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Orthogonal Matrix-Valued Wavelet Packets[†]

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Abstract. In this paper, we introduce matrix-valued multiresolution analysis and matrix-valued wavelet packets. A procedure for the construction of the orthogonal matrix-valued wavelet packets is presented. The properties of the matrix-valued wavelet packets are investigated. In particular, a new orthonormal basis of $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is obtained from the matrix-valued wavelet packets.

Key words: Matrix-valued multiresolution analysis; matrix-valued scaling functions; matrix-valued wavelet packets; refinement equation.

AMS subject classifications: 42C40, 65T60

1 Introduction

Wavelet packets, due to their nice characteristics, have been applied to signal processing [1], image compression [2], integral equations [3] and so on. Coifman and Meyer [4] firstly introduced the concept of orthogonal wavelet packets. The introduction for biorthogonal wavelet packets was attributable to Cohen and Daubechies [5]. Furthermore, Yang and Cheng [6] constructed a-scale orthogonal multiwavelet packets which are more flexible in applications. Recently, the multiwavelets have become the focus of active research both in theory and application, such as signal processing [7], mainly because of their ability to offer properties like orthogonality and symmetry simultaneously. The matrix-valued wavelets are a class of generalized multiwavelets. Xia and Suter [8] introduced the concept of the matrix-valued wavelets and investigated its construction. Moreover, they showed that multiwavelets can be generated from the component functions of matrix-valued wavelets. However, the multiwavelets and matrix-valued wavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms [9] but not necessary for discrete matrix-valued wavelet transforms. A typical example of such matrix-valued signals is video images. Hence, studying the matrix-valued wavelets is useful in representations of signals. It is necessary to extend the concept of orthogonal wavelet packets to the case of orthogonal matrix-valued wavelets. Based on an observation in

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[8] and some ideas from [5,6], we will give the definition for 3-scale orthogonal matrix-valued wavelet packets and investigate the properties of the orthogonal matrix-valued wavelet packets by using matrix theory and integral transform.

Throughout the paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be sets of all real and complex numbers, respectively. $\mathbb{Z}$ stands for all integers. Set $s \in \mathbb{Z}$, $s \geq 2$, and $\mathbb{Z}_+ = \{z : z \geq 0, z \in \mathbb{Z}\}$. By $\mathbf{I}_s$ and $\mathbf{O}$, we denote the $s \times s$ identity matrix and zero matrix, respectively.

$$L^2(\mathbb{R}, \mathbb{C}^{s\times s}) := \left\{ \hbar(t) := \begin{pmatrix} h_{11}(t) & h_{12}(t) & \cdots & h_{1\,s}(t) \\ h_{21}(t) & h_{22}(t) & \cdots & h_{2\,s}(t) \\ \cdots & \cdots & \cdots & \cdots \\ h_{s1}(t) & h_{s2}(t) & \cdots & h_{s\,s}(t) \end{pmatrix} : \begin{array}{c} t \in \mathbb{R},\ h_{k\,l}(t) \in L^2(\mathbb{R}), \\ k, l = 1, 2, \cdots, s \end{array} \right\}$$

The signal space $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called a matrix-valued function space. Examples of matrix-valued signals are video images where $h_{k\,l}(t)$ is the pixel on the kth row and the lth column at time t.

For each $\hbar \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, $||\hbar||$ represents the norm of operator $\hbar$ as

$$||\,\hbar|| := \left(\sum_{k,\,l=1}^{s} \int_{\mathbb{R}} |h_{k,\,l}(t)|^2 dt \right)^{1/2}. \tag{1}$$

which is the norm used in this paper for the matrix-valued function spaces $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$.

For $\hbar \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, its integration $\int_{\mathbb{R}} \hbar(t)dt$ is defined as $\int_{\mathbb{R}} \hbar(t)dt := (\int_{\mathbb{R}} h_{k,l}(t)\,dt\,)_{k,l=1}^{s}$, where $\hbar(t)$ is the matrix-valued functions $(\,h_{k,\,l}(t)\,)_{k,\,l=1}^{s}$ to be defined below. The Fourier transform of $\hbar(t)$ is defined by $\widehat{\hbar}(\omega) := \int_{\mathbb{R}} \hbar(t) \exp\{-i\omega t\}\,dt,\ \omega \in \mathbb{R}$.

For two matrix-valued functions $\hbar,\ \Upsilon \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, their *symbol inner product* is defined by $[\,\hbar, \Upsilon\,] := \int_{\mathbb{R}} \hbar(t)\Upsilon(t)^*\,dt$. Here and afterwards, * means the transpose and the complex conjugate.

Definition 1.1. A sequence $\{\hbar_k(t)\}_{k\in\mathbb{Z}} \subset \mathbf{X} \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called an orthonormal set in $\mathbf{X}$, if it satisfies

$$[\,\hbar_k, \hbar_l\,] = \delta_{k,\,l}\mathbf{I}_s, \qquad k,\, l \in \mathbb{Z} \tag{2}$$

where $\delta_{k,\,l}$ is the Kronecker symbol, i.e., $\delta_{k,\,l} = 1$ as $k = l$ and $\delta_{k,\,l} = 0$ otherwise.

Definition 1.2. A matrix-valued function $\hbar(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is said to be orthonormal, if $\{\hbar(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal set.

Definition 1.3. A sequence of matrix-valued functions $\{\hbar_k(t)\}_{k\in\mathbb{Z}} \subset \mathbf{X} \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called an orthonormal basis of $\mathbf{X}$ if it satisfies (2) and for any $\Upsilon(t) \in \mathbf{X}$, there exists a unique matrix sequence $\{P_k\}_{k\in\mathbb{Z}}$ such that $\Upsilon(t) = \sum_{k\in\mathbb{Z}} P_k \hbar_k(t),\ \ t \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we briefly recall the concepts relevant to the matrix-valued multiresolution analysis. In Section 3, we give our main result, and some properties of the matrix-valued wavelet packets.

2 Matrix-valued multiresolution analysis and wavelets

We begin with the generic setting of a matrix-valued multiresolution analysis of $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$. Let $\mathbf{S}\,(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ satisfy the following refinement equation:

$$\mathbf{S}\,(t) = 3 \cdot \sum_{k\in\mathbb{Z}} A_k\, \mathbf{S}(3\,t - k), \tag{3}$$

where $\{A_k\}_{k\in\mathbb{Z}}$ is a finitely supported sequence of $s\times s$ constant matrix.

Define a closed subspace $\mathbf{V}_j \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ by

$$\mathbf{V}_j = \mathbf{clos}_{L^2(\mathbb{R},\mathbb{C}^{s\times s})}\langle \mathbf{S}(3^j\ \cdot -k) : k\in\mathbb{Z}\rangle,\ j\in\mathbb{Z}. \tag{4}$$

Definition 2.1. We say that $\mathbf{S}\,(t)$ in (3) generates a matrix-valued multiresolution analysis $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, if the sequence $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ defined by (4) satisfies:

(1). $\cdots \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \cdots$;

(2). $\bigcap_{j\in\mathbb{Z}} \mathbf{V}_j = \{\mathbf{O}\}$; $\bigcup_{\mathbf{j}\in\mathbb{Z}} \mathbf{V_j}$ is dense in $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$;

(3). $\hbar(\cdot) \in \mathbf{V}_0 \Longleftrightarrow \hbar(3^j\cdot) \in \mathbf{V}_j,\ \forall j\in\mathbb{Z}$;

(4). $\exists\ \mathbf{S}\,(t) \in \mathbf{V}_0$ such that $\mathbf{S}_k(t) := \mathbf{S}(t-k), k\in\mathbb{Z}$, form an orthonarmal basis for $\mathbf{V}_0$.

A matrix-valued functions $\mathbf{S}(t)$ in (3) is said to be a matrix-valued scaling function if it generates a matrix-valued multiresolution analysis. Equation (3) is called a refinement equation. Set $\mathcal{A}(\omega) = \sum_{k\in\mathbb{Z}} A_k \cdot \exp\{-ik\omega\},\ \ \omega\in\mathbb{R}$. Then, the frequency form of (3) is

$$\widehat{\mathbf{S}}(\omega) = \mathcal{A}(\omega/3)\,\widehat{\mathbf{S}}(\omega/3), \quad \omega\in\mathbb{R}. \tag{5}$$

In the following, without loss of generality we assume $\widehat{\mathbf{S}}(\omega)$ is continuous at the origin and $\widehat{\mathbf{S}}(0) = \mathbf{I}_s$.

Let $\mathbf{U}_j$, $j\in\mathbb{Z}$ be the orthocomplement space of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$. Assume there exist two matrix-valued functions $W_1(t),\ W_2(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, such that their translates and dilates form a *Riesz* basis of $\mathbf{U}_j$, i.e.,

$$\mathbf{U}_j = \mathbf{clos}_{L^2(\mathbb{R},\mathbb{C}^{s\times s})}\langle W_\imath(3^j\ \cdot -k) : \imath = 1,2,\ \ k\in\mathbb{Z}\rangle, \quad j\in\mathbb{Z}. \tag{6}$$

Since $W_1(t), W_2(t) \in \mathbf{U}_0 \subset \mathbf{V}_1$, there exist two finitely supported sequences of $s\times s$ matrix $\{B_k^{(\imath)}\}_{k\in\mathbb{Z}},\ \imath = 1,2$ such that $W_\imath(t) = 3\sum_{k\in\mathbb{Z}} B_k^{(\imath)}\mathbf{S}\,(3t-k)$. Taking Fourier transform for (6) gives

$$\widehat{W_\imath}(\omega) = \mathcal{B}^{(\imath)}(\omega/3)\,\widehat{\mathbf{S}}(\omega/3),\ \imath = 1,2, \quad \omega\in\mathbb{R}, \tag{7}$$

where

$$\mathcal{B}^{(\imath)}(\omega) = \sum_{k\in\mathbb{Z}} B_k \cdot \exp\{-ik\omega\}, \quad \imath = 1,2. \tag{8}$$

We call $\mathbf{S}(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ an orthonormal matrix-valued scaling function if it is a scaling function and satisfies

$$[\,\mathbf{S}(\cdot),\, \mathbf{S}(\cdot - n)\,] = \delta_{0,\,n}\mathbf{I}_s, \quad n\in\mathbb{Z}. \tag{9}$$

We say that $W_1(t),\ W_2(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ are two orthonormal matrix-valued wavelet functions associated with an orthonormal matrix-valued scaling functions if it satisfies

$$[\,\mathbf{S}(\cdot),\, W_\imath(\cdot - n)\,] = \mathbf{O}, \qquad \imath = 1,2, \qquad n\in\mathbb{Z}; \tag{10}$$

$$[\,W_\imath(\cdot),\, W_\jmath(\cdot - n)\,] = \delta_{\imath,\,\jmath}\,\delta_{0,\,n}\mathbf{I}_s,\ \ \imath,\,\jmath \in \{1,2\},\ \ n\in\mathbb{Z}. \tag{11}$$

Lemma 2.1. *Let $\hbar(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$. Then $\hbar(t)$ is orthonormal if and only if*

$$\sum_{k\in\mathbb{Z}} \widehat{\hbar}(\omega + 2k\pi)\widehat{\hbar}(\omega + 2k\pi)^* = \mathbf{I}_s. \tag{12}$$

Proof If $\hbar(t)$ is an orthonormal matrix-valued functions, then we get from (2) that

$$\begin{aligned}\delta_{0,k}\mathbf{I}_s &= [\,\hbar(\cdot),\hbar(\cdot-k)\,] = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\hbar}(\omega)\widehat{\hbar}(\omega)^*\cdot\exp\{ik\omega\}d\omega\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{l\in\mathbb{Z}}\widehat{\hbar}(\omega+2l\pi)\widehat{\hbar}(\omega+2l\pi)^*\cdot\exp\{ik\omega\}d\omega,\end{aligned}$$

which implies that (12) holds. The converse is obvious. ■

By Lemma 2.1 and (5), (7), (9)-(11), we can obtain the following lemma.

Lemma 2.2. ([8]) *Let* $\mathbf{S}(t)\in L^2(\mathbb{R},\mathbb{C}^{s\times s})$ *be an orthonormal matrix-valued scaling function. Assume* $W_1(t)$, $W_2(t)\in L^2(\mathbb{R},\mathbb{C}^{s\times s})$ *are orthogonal matrix-valued wavelet functions associated with* $\mathbf{S}(t)$*. Then we have*

$$\mathcal{A}(\omega)\,\mathcal{A}(\omega)^*+\mathcal{A}(\omega_1)\,\mathcal{A}(\omega_1)^*+\mathcal{A}(\omega_2)\,\mathcal{A}(\omega_2)^*=\mathbf{I}_s,\quad \omega\in\mathbb{R}, \tag{13}$$

$$\mathcal{A}(\omega)\,\mathcal{B}^{(\imath)}(\omega)^*+\mathcal{A}(\omega_1)\,\mathcal{B}^{(\imath)}(\omega_1)^*+\mathcal{A}(\omega_2)\,\mathcal{B}^{(\imath)}(\omega_2)^*=\mathbf{O},\quad \imath=1,2,\ \ \omega\in\mathbb{R}, \tag{14}$$

$$\mathcal{B}^{(\imath)}(\omega)\,\mathcal{B}^{(\jmath)}(\omega)^*+\mathcal{B}^{(\imath)}(\omega_1)\,\mathcal{B}^{(\jmath)}(\omega_1)^*+\mathcal{B}^{(\imath)}(\omega_2)\,\mathcal{B}^{(\jmath)}(\omega_2)^*=\delta_{\imath,\jmath}\mathbf{I}_s,\quad \imath,\,\jmath\in\{1,2\}, \tag{15}$$

where $\omega_1=\omega+2\pi/3$ *and* $\omega_2=\omega+4\pi/3$.

We now present matrix-valued Meyer wavelets as a special family of the matrix-valued wavelets. For more about scalar-valued Meyer wavelets, see [10]. Let

$$\widehat{\mathbf{S}}(\omega)=\begin{cases}\mathbf{I}_s, & |\omega|<\dfrac{2\pi}{3},\\ \cos\left[\dfrac{\pi}{2}f\left(\dfrac{3}{2\pi}\right)|\omega|-1\right]\Gamma(\omega), & \dfrac{2\pi}{3}\le|\omega|\le\dfrac{4\pi}{3},\\ 0, & \text{otherwise},\end{cases} \tag{16}$$

where $\Gamma(\omega)$ is paraunitary and $\Gamma(2\pi/3)=\Gamma(-2\pi/3)=\mathbf{I}_s$, and $f(t)$ is a scalar-valued smooth function such that

$$f(t)=\begin{cases}1, & t\ge1,\\ 0, & t\le0,\end{cases}\qquad \text{and}\ \ f(t)+f(1-t)=1,\quad \text{for} t\in(0,\,1).$$

Then, after some computation, for $\omega\in\mathbb{R}$, we get that $\sum_{k\in\mathbb{Z}}\widehat{\mathbf{S}}(\omega+2k\pi)\widehat{\mathbf{S}}(\omega+2k\pi)^*=\mathbf{I}_s$.

By Lemma 2.1, $\mathbf{S}(t)$ is an orthonormal matrix-valued scaling function. This implies that $\mathbf{S}(t)$ defined by (16) is a matrix-valued scaling function. Similar to the scalar-valued Meyer wavelets ([10, p. 138]), the corresponding lowpass filter $\mathcal{A}(\omega)$ is $\mathcal{A}(\omega)=\sum_{k\in\mathbb{Z}}\widehat{\mathbf{S}}(2(\omega+2k\pi))$.

By using paraunitary vector filter theory [11], we can obtain two filter functions $\mathcal{B}^{(1)}(\omega)$ and $\mathcal{B}^{(2)}(\omega)$ satisfying (14) and (15). Let $\widehat{W_\imath}(\omega)=\mathcal{B}^{(\imath)}(\omega/3)\,\widehat{\mathbf{S}}(\omega/3)$, $\imath=1,2$. Then, $W_1(t)$ and $W_2(t)$ are two matrix-valued Meyer wavelets [8].

3 Orthogonal matrix-valued wavelet packets

Xia and Suter [8] introduced the notion of matrix-valued wavelets and investigated their construction. In this section, we will give the definition of the matrix-valued wavelet packets and discuss some of their properties. First, we set

$$\mathbf{\Psi}_0(t)=\mathbf{S}(t),\quad \mathbf{\Psi}_\imath(t)=W_\imath(t);\quad \Omega_k^{(0)}=A_k,\quad \Omega_k^{(\imath)}=B_k^{(\imath)},\ \imath=1,2,\quad k\in\mathbb{Z}.$$

Definition 3.1. The collection of the matrix-valued functions $\{\boldsymbol{\Psi}_{3n+\lambda}(t),\ n=0,1,\cdots,\lambda=0,1,2\}$ is called a matrix-valued wavelet packet with respect to the orthogonal matrix-valued scaling function $\mathbf{S}(t)$, where

$$\boldsymbol{\Psi}_{3n+\lambda}(t)=3\cdot\sum_{k\in\mathbb{Z}}\Omega_k^{(\lambda)}\,\boldsymbol{\Psi}_n(3t-k),\quad \lambda=0,1,2. \tag{17}$$

By implementing the Fourier transform for both sides of (17), we have

$$\widehat{\boldsymbol{\Psi}}_{3n+\lambda}(\omega)=\boldsymbol{\Omega}^{(\lambda)}(\omega/3)\,\widehat{\boldsymbol{\Psi}}_n(\omega/3),\quad \lambda=0,1,2, \tag{18}$$

where

$$\boldsymbol{\Omega}^{(\lambda)}(\omega)=\sum_{k\in\mathbb{Z}}\Omega_k^{(\lambda)}\cdot\exp\{-ik\omega\},\quad \lambda=0,1,2,\quad \omega\in\mathbb{R}. \tag{19}$$

Thus, $\boldsymbol{\Omega}^{(0)}(\omega)=\mathcal{A}(\omega),\ \boldsymbol{\Omega}^{(\imath)}(\omega)=\mathcal{B}^{(\imath)}(\omega),\ \imath=1,2$. Formulas (13)-(15) can be written as

$$\sum_{\sigma=0}^{2}\boldsymbol{\Omega}^{(\lambda)}\left(\omega+\frac{2\pi\sigma}{3}\right)\boldsymbol{\Omega}^{(\mu)}\left(\omega+\frac{2\pi\sigma}{3}\right)^*=\delta_{\lambda,\mu}\mathbf{I}_s,\quad \lambda,\mu\in\{0,1,2\},\quad \omega\in\mathbb{R}. \tag{20}$$

It is evident that (20) is equivalent to

$$\sum_{\sigma\in\mathbb{Z}}\Omega_{\sigma+3k}^{(\lambda)}\,(\Omega_{\sigma+3l}^{(\mu)})^*=\frac{1}{3}\delta_{\lambda,\mu}\delta_{k,l}\mathbf{I}_s,\ \lambda,\quad \mu=0,\,1,\,2,\quad k,\,l\in\mathbb{Z}. \tag{21}$$

In the following, we will investigate the properties of the matrix-valued wavelet packets.

Theorem 3.1. *If $\{\boldsymbol{\Psi}_n(t)\}$ is a matrix-valued wavelet packets with respect to the orthogonal matrix-valued scaling function $\mathbf{S}(t)$, then for every $n\in\mathbb{Z}_+$, we have*

$$[\,\boldsymbol{\Psi}_n(\cdot-j\,),\,\boldsymbol{\Psi}_n(\cdot-k)\,]\;=\;\delta_{j,k}\,\mathbf{I}_s,\quad j,\,k\in\mathbb{Z}. \tag{22}$$

Proof (Induction) (i) The result (22) follows from (9) as $n=0$. (ii) Assume that (22) holds when $0\le n<3^{\mathcal{L}}$, where $\mathcal{L}$ is a positive integer. Then, as $3^{\mathcal{L}}\le n<3^{\mathcal{L}+1}$, we have $3^{\mathcal{L}-1}\le[n/3]<3^{\mathcal{L}}$ where $[\rho]=\max\{\nu\in\mathbb{Z},\,\nu\le\rho\}$. Thus, order $n=3[n/3]+\lambda,\ \lambda=0,1,2$. By the induction assumption and Lemma 2.1, we obtain

$$[\,\boldsymbol{\Psi}_{[\frac{n}{3}]}(\cdot-j),\,\boldsymbol{\Psi}_{[\frac{n}{3}]}(\cdot-k)\,]\;=\;\delta_{j,k}\mathbf{I}_s\Longleftrightarrow\sum_{l\in\mathbb{Z}}\widehat{\boldsymbol{\Psi}}_{[\frac{n}{3}]}(\omega+2l\pi)\widehat{\boldsymbol{\Psi}}_{[\frac{n}{3}]}(\omega+2l\pi)^*=\mathbf{I}_s. \tag{23}$$

It follows from (18), (20) and (23) that

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}}\widehat{\boldsymbol{\Psi}}_n(\,\omega\,+\,2l\pi\,)\,\widehat{\boldsymbol{\Psi}}_n(\,\omega\,+\,2l\pi\,)^*\\
&=\sum_{l\in\mathbb{Z}}\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2l\pi}{3}\right)\widehat{\boldsymbol{\Psi}}_{[\frac{n}{3}]}\left(\frac{\omega+2l\pi}{3}\right)\widehat{\boldsymbol{\Psi}}_{[\frac{n}{3}]}\left(\frac{\omega+2l\pi}{3}\right)^*\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2l\pi}{3}\right)^*\\
&=\sum_{\sigma=0}^{2}\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2\sigma\pi}{3}\right)\left\{\sum_{\kappa\in\mathbb{Z}}\widehat{\boldsymbol{\Psi}}_{[\frac{n}{2}]}\left(\frac{\omega+2\sigma\pi}{3}+2\kappa\pi\right)\widehat{\boldsymbol{\Psi}}_{[\frac{n}{2}]}\left(\frac{\omega+2\sigma\pi}{3}+2\kappa\pi\right)^*\right\}\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2\sigma\pi}{3}\right)^*\\
&=\sum_{\sigma=0}^{2}\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2\sigma\pi}{3}\right)\boldsymbol{\Omega}^{(\lambda)}\left(\frac{\omega+2\sigma\pi}{3}\right)^*=\mathbf{I}_s
\end{aligned}$$

Therefore, by Lemma 2.1, the result (22) follows. ■

Theorem 3.2. *If $\{\mathbf{\Psi}_n(t)\}$ is a matrix-valued wavelet packets with respect to the orthogonal matrix-valued scaling function $\mathbf{S}(t)$, then for every $n \in \mathbb{Z}_+$, we have*

$$[\,\mathbf{\Psi}_{3n+\lambda}(\cdot)\,,\,\mathbf{\Psi}_{3n+\mu}(\cdot-k)\,] = \delta_{\lambda,\,\mu}\,\delta_{0,\,k}\,\mathbf{I}_s,\quad \lambda,\,\mu\in\{0,1,2\},\;\; k\in\mathbb{Z}. \tag{24}$$

Proof By (18) and (21) and Theorem 3.1, we obtain

$$\begin{aligned}
&[\,\mathbf{\Psi}_{3n+\lambda}(\,\cdot\,)\;,\;\mathbf{\Psi}_{3n+\mu}(\,\cdot\,-\,k\,)\,] = \frac{1}{2\pi}\int_{\mathbb{R}} \mathbf{\Omega}^{(\lambda)}\left(\frac{\omega}{3}\right)\widehat{\mathbf{\Psi}}_n\left(\frac{\omega}{3}\right)\,\widehat{\mathbf{\Psi}}_n\left(\frac{\omega}{3}\right)^{*}\mathbf{\Omega}^{(\mu)}\left(\frac{\omega}{3}\right)^{*}\cdot e^{ik\omega}\,d\omega\\
&= \frac{1}{2\pi}\int_0^{6\pi}\mathbf{\Omega}^{(\lambda)}\left(\frac{\omega}{3}\right)\left\{\sum_{l\in\mathbb{Z}}\widehat{\mathbf{\Psi}}_n\left(\frac{\omega}{3}+2l\pi\right)\,\widehat{\mathbf{\Psi}}_n\left(\frac{\omega}{3}+2l\pi\right)^{*}\right\}\mathbf{\Omega}^{(\mu)}\left(\frac{\omega}{3}\right)^{*}\cdot\exp\{ik\omega\}d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{\sigma=0}^{2}\mathbf{\Omega}^{(\lambda)}\left(\frac{\omega+2\pi\sigma}{3}\right)\,\mathbf{\Omega}^{(\mu)}\left(\frac{\omega+2\pi\sigma}{3}\right)^{*}\exp\{ik\omega\}d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\delta_{\lambda,\mu}\mathbf{I}_s\cdot\,\exp\{ik\omega\}\;d\omega = \delta_{\lambda,\mu}\delta_{0,k}\mathbf{I}_s.
\end{aligned}$$

This completes the proof of this theorem. ■

Theorem 3.3. *For any m, $n\in\mathbb{Z}_+$ and $k\in\mathbb{Z}$, we have*

$$[\,\mathbf{\Psi}_m(\cdot)\,,\,\mathbf{\Psi}_n(\cdot-k)\,] = \delta_{m,\,n}\,\delta_{0,\,k}\mathbf{I}_s. \tag{25}$$

Proof For $m=n$, (25) follows by Theorem 3.1. Without loss of generality, we suppose $m>n$ in case of $m\neq n$. Rewrite m, n as $m=3[m/3]+\lambda_1,\; n=3/[n/3]+\mu_1$, where $\lambda_1,\,\mu_1\in\{0,\,1,\,2\}$.

Case 1. If $[m/3]=[n/3]$, then $\lambda_1\neq\mu_1$. By (18), (20) and (23),

$$\begin{aligned}
&[\,\mathbf{\Psi}_m\,(\cdot)\,,\,\mathbf{\Psi}_n\,(\cdot-k)\,] = \frac{1}{2\pi}\int_{\mathbb{R}}\mathbf{\Omega}^{(\lambda_1)}\left(\frac{\omega}{3}\right)\widehat{\mathbf{\Psi}}_{[\frac{m}{3}]}\left(\frac{\omega}{3}\right)\,\widehat{\mathbf{\Psi}}_{[\frac{n}{3}]}\left(\frac{\omega}{3}\right)^{*}\mathbf{\Omega}^{(\mu_1)}\left(\frac{\omega}{3}\right)^{*}\cdot\exp\{ik\omega\}\,d\omega\\
&= \frac{3}{2\pi}\int_0^{2\pi}\mathbf{\Omega}^{(\lambda_1)}(\omega)\left\{\sum_{l\in\mathbb{Z}}\widehat{\mathbf{\Psi}}_{[\frac{n}{3}]}(\omega+2l\pi)\,\widehat{\mathbf{\Psi}}_{[\frac{n}{3}]}(\omega+2l\pi)^{*}\right\}\,\mathbf{\Omega}^{(\mu_1)}(\omega)^{*}\cdot\exp\{3ik\omega\}\,d\omega\\
&= \frac{3}{2\pi}\int_0^{\frac{2\pi}{3}}\sum_{\sigma=0}^{2}\mathbf{\Omega}^{(\lambda_1)}\left(\omega+\frac{2\pi\sigma}{3}\right)\,\mathbf{\Omega}^{(\mu_1)}\left(\omega+\frac{2\pi\sigma}{3}\right)\cdot\exp\{3ik\omega\}\,d\omega\\
&= \frac{3}{2\pi}\int_0^{\frac{2\pi}{3}}\delta_{\lambda_1,\,\mu_1}\mathbf{I}_s\cdot\exp\{3ik\omega\}\,d\omega = \mathbf{O},
\end{aligned}$$

which implies that (25) holds in this case.

Case 2. If $[\frac{m}{3}]\neq[\frac{n}{3}]$, then set $[m/3]=3[\,[m/3]/3]+\lambda_2,\; [n/3]=3[\,[n/3]/3]+\mu_2,\;\; \lambda_2,\,\mu_2\in\{0,1,2\}$. If $[\,[m/3]/3]=[\,[n/3]/3]$, then (25) can be established similar to Case 1. If $[\,[m/3]/3]\neq[\,[n/3]/3]$, then we again set $[\,[m/3]/3]=3[\,[\,[m/3]/3]/3]+\lambda_3,\; [\,[n/3]/3]=3[\,[\,[n/3]/3]/3]+\mu_3,\;\; \lambda_3,\,\mu_3\in\{0,1,2\}$. Thus, after taking finite times steps (denoted by κ), we obtain

$$a_\kappa=b_\kappa=1,\quad\text{or}\quad a_\kappa=b_\kappa=2, \tag{26}$$

where

$$a_\kappa=\overbrace{[\,[\cdots[}^{\kappa}m/2]\cdots]/2],\quad b_\kappa=\overbrace{[\,[\cdots[}^{\kappa}n/2]\cdots]/2].$$

$$a_\kappa = 1,\ b_\kappa = 0, \quad \text{or} \quad a_\kappa = 2,\ b_\kappa = 1, \quad \text{or} \quad a_\kappa = 2,\ b_\kappa = 0, \quad \lambda_\kappa,\, \mu_\kappa \in \{0,1,2\}. \tag{27}$$

For the case (26), the result (25) follows similarly to Case 1. For the case (27), we have from (10) and (11) that

$$\sum_{l\in\mathbb{Z}} \widehat{\mathbf{\Psi}}_{a_\kappa}(\omega+2l\pi)\,\widehat{\mathbf{\Psi}}_{b_\kappa}(\omega+2l\pi)^* = \mathbf{O}, \qquad \omega\in\mathbb{R}.$$

Consequently,

$$\begin{aligned}
&[\,\mathbf{\Psi}_m(\cdot)\,,\,\mathbf{\Psi}_n(\cdot-k)\,] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\mathbf{\Psi}}_m(\omega)\ \widehat{\mathbf{\Psi}}_n(\omega)^*\cdot\exp\{ik\omega\}\,d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathbf{\Omega}^{(\lambda_1)}\left(\frac{\omega}{3}\right)\ \widehat{\mathbf{\Psi}}_{[m/3]}\left(\frac{\omega}{3}\right)\ \widehat{\mathbf{\Psi}}_{[n/3]}\left(\frac{\omega}{3}\right)^*\ \mathbf{\Omega}^{(\mu_1)}\left(\frac{\omega}{3}\right)^*\cdot\exp\{ik\omega\}\,d\omega = \cdots\cdots \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\lambda_\sigma)}\left(\frac{\omega}{3^\sigma}\right)\ \widehat{\mathbf{\Psi}}_{a_\kappa}\left(\frac{\omega}{3^\kappa}\right)\ \widehat{\mathbf{\Psi}}_{b_\kappa}\left(\frac{\omega}{3^\kappa}\right)^*\ \left(\prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\mu_\sigma)}\left(\frac{\omega}{3^\sigma}\right)\right)^*\cdot\exp\{ik\omega\}\,d\omega \\
&= \frac{1}{2\pi}\int_0^{3^{\kappa+1}\pi} \prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\lambda_\sigma)}\left(\frac{\omega}{3^\sigma}\right)\left(\sum_{l\in\mathbb{Z}}\widehat{\mathbf{\Psi}}_{a_\kappa}\left(\frac{\omega}{3^\kappa}+2l\pi\right)\ \widehat{\mathbf{\Psi}}_{b_\kappa}\left(\frac{\omega}{3^\kappa}+2l\pi\right)^*\right) \\
&\qquad \cdot\left(\prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\mu_\sigma)}\left(\frac{\omega}{3^\sigma}\right)\right)^*\cdot e^{ik\omega}d\omega \\
&= \frac{1}{2\pi}\int_0^{3^{\kappa+1}\pi} \prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\lambda_\sigma)}\left(\frac{\omega}{2^\sigma}\right)\cdot\mathbf{O}\cdot\left(\prod_{\sigma=1}^{\kappa}\mathbf{\Omega}^{(\mu_\sigma)}\left(\frac{\omega}{3^\sigma}\right)\right)^*\cdot\exp\{ik\omega\}\,d\omega = \mathbf{O}.
\end{aligned}$$

Therefore, for any $m,\, n\in\mathbb{Z}_+$ and $k\in\mathbb{Z}$, (25) holds. ■

Lemma 3.1. *If $\{\mathbf{\Psi}_n(t),\ n=0,1,2,\cdots\}$ is a matrix-valued wavelet packets with respect to the orthonormal matrix-valued scaling functions $\mathbf{S}(t)$, then for every $n\in\mathbb{Z}+$, we have*

$$\mathbf{\Psi}_n(3t-k) = \frac{1}{3}\sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}}(\Omega_{k-3l}^{(\sigma)})^*\mathbf{\Psi}_{3n+\sigma}(t-l),\ \ k\in\mathbb{Z}. \tag{28}$$

Proof Observe

$$\begin{aligned}
&\frac{1}{3}\sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}}(\Omega_{k-3l}^{(\sigma)})^*\mathbf{\Psi}_{3n+\sigma}(t-l) = \sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}}(\Omega_{k-3l}^{(\sigma)})^*\sum_{j\in\mathbb{Z}}\Omega_j^{(\sigma)}\,\mathbf{\Psi}_n(3t-3l-j) \\
&= \sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}(\Omega_{k-3l}^{(\sigma)})^*\ \Omega_{m-3l}^{(\sigma)}\,\mathbf{\Psi}_n(3t-m) = \sum_{m\in\mathbb{Z}}\left\{\sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}}(\Omega_{k-3l}^{(\sigma)})^*\ \Omega_{m-3l}^{(\sigma)}\right\}\mathbf{\Psi}_n(3t-m) \\
&= \sum_{m\in\mathbb{Z}}\delta_{k,\,m}I_s\,\mathbf{\Psi}_n(3t-m) = \mathbf{\Psi}_n(3t-k).
\end{aligned}$$

This completes the proof of Lemma 3.1. ■

We shall discuss the orthogonal decomposition relation for $L^2(\mathbb{R},\,\mathbb{C}^{s\times s})$. Let

$$\mathbf{Y}_j^n = \mathbf{clos}_{L^2(\mathbb{R},\,\mathbb{C}^{s\times s})}\langle\,\mathbf{\Psi}_n(3^j\ \cdot -k)\,:\,k\in\mathbb{Z}\rangle,\ \ n\in\mathbb{Z}_+,\ j\in\mathbb{Z}. \tag{29}$$

Theorem 3.4. *Let $n \in \mathbb{Z}_+$ and $\bigoplus$ denote orthogonal direct sum. We have*

$$\mathbf{Y}_{j+1}^{n} = \mathbf{Y}_{j}^{3n} \bigoplus \mathbf{Y}_{j}^{3n+1} \bigoplus \mathbf{Y}_{j}^{3n+2}, \ \ j \in \mathbb{Z}. \tag{30}$$

Proof According to (17) and (29), $\mathbf{Y}_{j}^{3n} \bigoplus \mathbf{Y}_{j}^{3n+1} \bigoplus \mathbf{Y}_{j}^{3n+2} \subset \mathbf{Y}_{j+1}^{n}$. On the other hand, $\mathbf{Y}_{j}^{3n}$, $\mathbf{Y}_{j}^{3n+1}$ and $\mathbf{Y}_{j}^{3n+2}$ are orthogonal to each other by Theorem 3.2. By Lemma 3.1, we have

$$\mathbf{\Psi}_n(3^{j+1}\, t - k) = \frac{1}{3}\sum_{\sigma=0}^{2}\sum_{l\in\mathbb{Z}} (\Omega_{k-3l}^{(\sigma)})^{*}\mathbf{\Psi}_{n+\sigma}(3^{j}\, t - l), \ \ j,\, k \in \mathbb{Z}.$$

Hence, the basis of the space $\mathbf{Y}_{j+1}^{n}$ can be linearly represented by the basis of the space $\mathbf{Y}_{j}^{3n}$, $\mathbf{Y}_{j}^{3n+1}$ and $\mathbf{Y}_{j}^{3n+2}$. Then, we have $\mathbf{Y}_{j+1}^{n} \subset \mathbf{Y}_{j}^{3n} \bigoplus \mathbf{Y}_{j}^{3n+1} \bigoplus \mathbf{Y}_{j}^{3n+2}$. This implies that (30) holds for every $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$. ■

Corollary 3.1. *For every $j \geq 1$ and $1 \leq k \leq j$, we have*

$$\mathbf{U}_j = \mathbf{Y}_{j-k}^{3^k} \bigoplus \mathbf{Y}_{j-k}^{3^k+1} \bigoplus \cdots \bigoplus \mathbf{Y}_{j-k}^{3^{k+1}-1}. \tag{31}$$

Moreover,

$$L^2(\mathbb{R},\, \mathbb{C}^{s\times s}) = \bigoplus_{j\in\mathbb{Z}} \mathbf{U}_j = \cdots \bigoplus \mathbf{U}_{-2} \bigoplus \mathbf{U}_{-1} \bigoplus \mathbf{U}_0 \bigoplus_{\kappa=3}^{\infty} \mathbf{Y}_0^{\kappa}. \tag{32}$$

Finally, the family of matrix-valued functions

$$\{\mathbf{\Psi}_1(3^j - k),\ \mathbf{\Psi}_n(\cdot - k) : \ j = \cdots, -2, -1, 0; \ \ n = 3, 4, \cdots,\ \ k \in \mathbb{Z}\}$$

is an orthogonal basis of $L^2(\mathbb{R},\, \mathbb{C}^{s\times s})$.

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