is the symmetric branched continued fraction interpolant on the Π_{mn}, then one has

$$T(x,y)=I_{00}(x,y). \tag{12}$$

This is to say that the above block based bivariate blending rational interpolation includes the classical symmetric branched continued fraction interpolation as its special case.

Case 2: $u=0$, $v=1,2,\ldots,n$. In this case, it is easy to know that

$$\begin{aligned} T(x,y)&=Z_0(x,y)\\ &=I_{00}(x,y)+I_{01}(x,y)\omega_0^*(y)+\cdots+I_{0v}(x,y)\omega_0^*(y)\cdots\omega_{v-1}^*(y). \end{aligned}\tag{13}$$

Let us consider the case of $v=n$. It is clear that the set Π_{mn} is divided into $n+1$ subsets $\Pi_{mn}^{0t}=\{(x_i,y_j)\mid i=0,1,\ldots,m; j=t\}$ $(t=0,1,\ldots,n)$. In this case, the symmetric branched continued fraction interpolants $I_{0t}(x,y)(t = 0,1,\dots,n)$ on the subsets Π_{mn}^{0t} degenerate into univariate Thiele-type continued fraction interpolants ([7]). Therefore, when $u = 0$ and $v = n$, the above block based bivariate blending rational interpolation degenerates into the classical Thiele-Newton interpolation ([11]).

Case 3: $u = m, v = 0$. Then we have

$$Z_s(x,y) = I_{s0}(x,y), \quad (s = 0,1,\dots,m) \tag{14}$$

and

$$T(x,y) = I_{00}(x,y) + I_{10}(x,y)\omega_0(x) + \cdots + I_{m0}(x,y)\omega_0(x)\cdots\omega_{m-1}(x), \tag{15}$$

where all the $I_{s0}(x,y)$ $(s = 0,1,\dots,m)$ also degenerate into univariate Thiele-type continued fraction interpolants ([7]). In this case, the above block based bivariate blending rational interpolant becomes the classical Newton-Thiele interpolant ([11]).

Case 4: $u = m$, $v = n$, i.e., each block (subset) contains only one point. Then the block based bivariate partial divided differences degenerate into the classical bivariate partial divided differences and the above block based bivariate blending rational interpolant degenerates into the classical bivariate Newton interpolant .

The above special cases tell us that the block based bivariate blending rational interpolation is a kind of tradeoff between the purely linear interpolation (the bivariate Newton interpolation) and the purely nonlinear interpolation (the symmetric branched continued fraction interpolation) in the block blending manner. In particular case when $u = 0$ and $v = 0$ (no partition of block), the block based bivariate blending rational interpolant is the purely linear one. As u or v becomes larger (finer partition of block), the linear weight in the blending rational interpolant becomes larger. Especially, when $u = m$ and $v = n$ (the finest partition of block), the linear weight in the blending rational interpolant becomes the largest, which means the above blending rational interpolant degenerates into the purely linear bivariate Newton interpolant.

3 Error estimate

In this section, we discuss the error in the approximation of a function $f(x,y)$ by its block based bivariate blending rational interpolants via the symmetric branched continued fractions.

Theorem 3.1. *Suppose $D = [a,b]\times[c,d]$ is a rectangular domain containing Π_{mn} and $f(x,y) \in C^{(m+n+2)}(D)$. Let*

$$\begin{aligned} T(x,y) &= Z_0(x,y) + Z_1(x,y)\omega_0(x) + \cdots + Z_u(x,y)\omega_0(x)\cdots\omega_{u-1}(x) \\ &= \frac{P(x,y)}{Q(x,y)} \end{aligned} \tag{16}$$

be the block based bivariate blending rational interpolant on Π_{mn}. Then $\forall (x,y) \in D$, we have

$$\begin{aligned} f(x,y) - T(x,y) = {} & \frac{\omega(x)}{Q(x,y)} \frac{\frac{\partial^{m+1}}{\partial x^{m+1}}[fQ-P]_{x=\xi}}{(m+1)!} + \frac{\omega^*(y)}{Q(x,y)} \frac{\frac{\partial^{n+1}}{\partial y^{n+1}}[fQ-P]_{y=\eta}}{(n+1)!} \\ & - \frac{\omega(x)\omega^*(y)}{Q(x,y)} \frac{\frac{\partial^{n+m+2}}{\partial x^{m+1}\partial y^{n+1}}[fQ-P]_{x=\overline{\xi},y=\overline{\eta}}}{(m+1)!(n+1)!}, \end{aligned}$$

with $\xi,\overline{\xi} \in (a,b)$ and $\eta,\overline{\eta} \in (c,d)$, where

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_m), \quad \omega^*(y) = (y-y_0)(y-y_1)\cdots(y-y_n).$$

Proof Let $E(x,y) = f(x,y)Q(x,y) - P(x,y)$. It follows from Theorem 2.1 and (16) that

$$E(x_i, y_j) = 0, \quad (i = 0, 1, \ldots, m; \quad j = 0, 1, \ldots, n).$$

Using the bivariate Newton formula ([5]) gives

$$\begin{aligned} E(x,y) = {} & \sum_{i=0}^{m}\sum_{j=0}^{n} E[x_0,\ldots,x_i; y_0,\ldots,y_j] \prod_{h=0}^{i-1}(x-x_h) \prod_{k=0}^{j-1}(y-y_k) \\ & + \frac{\partial^{m+1}E(\xi,y)}{\partial x^{m+1}} \frac{\prod_{h=0}^{m}(x-x_h)}{(m+1)!} + \frac{\partial^{n+1}E(x,\eta)}{\partial y^{n+1}} \frac{\prod_{k=0}^{n}(y-y_k)}{(n+1)!} \\ & - \frac{\partial^{m+n+2}E(\overline{\xi},\overline{\eta})}{\partial x^{m+1}\partial y^{n+1}} \frac{\prod_{h=0}^{m}(x-x_h)\prod_{k=0}^{n}(y-y_k)}{(m+1)!(n+1)!}, \end{aligned}$$

where $\xi, \overline{\xi} \in (a,b)$ and $\eta, \overline{\eta} \in (c,d)$. It is easy to verify that

$$\begin{aligned} f(x,y) - T(x,y) &= \frac{E(x,y)}{Q(x,y)} \\ &= \frac{\omega(x)}{Q(x,y)} \frac{\frac{\partial^{m+1}}{\partial x^{m+1}}[fQ-P]_{x=\xi}}{(m+1)!} + \frac{\omega^*(y)}{Q(x,y)} \frac{\frac{\partial^{n+1}}{\partial y^{n+1}}[fQ-P]_{y=\eta}}{(n+1)!} \\ &\quad - \frac{\omega(x)\omega^*(y)}{Q(x,y)} \frac{\frac{\partial^{n+m+2}}{\partial x^{m+1}\partial y^{n+1}}[fQ-P]_{x=\overline{\xi},y=\overline{\eta}}}{(m+1)!(n+1)!}, \end{aligned}$$

with $\xi, \overline{\xi} \in (a,b)$ and $\eta, \overline{\eta} \in (c,d)$, where

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_m), \quad \omega^*(y) = (y-y_0)(y-y_1)\cdots(y-y_n).$$

The proof is complete. ■

4 Numerical examples

In this section, some numerical examples are given to show how the algorithms are implemented.

Example 1: Suppose the interpolating points and the prescribed values of $f(x,y)$ at the support abscissas (x_i, y_j) are given in the following table

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$
$x_0 = 0$	0	-3	-4	1
$x_1 = 1$	1	0	-1	-3
$x_2 = 2$	3	1	0	-1
$x_3 = 3$	4	-4	3	0

For convenience, we only present a few schemes.

Scheme 1: Classical symmetric branched continued fraction interpolant
Let us consider $u = v = 0$ (see *Case 1*). Then the above block based bivariate blending rational interpolant degenerates into the following classical symmetric branched continued fraction interpolant on the whole set Π_{33}

$$\begin{aligned} T(x,y) = {} & \frac{63x + 114xy - 45x^2 + 10x^2y - 192y + 66y^2 - 44xy^2}{54 - 36x - 12y + 8xy} \\ & + \frac{xy}{\frac{2157-237y+27x-283xy}{9858-4346x-3906y+1722xy} + \frac{(x-1)(y-1)}{\frac{10}{3}+\frac{2}{7}x-\frac{34}{21}y+\frac{(x-2)(y-2)}{-\frac{3}{118}}}}. \end{aligned}$$

It is easy to check that

$$T(x_i, y_j) = f_{ij}, \quad i, j = 0, 1, 2, 3.$$

Scheme 2: Block based bivariate blending rational interpolant
Let $u = 0$, $v = 1$ and $h_0 = 0$, $r_0 = 1$; $h_1 = 2$, $r_1 = 3$ (see *Case 2*). The whole set Π_{33} is divided into the following two subsets Π_{33}^{00} and Π_{33}^{01}:

(0,0)	(0,1)	(0,2)	(0,3)
(1,0)	(1,1)	(1,2)	(1,3)
(2,0)	(2,1)	(2,2)	(2,3)
(3,0)	(3,1)	(3,2)	(3,3)

Suppose $I_{00}(x, y)$ is the symmetric branched continued fraction interpolant on the subset Π_{33}^{00}. Then we have

$$I_{00}(x, y) = \frac{x}{1 + \frac{x-1}{7-5x}} + \frac{y}{-\frac{1}{3}} + \frac{xy}{\frac{1}{2} + \frac{x-1}{\frac{2}{3} - \frac{82}{33}(x-2)}}.$$

By (7), we have

$$f_{02}^{01} = 1 \quad f_{03}^{01} = \frac{5}{3} \quad f_{12}^{01} = 0 \quad f_{13}^{01} = -\frac{1}{6}$$
$$f_{22}^{01} = \frac{1}{2} \quad f_{23}^{01} = \frac{1}{3} \quad f_{32}^{01} = \frac{15}{2} \quad f_{33}^{01} = \frac{10}{3}$$

Let $I_{01}(x, y)$ be the symmetric branched continued fraction interpolant on the subset Π_{33}^{01}. Then it follows from (8) that

$$I_{01}(x, y) = \frac{1}{-33696 + 22212x - 3480x^2} \times (-23016x + 5557x^2 + 11232$$
$$+1239x^3 + 45618xy - 24260x^2y - 22464y + 3600x^3y).$$

By (13) and (4), one finally obtains

$$T(x, y) = \frac{x}{1 + \frac{x-1}{7-5x}} + \frac{y}{-\frac{1}{3}} + \frac{xy}{\frac{1}{2} + \frac{x-1}{\frac{2}{3} - \frac{82}{33}(x-2)}} +$$
$$\frac{y(y-1)}{-33696 + 22212x - 3480x^2} \times (-23016x + 5557x^2$$
$$+11232 + 1239x^3 + 45618xy - 24260x^2y - 22464y + 3600x^3y).$$

It is easy to verify that

$$T(x_i, y_j) = f_{ij}, \quad i, j = 0, 1, 2, 3.$$

Scheme 3: Classical Newton-Thiele interpolant
Suppose $u = 3$ and $v = 0$ (see *Case 3*). Then the above block based bivariate blending rational interpolant becomes the following classical Newton-Thiele interpolant on the whole set Π_{33}

$$T(x, y) = \frac{11y^2 - 32y}{-2y + 9} + \frac{21y^2 - 48y - 21}{5y - 21}x$$
$$+\frac{18y^2 - 49y + 12}{24 - 5y}x(x-1) + \frac{53y^2 - 51y - 114}{342 - 174y}x(x-1)(x-2).$$

It is easy to verify that

$$T(x_i, y_j) = f_{ij}, \quad i, j = 0, 1, 2, 3.$$

Scheme 4: Block based bivariate blending rational interpolant
Let $u = 1, v = 0$ and $c_0 = 0, d_0 = 1; c_1 = 2, d_1 = 3$ (see *Case 3*). The whole set Π_{33} is divided into the following two subsets Π_{33}^{00} and Π_{33}^{10}:

$$\begin{array}{cccc} (0,0) & (0,1) & (0,2) & (0,3) \\ (1,0) & (1,1) & (1,2) & (1,3) \\[4pt] (2,0) & (2,1) & (2,2) & (2,3) \\ (3,0) & (3,1) & (3,2) & (3,3) \end{array}$$

Suppose $I_{00}(x, y)$ is the symmetric branched continued fraction interpolant on the subset Π_{33}^{00}. Then we have

$$I_{00}(x, y) = \frac{-9x + 2xy + 32y - 11y^2}{2y - 9} + \frac{xy(21y - 53)}{5y - 21}.$$

By (9), we have

$$\begin{array}{cccc} f_{20}^{10} = \frac{1}{2} & f_{21}^{10} = -1 & f_{22}^{10} = -1 & f_{23}^{10} = 3 \\ f_{30}^{10} = \frac{1}{6} & f_{31}^{10} = -\frac{5}{3} & f_{32}^{10} = -\frac{1}{3} & f_{33}^{10} = \frac{11}{6} \end{array}$$

Let $I_{10}(x, y)$ denote the symmetric branched continued fraction interpolant on the subset Π_{33}^{10}. Then it follows from (11) that

$$I_{01}(x, y) = \frac{84 - 157y - 24x + 5xy + 54y^2}{72 - 15y} + \frac{y(x - 2)(109 - 53y)}{6(29y - 57)}.$$

By (15) and (3), we finally obtain

$$\begin{aligned} T(x, y) &= \frac{-9x + 2xy + 32y - 11y^2}{2y - 9} + \frac{xy(21y - 53)}{5y - 21} \\ &+ \left(\frac{84 - 157y - 24x + 5xy + 54y^2}{72 - 15y} + \frac{y(x - 2)(109 - 53y)}{6(29y - 57)} \right) x(x - 1). \end{aligned}$$

It is easy to check that

$$T(x_i, y_j) = f_{ij}, \quad i, j = 0, 1, 2, 3.$$

Scheme 5: Classical Thiele-Newton interpolant
Suppose $u = 0$ and $v = 3$ (see *Case 3*). Then the block based bivariate blending rational interpolant degenerates into the following classical Thiele-Newton interpolant on the whole set Π_{33}

$$\begin{aligned} T(x, y) = -\frac{8}{3}y - y^2 + \frac{2}{3}y^3 + \frac{x}{\left| 1 - \frac{17}{12}y + \frac{23}{24}y^2 - \frac{5}{24}y^3 \right.} \\ + \frac{x - 1}{\left| -3 + \frac{253}{18}y - \frac{16}{3}y^2 + \frac{5}{18}y^3 \right.} + \frac{x - 2}{\left| -\frac{1}{5} + \frac{239}{660}y - \frac{677}{1320}y^2 + \frac{263}{1320}y^3 \right.}. \end{aligned}$$

It is easy to verify that

$$T(x_i, y_j) = f_{ij}, \quad i, j = 0, 1, 2, 3.$$

Scheme 6: Classical bivariate Newton polynomial interpolant
Let $u = 3$ and $v = 3$ (see *Case 4*). Then the block based bivariate blending rational interpolant

degenerates into the following classical bivariate Newton polynomial interpolant on the whole set Π_{33}

$$\begin{aligned}T(x,y) = {} & x - 3y + \frac{1}{2}x(x-1) + 2xy + y(y-1) - \frac{1}{3}x(x-1)(x-2)\\ & -\frac{3}{2}x(x-1)y - xy(y-1) + \frac{2}{3}y(y-1)(y-2)\\ & -\frac{1}{3}x(x-1)(x-2)y + \frac{3}{4}x(x-1)y(y-1)\\ & -\frac{5}{6}xy(y-1)(y-2) + \frac{5}{6}x(x-1)(x-2)y(y-1)\\ & +\frac{5}{12}x(x-1)y(y-1)(y-2) - \frac{29}{36}x(x-1)(x-2)y(y-1)(y-2).\end{aligned}$$

It is easy to verify that

$$T(x_i, y_j) = f_{ij}, \quad i,j = 0,1,2,3.$$

Example 2: In order to compare the stability of block based bivariate blending rational interpolation and the classical bivariate Newton's polynomial interpolation, we have chosen the function $f(x,y) = \ln(1+x^2+y^2) + x(x-1)(x-2)\exp((1+x^2+y^2)^{-1})$. As a set of the interpolating points we take $\Pi_{44} = \{(x_i, y_j) \mid i = 0,1,\ldots,4; j = 0,1,\ldots,4\}$ and the values of $f(x,y)$ at the support abscissas (x_i, y_j) are given in the following table

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	$y_4 = 4$
$x_0 = 0$	0	0.6931471806	1.609437912	2.302585093	2.833213344
$x_1 = 1$	0.6931471806	1.098612289	1.791759469	2.397895273	2.890371758
$x_2 = 2$	1.609437912	1.791759469	2.197224577	2.639057330	3.044522438
$x_3 = 3$	8.933610601	8.968911913	9.083305916	9.268686437	9.493361086
$x_4 = 4$	28.28732683	28.26143764	28.21502757	28.19915473	28.23491172

The classical bivariate Newton polynomial interpolant on the whole set Π_{44} is obtained and a block based bivariate blending rational interpolant is computed when the whole set Π_{44} is divided into the following two subsets Π_{44}^{00} and Π_{44}^{10}:

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)

(3,0)	(3,1)	(3,2)	(3,3)	(3,4)
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)

We have calculated the errors at $(1.5, 1.5)$ and $(3.5, 3.5)$. The results are displayed in the following table

	(1.5,1.5)	(3.5,3.5)
Bivariate Newton interpolant	0.395318059	0.28624459
Block based blending interpolant	0.025869493	0.00200955

5 Conclusions

This paper presents a new kind of block based bivariate blending rational interpolation which can be computed recursively based on the block blending method via the symmetric branched continued fractions. It is demonstrated that our method provides many flexible bivariate blending rational interpolation schemes. We give a brief discussion of the algorithm for the block based bivariate blending rational interpolation, illustrate some special cases and investigate the error estimates. Numerical examples show the flexibility and effectiveness of our method. Finally we point out that the block based bivariate blending rational interpolation method can be easily generalized to vector-valued cases or matrix-valued cases ([6, 10, 16]).

References

[1] Chaffy C. Fractions continues a deux variables et calcul formel. Technical Report, IMAG, 1986.

[2] Cuyt A, Verdonk B. Multivariate rational interpolation. Computing, 1985, 34: 41–61.

[3] Cuyt A, Verdonk B. A review of branched continued fraction theory for the construction of multivariate rational approximants. Appl. Numer. Math., 1988, 4: 263–271.

[4] Cuyt A. A recursive computation scheme for multivariate rational interpolants. SIAM J. Numer. Anal., 1987, 24: 228–239.

[5] Gasca M, Sauer T., On the history of multivariate polynomial interpolation. J. Comput. Appl. Math., 2000, 122: 23–35.

[6] Graves-Morris P R. Vector valued rational interpolants I. Numer. Math., 1983, 42: 331–348.

[7] Kuchminskaya K, Siemaszko W. Rational approximation and interpolation of functions by branched continued fractions. In: Rational Approximation and its Applications in Mathematics and Physics, Gilewicz J, Pindor M, Siemaszko W (Eds.), Lecture Notes Math., volume 1237, Springer, Berlin, 1987, pages 24–40.

[8] Siemaszko W. Thiele-type branched continued fractions for two variable functions. J. Comput. Appl. Math., 1983, 9: 137–153.

[9] Tan J. Bivariate blending rational interpolants. Approx. Theory Appl., 1999, 15(2): 74–83.

[10] Tan J, Zhu G. General framework for vector-valued interpolants. In: Proc. of the 3rd China-Japan Seminar on Numerical Mathematics, Shi Z (Ed.), Science Press, Beijing/New York, 1998, pages 273–278.

[11] Tan J, Fang Y. Newton-Thiele's rational interpolants. Numer. Algorithms, 2000, 24: 141–157.

[12] Tan J, Jiang P. A Neville-like method via continued fractions. J. Comput. Appl. Math., 2004, 163: 219–232.

[13] Tan J, Tang S. Bivariate composite vector valued rational interpolation. Math. Comp., 1999, 69: 1521–1532.

[14] Tan J, Tang S. Composite schemes for multivariate blending rational interpolation. J. Comput. Appl. Math., 2002, 144: 263–275.

[15] Wang J, Gu C. Vector valued Thiele-Werner-type osculatory rational interpolants. J. Comput. Appl. Math., 2004, 163: 241–252.

[16] Zhu G, Tan J. A note on matrix-valued rational interpolants. J. Comput. Appl. Math., 1999, 110: 129-140.