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Dynamical System Method for Solving Ill-Posed Operator Equations[†]

Xingjun Luo[*] and Suhua Yang

Department of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China.

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Abstract. Two dynamical system methods are studied for solving linear ill-posed problems with both operator and right-hand nonexact. The methods solve a Cauchy problem for a linear operator equation which possesses a global solution. The limit of the global solution at infinity solves the original linear equation. Moreover, we also present a convergent iterative process for solving the Cauchy problem.

Key words: Ill-posed problems; dynamical system method; operator equations.

AMS subject classifications: 65J20, 65R30

1 Introduction

Dynamical systems method (DSM) is a general method for solving operator equations, especially for non-linear, ill-posed as well as well-posed operator equations [1-6]. In [5, 6], Ramm proposed a DSM for linear ill-posed problem with right hand nonexact. However, in practice, not only the right-hand side of equations but also the operators are approximately given. This paper is to provide a DSM for linear operator equation with not only noisy data but also perturbed operators.

We first briefly describe the dynamical systems method for solving operator equations. Consider an operator equation

$$\mathcal{A}u = f, \quad f \in H \tag{1}$$

Let us denote by $(\mathbf{\Sigma})$ the following assumption:

$(\mathbf{\Sigma})$: $\mathcal{A}$ is a linear, bounded operator in H, defined on all of H; the range $R(\mathcal{A})$ is not closed, so (1) is ill-posed problem. There is a y such that $\mathcal{A}y = f, \quad y \in N(\mathcal{A})^{\perp}$, where $N(\mathcal{A})$ is the null-space of $\mathcal{A}$.

Let $\dot{u}$ denote the derivative of u with respect to time. Consider the following dynamical system (the Cauchy problem):

$$\dot{u} = \Phi(t, u), \quad t > 0, \quad u(0) = u_0 \tag{2}$$

[*]Correspondence to: Xingjun Luo, Department of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China. Email: `luoxingjun98@yahoo.com.cn`
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where $\Phi(t,u)$ is globally *Lipchitz* with respect to $u \in H$ and continuous with respect to $t \geq 0$:

$$\sup_{u,v\in H, t\in[0,\infty)} \|\Phi(t,u) - \Phi(t,v)\| \leq c\|u - v\|, \quad c = const > 0. \tag{3}$$

Problem (2) has a global solution if (3) holds. The DSM for solving (1) consists of solving (2), where Φ is so chosen that the following three conditions hold:

$$\exists u(t)\ \forall t > 0; \quad \exists y := u(\infty) := \lim_{t\to\infty} u(t); \quad \mathcal{A}y = f. \tag{4}$$

For real number $h > 0$, let $\mathcal{A}_h$ be a bounded linear operator in a real Hilbert space H such that

$$\|\mathcal{A} - \mathcal{A}_h\| \leq h. \tag{5}$$

Problem (1) with noisy data $f^\delta, \|f - f^\delta\| \leq \delta$ and perturbed operator $\mathcal{A}_h$, satisfying (5), given in place of f and $\mathcal{A}$, respectively, generates the problem:

$$\dot{u}_{\delta,h} = \Phi_{\delta,h}(t,u), \quad t > 0, \quad u_{\delta,h}(0) = u_0. \tag{6}$$

The solution $u_{\delta,h}$ to (6) at $t = t_{\delta,h}$, will have the property

$$\lim_{r\to 0} \|u_{\delta,h}(t_{\delta,h}) - y\| = 0, \tag{7}$$

where $r = \sqrt{\delta^2 + h^2}$. The choice of $t_{\delta,h}$ with this property is called the stopping rule. One has usually $\lim_{r\to 0} t_{\delta,h} = \infty$.

We organize this paper into four sections. In Section 2, we describe one DSM for solving linear problem. In Section 3, we present another version of DSM. In Section 4, we propose two convergent iterative processes to solve the two Cauchy problems.

2 DSM I for solving the linear problem

Consider the Cauchy problem

$$\dot{u}_{\delta,h}(t) = \Phi_{\delta,h}(t, u_{\delta,h}(t)), \quad t > 0, \quad u_{\delta,h}(0) = u_0 \tag{8}$$

where $\Phi_{\delta,h}(t, u_{\delta,h}(t)) = -[\mathcal{B}_h u_{\delta,h}(t) + \varepsilon(t) u_{\delta,h}(t) - \mathcal{F}_{\delta,h}]$, $\mathcal{B}_h := \mathcal{A}_h^* \mathcal{A}_h$, $\quad \mathcal{F}_{\delta,h} = \mathcal{A}_h^* f^\delta$ and

$$\varepsilon(t) \in C^1[0,\infty), \quad \varepsilon(t) > 0, \quad \varepsilon(t) \searrow 0\ (t\to\infty), \tag{9}$$

$$\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)^{\frac{5}{2}}} \to 0\ (t\to\infty). \tag{10}$$

Lemma 2.1. [5] *Let $\mathcal{A}$ and $\mathcal{A}_h$ are linear operator in a real Hilbert space H, $\mathcal{B} = \mathcal{A}^*\mathcal{A}$, $\quad \mathcal{B}_h = \mathcal{A}_h^*\mathcal{A}_h$, $\varepsilon(t) \in C[0,\infty)$ and $\varepsilon(t) > 0$. Then the following inequalities hold*

(i). $\|(\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^*\| \leq \frac{1}{2\sqrt{\varepsilon(t)}}$,

(ii). $\|(\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^*\mathcal{A}\| \leq 1$,

(iii). $\|\varepsilon(t)(\varepsilon(t) + \mathcal{B})^{-1}\| \leq 1$.

If $\mathcal{A}, \mathcal{B}$ are replaced by $\mathcal{A}_h, \mathcal{B}_h$, respectively, the above conclusions are still correct.

Theorem 2.1. *Assume that assumption* $(\mathbf{\Sigma})$ *on* $\mathcal{A}$ *and* (9), (10) *on* $\varepsilon(t)$ *satisfied. Then, for any* $u_0 \in H$, *problem* (8) *has a unique global solution* $u_{\delta,h}(t)$. *Moreover, there exists* $t_{\delta,h}$ *such that*

$$\lim_{r\to 0} \|u_{\delta,h}(t_{\delta,h}) - y\| = 0.$$

Proof Eq. (8) with bounded operators has unique global solutions. Consider the problem

$$\mathcal{B}_h\omega_{\delta,h}(t) + \varepsilon(t)\omega_{\delta,h}(t) - \mathcal{F}_{\delta,h} = 0, \tag{11}$$

where $\mathcal{B}_h \geq 0$ and $\varepsilon(t) > 0$, so the solution $\omega_{\delta,h}(t)$ to (11) exists, is unique and admits the estimate

$$\|\omega_{\delta,h}(t)\| = \|(\mathcal{B}_h + \varepsilon(t))^{-1}\mathcal{F}_{\delta,h}\| \leq \frac{1}{2\sqrt{\varepsilon(t)}}(\delta + \|f\|).$$

We differentiate (11) with respect to t and get

$$\mathcal{B}_h\dot{\omega}_{\delta,h}(t) + \dot{\varepsilon}(t)\omega_{\delta,h}(t) + \varepsilon(t)\dot{\omega}_{\delta,h}(t) = 0. \tag{12}$$

It follows from (12) that

$$\|\dot{\omega}_{\delta,h}(t)\| = \|(\mathcal{B}_h + \varepsilon(t))^{-1}\dot{\varepsilon}(t)\omega_{\delta,h}(t)\| \leq \frac{|\dot{\varepsilon}(t)|}{2\varepsilon^{\frac{3}{2}}(t)}(\delta + \|f\|). \tag{13}$$

Denote $z_{\delta,h}(t) := u_{\delta,h}(t) - \omega_{\delta,h}(t)$. Then

$$\dot{z}_{\delta,h}(t) = -\dot{\omega}_{\delta,h}(t) - [\mathcal{B}_h z_{\delta,h}(t) + \varepsilon(t)z_{\delta,h}(t)], \quad z_{\delta,h}(0) = u_{\delta,h}(0) - \omega_{\delta,h}(0).$$

Denote $g_{\delta,h}(t) := \|z_{\delta,h}(t)\|$. Then we have

$$g_{\delta,h}(t)\dot{g}_{\delta,h}(t) = (\dot{z}_{\delta,h}(t), z_{\delta,h}(t)) \leq g_{\delta,h}(t)\|\dot{\omega}_{\delta,h}(t)\| - \varepsilon(t)g^2_{\delta,h}(t). \tag{14}$$

Since $g_{\delta,h}(t) \geq 0$, it follows from (13) and (14) that

$$\dot{g}_{\delta,h}(t) \leq C_1 - C_2 g_{\delta,h}(t), \quad g_{\delta,h}(0) = \|u_{\delta,h}(0) - \omega_{\delta,h}(0)\|,$$

where $C_1 = \frac{|\dot{\varepsilon}(t)|}{2\varepsilon^{\frac{3}{2}}(t)}(\delta + \|f\|) > 0$ and $C_2 = \varepsilon(t) > 0$. This gives

$$g_{\delta,h}(t) \leq e^{-\int_0^t \varepsilon(s)ds}\Big[g_{\delta,h}(0) + \int_0^t e^{\int_0^\tau \varepsilon(s)ds}\frac{|\dot{\varepsilon}(\tau)|}{2\varepsilon^{\frac{3}{2}}(\tau)}(\delta + \|f\|)d\tau\Big]. \tag{15}$$

It follows from (10) that $\int_0^\infty \varepsilon(s)ds = +\infty$, see, e.g., [3]. Using L'Hospital's rule gives

$$\lim_{t\to\infty}\frac{\int_0^t e^{\int_0^\tau \varepsilon(s)ds}\frac{|\dot{\varepsilon}(\tau)|}{2\varepsilon^{\frac{3}{2}}(\tau)}(\delta + \|f\|)d\tau}{e^{\int_0^t \varepsilon(s)ds}} = \lim_{t\to\infty}\frac{|\dot{\varepsilon}(t)|}{2\varepsilon^{\frac{5}{2}}(t)}(\delta + \|f\|) = 0,$$

which yields

$$\lim_{t\to\infty} g_{\delta,h}(t) = 0, \quad \forall\delta, h > 0. \tag{16}$$

Next, let us estimate $\|\omega_{\delta,h}(t) - y\|$. By the triangle inequality and Lemma 2.1, one gets

$$\begin{aligned}
\|\omega_{\delta,h}(t) - y\| &= \|(\varepsilon(t) + \mathcal{B}_h)^{-1}\mathcal{A}_h^* f^\delta - y\| \\
&\leq \|(\varepsilon(t) + \mathcal{B}_h)^{-1}\mathcal{A}_h^*(f^\delta - f)\| + \|(\varepsilon(t) + \mathcal{B}_h)^{-1}\mathcal{A}_h^* f - (\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^* f\| \\
&\quad + \|(\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^* f - y\| \\
&\leq \frac{\delta}{2\sqrt{\varepsilon(t)}} + I_1 + I_2.
\end{aligned}$$

It follows from Lemma 2.1 and (5) that

$$\begin{aligned} I_1 &= \|(\varepsilon(t)+\mathcal{B}_h)^{-1}\mathcal{A}_h^* f - (\varepsilon(t)+\mathcal{B})^{-1}\mathcal{A}^* f\| \\ &= \|(\varepsilon(t)+\mathcal{B}_h)^{-1}[\varepsilon(t)(\mathcal{A}_h^*-\mathcal{A}^*)+\mathcal{A}_h^*(\mathcal{A}-\mathcal{A}_h)\mathcal{A}^*](\varepsilon(t)+\mathcal{A}\mathcal{A}^*)^{-1}\mathcal{A}y\| \\ &\le \frac{\|y\|h}{\sqrt{\varepsilon(t)}}. \end{aligned}$$

Since $f=\mathcal{A}y$, one gets

$$I_2 = \|(\varepsilon(t)+\mathcal{B})^{-1}\mathcal{A}^* f - y\| = \|\varepsilon(t)(\varepsilon(t)+\mathcal{B})^{-1}y\| := \phi(\varepsilon(t),y) := \phi(\varepsilon(t)). \tag{17}$$

Consequently,

$$\|u_{\delta,h}(t)-y\| \le g_{\delta,h}(t) + C\frac{\delta+h}{\sqrt{\varepsilon(t)}} + \phi(\varepsilon(t)), \tag{18}$$

where $C=\max\{1/2,\|y\|\}$. Let us proof that $\lim_{\beta\to 0}\phi(\beta)=0$. Suppose $\{E_\lambda\}$ is the spectral family generated by the operator $\mathcal{B}=\mathcal{A}^*\mathcal{A}$. Then we have

$$\phi(\beta)^2 = \phi(\beta,y)^2 = \int_0^{\|A\|^2}\Big(\frac{\beta}{\lambda+\beta}\Big)^2 d(E_\lambda y,y). \tag{19}$$

Noting that as $\beta\to 0$,

$$\frac{\beta}{\beta+\lambda}\begin{cases} \to 0 & \text{as } \lambda>0, \\ =1 & \text{as } \lambda=0, \end{cases}$$

and using the assumption $y\in N(A)^\perp$, one gets $\lim_{\beta\to 0}\phi(\beta)=0$. If the corresponding stopping time $t_{\delta,h}$ can be taken as the root of equation

$$\sqrt{\varepsilon(t)} = (\delta+h)^d, \quad d\in(0,1), \tag{20}$$

then it is obvious that $\lim_{r\to 0} t_{\delta,h}=\infty$. The conclusion holds following from (16), (18) and (20). ■

3 DSM II for solving the linear system

Consider the Cauchy problem

$$\dot{u}_{\delta,h}(t) = \Phi_{\delta,h}(t,u_{\delta,h}(t)), \quad t>0, \quad u_{\delta,h}(0)=u_0, \tag{21}$$

where

$$\begin{aligned} \Phi_{\delta,h}(t,u_{\delta,h}(t)) &= -(\mathcal{B}_h+\varepsilon(t))^{-1}\Big[\mathcal{B}_h u_{\delta,h}(t)+\varepsilon(t)u_{\delta,h}(t)-\mathcal{F}_{\delta,h}\Big] \\ &= -u_{\delta,h}(t) + (\mathcal{B}_h+\varepsilon(t))^{-1}\mathcal{F}_{\delta,h}. \end{aligned}$$

Theorem 3.1. *Let the assumption* (Σ) *holds. Assume* $\varepsilon(t)>0$ *is continuous, monotonically decaying to zero on* $[0,\infty)$*. Then, for any* $u_0\in H$*, problem* (21) *has a unique global solution* $u_{\delta,h}(t)$*. Moreover, there exists* $t_{\delta,h}$ *such that*

$$\lim_{r\to 0}\|u_{\delta,h}(t_{\delta,h})-y\| = 0.$$

Proof Denote $\rho_{\delta,h}(t) := u_{\delta,h}(t) - u_h(t)$ and $g^\delta := f^\delta - f$, where $u_h(t)$ is the solution to problem (21) with $\mathcal{F}_{\delta,h}$ replaced by $\mathcal{F}_\delta$. It is easy to obtain

$$\dot\rho_{\delta,h}(t) = -\rho_{\delta,h}(t) + (\mathcal{B}_h + \varepsilon(t))^{-1}\mathcal{A}_h^* g^\delta, \quad t > 0, \quad \rho_{\delta,h}(0) = 0. \tag{22}$$

Therefore, the solution to (21) is

$$\rho_{\delta,h}(t) = e^{-t}\int_0^t e^s(\mathcal{B}_h + \varepsilon(s))^{-1}\mathcal{A}_h^* g^\delta ds.$$

It follows from Lemma 2.1 that

$$\|\rho_{\delta,h}(t)\| \le \delta e^{-t}\int_0^t e^s\|(\mathcal{B}_h + \varepsilon(s))^{-1}\mathcal{A}_h^*\|ds \le \delta e^{-t}\int_0^t \frac{e^s}{2\sqrt{\varepsilon(s)}}ds \le \frac{\delta}{2\sqrt{\varepsilon(t)}}.$$

Let $u(t)$ be the solution to problem (21) with $\mathcal{F}_{\delta,h}$ and $\mathcal{B}_h$ replaced by $\mathcal{F}_h = \mathcal{A}_h^* f$ and $\mathcal{B} = \mathcal{A}^*\mathcal{A}$, respectively. Then one gets

$$u(t) = u_0 e^{-t} + e^{-t}\int_0^t e^s(\varepsilon(s) + \mathcal{B})^{-1}\mathcal{A}^* f ds,$$

$$u_h(t) = u_0 e^{-t} + e^{-t}\int_0^t e^s(\varepsilon(s) + \mathcal{B}_h)^{-1}\mathcal{A}_h^* f ds.$$

Since $\|[(\varepsilon(s) + \mathcal{B}_h)^{-1}\mathcal{A}_h^* - (\varepsilon(s) + \mathcal{B})^{-1}\mathcal{A}^*]f\| \le \frac{\|y\|h}{\sqrt{\varepsilon(s)}}$, one gets

$$\begin{aligned}\|u_h(t) - u(t)\| &\le e^{-t}\int_0^t e^s\|[(\varepsilon(s) + \mathcal{B}_h)^{-1}\mathcal{A}_h^* - (\varepsilon(s) + \mathcal{B})^{-1}\mathcal{A}^*]f\|ds \\ &\le e^{-t}\int_0^t e^s \times \frac{\|y\|h}{\sqrt{\varepsilon(s)}}ds \le \frac{\|y\|h}{\sqrt{\varepsilon(t)}}.\end{aligned}$$

Thus, by the triangle inequality, one gets

$$\begin{aligned}\|u_{\delta,h}(t) - y\| &\le \|u_{\delta,h} - u_h(t)\| + \|u_h - u(t)\| + \|u(t) - y\| \\ &\le \frac{\delta}{2\sqrt{\varepsilon(t)}} + \frac{\|y\|h}{\sqrt{\varepsilon(t)}} + \|u(t) - y\| \\ &\le \frac{C(\delta + h)}{\sqrt{\varepsilon(t)}} + \|u(t) - y\|.\end{aligned}$$

As a result, the corresponding stopping time $t_{\delta,h}$ can be taken as the root of equation:

$$\sqrt{\varepsilon(t)} = C(\delta + h)^b, \quad b \in (0, 1). \tag{23}$$

It follows from the condition on $\varepsilon(t)$ that $\lim_{r\to 0} t_{\delta,h} = \infty$. One also gets from [4] that $\lim_{t\to\infty}\|u(t) - y\| = 0$. Thus

$$\lim_{r\to 0}\|u_{\delta,h}(t_{\delta,h}) - y\| = 0. \quad \blacksquare$$

4 Convergence of the iterative process

Let us use Euler's method to solve the Cauchy problems (8) and (21) numerically. The numerical methods are

$$p_{\delta,h}^{n+1} = p_{\delta,h}^{n} - \omega_n\big[(\mathcal{B}_h + \varepsilon_n)p_{\delta,h}^{n} - \mathcal{F}_{\delta,h}\big], \quad n = 0, 1, \cdots, \tag{24}$$

$$p_{\delta,h}^{0} := u_0, \quad \varepsilon_n := \varepsilon(t_n), \quad t_n := \sum_{i=0}^{n} \omega_i, \quad \omega_i > 0 \tag{25}$$

$$q_{\delta,h}^{n+1} = (1-\omega_n)q_{\delta,h}^{n} + \omega_n(\mathcal{B}_h + \varepsilon_n)^{-1}\mathcal{F}_{\delta,h}, \quad n = 0, 1, \cdots, \tag{26}$$

$$q_{\delta,h}^{0} := u_0, \quad \varepsilon_n := \varepsilon(t_n), \quad t_n := \sum_{i=0}^{n} \omega_i, \quad \omega_i > 0. \tag{27}$$

In this section, it is proved that under certain conditions the iterative schemes (24) with (25) and (26) with (27) are convergent.

Lemma 4.1. [7] *Assume the sequence of positive number* ν_n *satisfies the inequality*

$$\nu_{n+1} \le (1-\alpha_n)\nu_n + \theta_n,$$

where

$$0 < \alpha_n \le 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0.$$

Then

$$\lim_{n\to\infty} \nu_n = 0.$$

Theorem 4.1. *Assume the conditions of Theorem* 2.1 *hold. Assume further that*

(i). δ *is the level of noise in* (24): $\|f - f^\delta\| \le \delta$, (5) *holds and* $\|\mathcal{A}\| \le N$*;*

(ii). $n(\delta, h)$ *is chosen in such a way that* $\lim_{r\to 0} n(\delta, h) = \infty$*;*

(iii). $\omega_{n(\delta,h)}$ *tends to zero monotonically as* $r \to 0$, *and* $0 < \omega_{n(\delta,h)}\varepsilon_{n(\delta,h)} - c\omega_{n(\delta,h)}^2 \le 1$ *with* c *defined by* (31)*;*

(iv). $\sum_{n=1}^{\infty} \omega_n\varepsilon_n = \infty, \quad \lim_{r\to 0} \frac{\omega_{n(\delta,h)}}{\varepsilon_{n(\delta,h)}} = 0$, *and* $\lim_{r\to 0} \frac{\delta+h}{\sqrt{\varepsilon_{n(\delta,h)}}} = 0$.

Then

$$\lim_{r\to 0} \|p_{\delta,h}^{n(\delta,h)} - y\| = 0. \tag{28}$$

Proof Note that $u_{\delta,h}(t)$ satisfies (8), one has

$$\begin{aligned} u_{\delta,h}(t_{n+1}) &= u_{\delta,h}(t_n) + \omega_n \dot{u}_{\delta,h}(t_n) + \frac{\omega_n^2}{2} u''_{\delta,h}(\xi_n), \qquad \xi_n \in (t_n, t_{n+1}) \\ &= u_{\delta,h}(t_n) - \omega_n[\mathcal{B}_h u_{\delta,h}(t_n) + \varepsilon(t_n)u_{\delta,h}(t_n) - \mathcal{F}_{\delta,h}] \\ &\quad + \frac{\omega_n^2}{2}\Big\{\big[\mathcal{B}_h + \varepsilon(\xi_n)\big]\big[\mathcal{B}_h u_{\delta,h}(\xi_n) + \varepsilon(\xi_n)u_{\delta,h}(\xi_n) - \mathcal{F}_{\delta,h}\big] - \dot{\varepsilon}(\xi_n)u_{\delta,h}(\xi_n)\Big\}. \end{aligned}$$

Then

$$\begin{aligned}\|p_{\delta,h}^{n+1}-u_{\delta,h}(t_{n+1})\| &\le \|\big(1-\omega_n\varepsilon(t_n)\big)\big(p_{\delta,h}^n-u_{\delta,h}(t_n)\big)-\omega_n\mathcal{B}_h(p_{\delta,h}^n-u_{\delta,h}(t_n))\| \\ &\quad +\frac{\omega_n^2}{2}\|\big[\mathcal{B}_h+\varepsilon(\xi_n)\big]\big[\mathcal{A}_h^*(\mathcal{A}_h u_{\delta,h}(\xi_n)-f^\delta)+\varepsilon(\xi_n)u_{\delta,h}(\xi_n)\big]-\dot\varepsilon(\xi_n)u_{\delta,h}(\xi_n)\| \\ &\le \|\big(1-\omega_n\varepsilon(t_n)\big)\big(p_{\delta,h}^n-u_{\delta,h}(t_n)\big)-\omega_n\mathcal{B}_h(p_{\delta,h}^n-u_{\delta,h}(t_n))\| \\ &\quad +\frac{\omega_n^2}{2}\Big\{\Big[(N+h)^2+\varepsilon(\xi_n)\Big]\Big[(N+h)\Big((N+h+\dot\varepsilon(\xi_n))\|u_{\delta,h}(\xi_n)-y\| \\ &\quad +(h+\dot\varepsilon(\xi_n))\|y\|+\delta\Big)\Big]+|\dot\varepsilon(\xi_n)|\big(\|u_{\delta,h}(\xi_n)-y\|+y\big)\Big\}.\end{aligned}$$

By introducing the notation

$$\begin{aligned}\lambda_n &:= \Big[(N+h)^2+\varepsilon(\xi_n)\Big]\Big[(N+h)\Big((N+h+\dot\varepsilon(\xi_n))\|u_{\delta,h}(\xi_n)-y\|+(h+\dot\varepsilon(\xi_n))\|y\|+\delta\Big)\Big] \\ &\quad +|\dot\varepsilon(\xi_n)|\big(\|u_{\delta,h}(\xi_n)-y\|+y\big),\end{aligned}$$

one obtains from Theorem 2.1 that

$$\lim_{r\to 0}\lambda_{n(\delta,h)}=0.$$

If $\omega_n\varepsilon_n<1$, then from the condition $\mathcal{B}_h\ge 0$ one obtains

$$\begin{aligned}&\|p_{\delta,h}^{n+1}-u_{\delta,h}(t_{n+1})\| \\ \le\ &\Big\{(1-\omega_n\varepsilon_n^2)^2\|p_{\delta,h}^n-u_{\delta,h}(t_n)\|^2+\omega_n^2\|\mathcal{B}_h(p_{\delta,h}^n-u_{\delta,h}(t_n))\|^2\Big\}^{\frac12}+\frac{\omega_n^2\lambda_n}{2}.\end{aligned}\tag{29}$$

Applying the elementary estimate

$$(a+b)^2\le(1+\omega_n\varepsilon_n)a^2+\Big(1+\frac{1}{\omega_n\varepsilon_n}\Big)b^2$$

to the right-hand sides of (29) with

$$a:=\Big\{(1-\omega_n\varepsilon_n)^2\|(p_{\delta,h}^n-u_{\delta,h}(t_n))\|^2+\omega_n^2\|\mathcal{B}_h(p_{\delta,h}^n-u_{\delta,h}(t_n))\|^2\Big\}^{\frac12},\quad b:=\frac{\omega_n^2\lambda_n}{2},$$

one gets

$$\|p_{\delta,h}^{n+1}-u_{\delta,h}(t_{n+1})\|^2\le(1-\omega_n\varepsilon_n+c\omega_n^2)\|p_{\delta,h}^n-u_{\delta,h}(t_n)\|^2+d\frac{\omega_n^3}{\varepsilon_n},\tag{30}$$

where we assume that ω_n tends to zero monotonically as $n\to\infty$, and

$$c:=(1+\omega_0\varepsilon_0)(\varepsilon_0^2+N_1),\quad N_1=(N+h)^2,\quad d:=\frac14\lambda_n^2(1+\varepsilon_0\omega_0).\tag{31}$$

Let

$$\nu_{n(\delta,h)}:=\|p_{\delta,h}^{n(\delta,h)}-u_{\delta,h}(t_{n(\delta,h)})\|^2,\quad \alpha_{n(\delta,h)}:=\omega_{n(\delta,h)}\varepsilon_{n(\delta,h)}-c\omega_{n(\delta,h)}^2,\quad \theta_{n(\delta,h)}:=d\frac{\omega_{n(\delta,h)}^3}{\varepsilon_{n(\delta,h)}}.$$

The desired result (28) is an immediate consequence of (30) and Lemma 4.1 ■

Theorem 4.2. *Assume the condition* (Σ) *holds. Assume further that* (i). *δ is the level of noise in* (26)*: $\|f - f^{\delta}\| \le \delta$ and* (5) *holds;*

(ii). *$n = n(\delta, h)$ is chosen in such a way that $\lim\limits_{r\to 0} n(\delta, h) = \infty$;*

(iii). *$\varepsilon(t) \in C[0,\infty)$, $\varepsilon(t) \searrow 0\,(t\to\infty)$, and $\frac{|\dot\varepsilon(t)|}{\varepsilon(t)^2} \to 0(t\to\infty)$;*

(iv). *$\sum_{n=1}^{\infty} \omega_n = \infty$, $0 < \omega_n < 1$ and $\lim_{r\to 0} \frac{\delta+h}{\sqrt{\varepsilon_{n(\delta,h)}}} = 0$.*

Then

$$\lim_{r\to 0} \|q^n_{\delta,h} - y\| = 0,$$

where $n := n(\delta, h)$.

Proof Note that $u_{\delta,h}(t)$ satisfies (21), one has

$$\begin{aligned} u_{\delta,h}(t_{n+1}) &= u_{\delta,h}(t_n) + \omega_n \dot u_{\delta,h}(t_n) + \frac{\omega_n^2}{2} u''_{\delta,h}(\xi_n), \quad \xi_n \in (t_n, t_{n+1}) \\ &= u_{\delta,h}(t_n) + \omega_n[(-u_{\delta,h}(t_n) + (\mathcal{B}_h + \varepsilon_n)^{-1}\mathcal{F}_{\delta,h}] \\ &\quad + \frac{\omega_n^2}{2}\left\{u(\xi_n) - \Big(I + \dot\varepsilon(\xi_n)(\mathcal{B}_h + \varepsilon(\xi_n))^{-1}\Big)(\mathcal{B}_h + \varepsilon(\xi_n))^{-1}\mathcal{F}_{\delta,h}\right\}. \end{aligned}$$

Then

$$\|q^{n+1}_{\delta,h} - u_{\delta,h}(t_{n+1})\| \le (1-\omega_n)\|q^n_{\delta,h} - u_{\delta,h}(t_n)\| + \frac{\omega_n^2\lambda_n}{2}, \tag{32}$$

where

$$\lambda_n = \|u_{\delta,h}(\xi_n) - y\| + \|y - (\mathcal{B}_h + \varepsilon(\xi_n))^{-1}\mathcal{F}_{\delta,h}\| + \|\dot\varepsilon(\xi_n)(\mathcal{B}_h + \varepsilon(\xi_n))^{-2}\mathcal{F}_{\delta,h}\|. \tag{33}$$

Since

$$\|y - (\mathcal{B}_h + \varepsilon(\xi_n))^{-1}\mathcal{F}_{\delta,h}\| \le \frac{C(\delta+h)}{\sqrt{\varepsilon(\xi_n)}} + \phi(\varepsilon(\xi_n)), \tag{34}$$

where the function $\phi(\beta)$ is as the same as in (17), $C = \max\{1/2, \|y\|\}$. By (33), (34) and Lemma 2.1 one gets

$$\begin{aligned} \lambda_n &= \|u_{\delta,h}(\xi_n) - y\| + \|y - (\mathcal{B}_h + \varepsilon(\xi_n))^{-1}\mathcal{F}_{\delta,h}\| + \|\dot\varepsilon(\xi_n)(\mathcal{B}_h + \varepsilon(\xi_n))^{-2}\mathcal{F}_{\delta,h}\| \\ &\le \|u_{\delta,h}(\xi_n) - y\| + \frac{C(\delta+h)}{\sqrt{\varepsilon(\xi_n)}} + \phi(\varepsilon(\xi_n) + \|\frac{\dot\varepsilon(\xi_n)}{\varepsilon^2(\xi_n)}\varepsilon^2(\xi_n)(\mathcal{B}_h + \varepsilon(\xi_n))^{-2}\mathcal{F}_{\delta,h}\| \\ &\le \|u_{\delta,h}(\xi_n) - y\| + \frac{C(\delta+h)}{\sqrt{\varepsilon(\xi_n)}} + \phi(\varepsilon(\xi_n) + \frac{|\dot\varepsilon(\xi_n)|}{\varepsilon^2(\xi_n)}\|\mathcal{F}_{\delta,h}\|. \end{aligned}$$

It follows from Theorem 3.1 and the conditions in Theorem 4.2 that $\lim_{r\to 0}\lambda_{n(\delta,h)} = 0$. Denote $g^n_{\delta,h} = \|q^n_{\delta,h} - u_{\delta,h}(t_n)\|$ and $\beta_n = \lambda_n\omega_n^2/2$. Then

$$g^{n+1}_{\delta,h} \le (1-\omega_n)g^n_{\delta,h} + \beta_n. \tag{35}$$

From Lemma 4.1 and the condition (iv), one has

$$\lim_{r\to 0} g^{n(\delta,h)}_{\delta,h} = 0. \tag{36}$$

The conclusion follows from Theorem 3.1 and (36). ■

5 Conclusion

Two dynamical system methods are presented for solving operator equations of the first kind with both operator and right-hand nonexact, which extended the methods introduced in [5, 6]. Moreover, we also present two convergent iterative processes for the dynamical systems. It is of further interest to investigate other types of dynamical system methods for solving operator equations of the first kind and to present some discrete methods for solving the relevant Cauchy problems more efficiently.

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