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CONVERGENCE OF THE CYCLIC REDUCTION ALGORITHM FOR A CLASS OF WEAKLY OVERDAMPED QUADRATICS*

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Abstract

In this paper, we establish a convergence result of the cyclic reduction (CR) algorithm for a class of weakly overdamped quadratic matrix polynomials without assumption that the partial multiplicities of the nth largest eigenvalue are all equal to 2. Our result can be regarded as a complement of that by Guo, Higham and Tisseur [SIAM J. Matrix Anal. Appl., 30 (2009), pp. 1593-1613]. The numerical example indicates that the convergence behavior of the CR algorithm is largely dictated by our theory.

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1. Introduction

The quadratic eigenvalue problems (QEPs) are to find scalars λ and nonzero vectors x and y satisfying $Q(\lambda)x = 0$ and $y^*Q(\lambda) = 0$, where

$$Q(\lambda) = \lambda^2 A + \lambda B + C \qquad \text{with} \quad A, B, C \in \mathbb{C}^{n\times n} \tag{1.1}$$

is a quadratic matrix polynomial (or the quadratic for the brevity). Vectors x and y are right and left eigenvectors corresponding to the eigenvalue λ. QEPs have extensive applications in practical engineering problems. We refer to [18] for a good review.

In this paper, we consider the overdamped QEP which belongs to a class of hyperbolic QEPs with the following definition [6].

Definition 1.1. *The quadratic $Q(\lambda)$ is called hyperbolic if A, B, C are all Hermitian, A is positive definite, and*

$$(x^*Bx)^2 > 4(x^*Ax)(x^*Cx) \quad \textit{for all nonzero} \quad x \in \mathbb{C}^n.$$

A hyperbolic QEP can be transformed into an overdamped one [11], so there is no loss of generality to consider only overdamped problems.

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Definition 1.2. *The quadratic $Q(\lambda)$ is called overdamped if it is hyperbolic with positive definite B and positive semidefinite C.*

It is known that an overdamped quadratic has $2n$ real, nonpositive and semisimple eigenvalues that can be ordered $0 \geq \lambda_1 \geq \cdots \geq \lambda_n > \lambda_{n+1} \geq \cdots \geq \lambda_{2n}$ [18]. When the nth largest and the nth smallest eigenvalues coalesce (i.e. $\lambda_n = \lambda_{n+1}$), the quadratic is called *weakly overdamped* (WO) in the terminology of Markus [16] (see also [8, Sec. 5]). The following lemma collects some properties of a weakly overdamped quadratic [6,16].

Lemma 1.1. *Let $Q(\lambda)$ be a WO quadratic.*

(a) $Q(\lambda)$ has $2n$ real eigenvalues that can be ordered $0 \geq \lambda_1 \geq \cdots \geq \lambda_n = \lambda_{n+1} \geq \cdots \geq \lambda_{2n}$. The partial multiplicities[1)] of λ_n are at most 2, and the eigenvalues other than λ_n are semisimple.

(b) The associated quadratic matrix equation (QME)

$$Q(S) = AS^2 + BS + C = 0 \tag{1.2}$$

admits two extremal solutions (also called solvents in the matrix polynomial theory) $S^{(1)}$ (the primary) and $S^{(2)}$ (the secondary), whose eigenvalues are the n largest and the n smallest roots of $Q(\lambda)$, respectively.

Recently, Guo, Higham and Tisseur [6] devised an efficient algorithm to detect and solve the overdamped QEPs. This algorithm was based on the cyclic reduction (CR) (stated in Section 3) with quadratic convergence. They also showed that, for WO QEPs, the convergence of the CR algorithm became linear with a constant at worst 1/2. We note that their analysis in that case needs the requirement that the partial multiplicities of the nth largest eigenvalue (i.e. λ_n) are all equal to 2. However, we know from Lemma 1.1 that the partial multiplicities of λ_n are at most 2. So one may wonder: (i) Are there any WO quadratics with the partial multiplicities of λ_n containing both 1 and 2? (ii) If such WO quadratics exist, what is the behavior of the CR algorithm for them?

The purpose of this paper is to investigate the above two issues. We give an example in the Section 3 to show that it does exist the WO quadratic with the partial multiplicities of λ_n containing both 1 and 2. However, the convergence behavior of the CR algorithm for such a quadratic is some different with that in [6]. We also try to establish a convergence of the CR algorithm for general WO quadratics (with no assumption on the partial multiplicities of λ_n). Unfortunately, the attempt seems not easy since the structure of the corresponding eigenspace is indefinite. So we instead construct a canonical diagonal quadratic in which the partial multiplicities of λ_n include 1 and 2. Then we extend the diagonal quadratic to a class of (not all) isospectral[2)] WO quadratics. Since the structure of eigenspace for such extended quadratics can be made out, we can obtain the convergence theorem of the CR algorithm by another equivalent doubling algorithm (see [2, 6, 15, 22]). The derived theorem (unlike that in [6]) indicates that some matrix sequences generated by the CR algorithm no longer converge to the zero matrix if the partial multiplicities of λ_n contain 1. Therefore, our result can be seen as a complement of convergence for the CR algorithm.

1) The partial multiplicities of an eigenvalue of $Q(\lambda)$ are the sizes of the Jordan blocks in which it appears in a Jordan matrix of $Q(\lambda)$ [5].

2) The term "isospectral" is in the sense that the eigenvalues and all their partial multiplicities are common to isospectral matrix polynomial [13].

The rest of this paper is organized as follows. We give some preliminaries in the next section. We review the CR algorithm in Section 3 and present a WO quadratic to show some different convergence of the algorithm if the partial multiplicities of λ_n are not all 2. In Section 4, we describe the main convergence theorem of the CR algorithm for a class of WO quadratics without any assumption on the partial multiplicities of λ_n. Section 5 is dedicated to the proof of the main theorem and Section 6 is devoted to the validation of the obtained convergence theorem via numerical experiments. At last, we conclude the paper by discussion in Section 7.

Throughout this paper, the matrix inequality $M_1 \geq M_2 (M_1 > M_2)$ for Hermitian matrices M_1 and M_2 means that matrix $M_1 - M_2$ is positive semidefinite (definite). The notation $M \oplus N$ stands for $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$. $I_n := I$ and $0_n := 0$ denote the identity and zero matrices of order n, respectively. $0_k (1 < k < n)$ and $0_{m\times l}$ denote the zero matrices with the dimension $k \times k$ and $m \times l$, respectively.

2. Preliminaries

In this section, we introduce some required concepts from matrix polynomial theory (see, e.g., [5, Chap. 1] and [13]).

Definition 2.1. *Let $L(\lambda) = \lambda^\tau I_n + \sum_{j=0}^{\tau-1} \lambda^j A_j$ be a monic matrix polynomial of degree τ. The nonzero vectors ϕ_0 $\phi_1, \cdots, \phi_k$ determined by*

$$\sum_{p=0}^{i} \frac{1}{p!} L^{(p)}(\lambda_0)\phi_{i-p} = 0 \quad (k \geq i \geq 0) \tag{2.1}$$

are called a Jordan chain of length $k+1$ for $L(\lambda)$ corresponding to λ_0, where $L^{(p)}(\lambda)$ denotes the pth derivative of $L(\lambda)$ with respect to λ.

Definition 2.2. *A Jordan matrix denotes the block diagonal matrix with diagonal blocks being Jordan blocks. Given the Jordan matrix J of a quadratic matrix polynomial $Q(\lambda)$ in (1.1). Denote by X an $n\times 2n$ matrix whose columns are formed by nonzero eigenvectors associated with each diagonal entry of J. We call the pair (X, J) forms a Jordan pair if $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is nonsingular.*

Definition 2.3. *A Jordan triple of a quadratic matrix polynomial $Q(\lambda)$ in (1.1) is a set of matrices (X, J, Y) for which (X, J) is a Jordan pair, Y is a $2n \times n$ matrix, and the matrix equation*

$$\begin{bmatrix} X \\ XJ \end{bmatrix} Y = \begin{bmatrix} 0 \\ A^{-1} \end{bmatrix}$$

holds for the nonsingular leading coefficient matrix A in $Q(\lambda)$.

The Jordan triple has a close relation with coefficient matrices of $Q(\lambda)$ [13, Thm. 1].

Lemma 2.1. *Let (X, J, Y) be a Jordan triple of a quadratic matrix polynomial $Q(\lambda)$ in (1.1). Denote $\Gamma_i = XJ^iY$, $i = 1, 2, 3$. The coefficient matrices of $Q(\lambda)$ can be defined recursively:*

$$A = \Gamma_1^{-1}, \quad B = -A\Gamma_2 A, \quad C = -A\Gamma_3 A + B\Gamma_1 B.$$

Moreover, if C is nonsingular, it can also be formulated as $C = -(XJ^{-1}Y)^{-1} =: -\Gamma_{-1}^{-1}$.

3. The CR Algorithm for WO QEPs

The cyclic reduction is a very efficient algorithm for solving some nonlinear matrix equations (see [1,17]). Attractive properties of the CR algorithm include its quadratic convergence rate, low computational cost per iteration (compared with Newton's method [10,23]) and nice numerical reliability. The iteration scheme of the CR algorithm for WO QEPs is as follows [6]:

$$\begin{cases} S_0 = B,\ A_0 = A,\ B_0 = B,\ C_0 = C, \\ S_{k+1} = S_k - A_k B_k^{-1} C_k, \\ A_{k+1} = -A_k B_k^{-1} A_k, \\ B_{k+1} = B_k - A_k B_k^{-1} C_k - C_k B_k^{-1} A_k, \\ C_{k+1} = -C_k B_k^{-1} C_k. \end{cases} \tag{3.1}$$

Guo, Higham and Tisseur [6] showed that the matrix sequences in (3.1) satisfied

$$A_k < 0, \quad C_k \le 0, \quad B_k \ge \mu^{2^k} A_k + \mu^{-2^k} C_k, \quad k \ge 0 \tag{3.2}$$

with some positive real constant μ. So the iterative process (3.1) is well defined for WO QEPs. The convergence of the CR algorithm in this case is given in [6, Thm 4.6, Cor 4.7]).

Theorem 3.1. *Let $Q(\lambda)$ be weakly overdamped with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n = \lambda_{n+1} \ge \cdots \ge \lambda_{2n}$, and the partial multiplicities of λ_n be all equal to 2. Let $S^{(1)}$ and $S^{(2)}$ be the primary and secondary solvents of $Q(S) = 0$, respectively, and λ_n be a semisimple eigenvalue of $S^{(1)}$ and $S^{(2)}$. Then for any matrix norm $\|\cdot\|$, the iterates $\{B_k\}$ and $\{S_k\}$ generated by (3.1) satisfy*

$$\limsup_{k\to\infty} \sqrt[k]{\|S_k - \widetilde{S}\|} \le \frac{1}{2}, \quad \limsup_{k\to\infty} \sqrt[k]{\|B_k - \widetilde{B}\|} \le \frac{1}{2},$$

where $\widetilde{S} = -S^{(2)^*} A$ is nonsingular and $\widetilde{B} = A(S^{(1)} - S^{(2)}) \ge 0$ is singular. Moreover, $\{A_k\}$ and $\{C_k\}$ are both $O(2^{-k})$ with

$$\limsup_{k\to\infty} \sqrt[k]{\|A_k\|\|C_k\|} \le \frac{1}{4}.$$

Theorem 3.1 shows that the matrix sequences $\{S_k\}$, $\{A_k\}$, $\{B_k\}$ and $\{C_k\}$ are convergent. In particular, both $\{A_k\}$ and $\{C_k\}$ converge to the zero matrix linearly. We note that the assumption on the partial multiplicities of λ_n all equaling to 2 is necessary in the above theorem. On the other hand, when the partial multiplicities of λ_n are all equal to 1, an example given in [6, Sec. 4] shows that the matrices sequences defined in (3.1) converge quadratically. However, in this case, the limit of $\{A_k\}$ and $\{C_k\}$ may not be the zero matrix. So what interest us is the convergence of the CR algorithm when the partial multiplicities of λ_n include both 1 and 2. The next example shows that such quadratic has some different convergence behavior from Theorem 3.1.

Example 3.1. Consider the quadratic

$$Q(\lambda) = \lambda^2 A + \lambda B + C = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & \sigma+1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix},$$

where $\sigma > 1$ is a real constant. It is easy to see that $Q(\lambda)$ is weakly overdamped with eigenvalues $\{-1, -1, -1, -\sigma\}$ and the partial multiplicities of -1 are 1 and 2. It is also not difficult to see that $S^{(1)} = -I_2$ and $S^{(2)} = \text{diag}(-1, -\sigma)$ are the primary and secondary solvents of $Q(S) = 0$, respectively. By direct computation, we get from iterates (3.1)

$$
\begin{aligned}
A_k &= \begin{bmatrix} \frac{1}{2^k} & 0 \\ 0 & \frac{1}{\Pi_{i=0}^{k-1}(\sigma^{2^i}+1)} \end{bmatrix}, \quad B_k = \begin{bmatrix} \frac{1}{2^{k-1}} & 0 \\ 0 & \frac{\sigma^{2^k}+1}{\Pi_{i=0}^{k-1}(\sigma^{2^i}+1)} \end{bmatrix}, \\
C_k &= \begin{bmatrix} \frac{1}{2^k} & 0 \\ 0 & \frac{\sigma^{2^k}}{\Pi_{i=0}^{k-1}(\sigma^{2^i}+1)} \end{bmatrix}, \quad S_k = \begin{bmatrix} \frac{2^k+1}{2^k} & 0 \\ 0 & \sigma + \frac{1}{\Pi_{i=0}^{k-1}(\sigma^{2^i}+1)} \end{bmatrix}.
\end{aligned}
$$

It is clear that $\{S_k\}$ converges with constant $1/2$ (i.e. $\|S_{k+1} - S_k\|_1 = \frac{1}{2}\|S_k - S_{k-1}\|_1$ for $k \geq 1$). However, unlike Theorem 3.1, $\{C_k\}$ does not converge to the zero matrix if $\sigma > 1$.

The remainder of the paper is to investigate the convergence of the CR algorithm for WO QEPs without any assumption on the partial multiplicities of λ_n. An accustomed idea is to directly employ some canonical form (such as the Kronecker form in [2]) to the general quadratic $Q(\lambda)$ with the partial multiplicities of λ_n containing both 1 and 2. However, such a strategy seems ineffectual for our case as the structure of the corresponding eigenspace is indefinite. So in this paper, we only focus on deriving the convergence of the CR algorithm for a class of WO quadratics $\widetilde{Q}(\lambda)$ isospectral with $Q(\lambda)$.

4. The Convergence of the CR Algorithm for a Class of WO QEPs

In this section, we first construct a class of WO quadratics $\widetilde{Q}(\lambda)$ and then give the main convergence theorem of the CR algorithm for $\widetilde{Q}(\lambda)$.

Our construction of $\widetilde{Q}(\lambda)$ starts with the following diagonal quadratic $Q_d(\lambda)$ with $\lambda_n = \lambda_{n+1} = -1$ (otherwise, a technique of scaling can always be used to shift the spectrum such that $\lambda_n = \lambda_{n+1} = -1$ [6,8])

$$
\begin{aligned}
Q_d(\lambda) &= \lambda^2 I_n + \lambda\left[(\Sigma_1 + I_{m_1}) \oplus (\Sigma_2 + \Delta_1) \oplus (\Delta_2 + I_{m_2}) \oplus 2I_r\right] \\
&\quad + \left[\Sigma_1 \oplus \Sigma_2\Delta_1 \oplus \Delta_2 \oplus I_r\right] \\
&=: \lambda^2 I_n + \lambda\widehat{B} + \widehat{C},
\end{aligned} \tag{4.1}
$$

where Σ_1 (Σ_2) is an $m_1 \times m_1$ ($l \times l$) diagonal matrix with its diagonal elements σ_t (σ_i) greater than 1, $m_1 \geq t \geq 0$ ($m_1 + l \geq i \geq m_1 + 1$), Δ_2 (Δ_1) is an $m_2 \times m_2$ ($l \times l$) diagonal matrix with its diagonal elements δ_s (δ_j) less than 1, $m_2 \geq s \geq 0$ ($m_2 + l \geq j \geq m_2 + 1$)), I_r is an identity matrix of order r, and $n = m_1 + m_2 + l + r$. To remove the assumption on the partial multiplicity of λ_n, we suppose that m_1 and m_2 do not equal zero simultaneously and $r \geq 1$.

It is obvious that the eigenvalues of $Q_d(\lambda)$ form a $2n \times 2n$ matrix

$$
J = \left[-\Sigma_1 \oplus -\Sigma_2 \oplus -I_{m_2} \oplus J_1 \oplus \cdots \oplus J_r \oplus -I_{m_1} \oplus -\Delta_1 \oplus -\Delta_2\right],
$$

where J_i ($r \geq i \geq 1$) are 2×2 Jordan blocks associated with the eigenvalue -1.

The *Jordan chains* (see Definition 2.1) of $Q_d(\lambda)$ form an $n \times 2n$ matrix

$$
\begin{aligned}
X_d = [&\phi^{(-\sigma_1)}, \cdots, \phi^{(-\sigma_{m_1+l})}, \phi_1^{(-1)}, \cdots, \phi_{m_2}^{(-1)}, \phi_{10}^{(-1)}, \phi_{11}^{(-1)}, \\
&\cdots, \phi_{r0}^{(-1)}, \phi_{r1}^{(-1)}, \hat{\phi}_1^{(-1)}, \cdots, \hat{\phi}_{m_1}^{(-1)}, \phi^{(-\delta_1)}, \cdots, \phi^{(-\delta_{m_2+l})}],
\end{aligned} \tag{4.2}
$$

where $\{\phi^{(-\sigma_t)}\}_{t=1}^{m_1+l}$ ($\{\phi^{(-\delta_s)}\}_{s=1}^{m_2+l}$) are eigenvectors corresponding to the eigenvalues in $-\Sigma_1$ and $-\Sigma_2$ ($-\Delta_1$ and $-\Delta_2$), $\{\phi_i^{(-1)}\}_{i=1}^{m_2}$, $\{\phi_{i0}^{(-1)}\}_{i=1}^{r}$ and $\{\hat{\phi}_i^{(-1)}\}_{i=1}^{m_1}$ are $m_1 + m_2 + r$ linearly independent eigenvectors corresponding to -1, $\{\phi_{i1}^{(-1)}\}_{i=1}^{r}$ are r generalized eigenvectors corresponding to -1. It follows from [5, Prop. 1.15] that the columns of X_d form a *canonical set of Jordan chains* (see [5, Chap. 1.6]), i.e, (X_d, J) is a Jordan pair of $Q_d(\lambda)$. Let (X_d, J, Y_d) be a Jordan triple of $Q_d(\lambda)$. By Lemma 2.1, the coefficient matrices can be formulated as

$$I_n = \Gamma_{d_1}^{-1}, \quad \widehat{B} = -\Gamma_{d_2}, \quad \widehat{C} = -\Gamma_{d_3} + \widehat{B}\Gamma_{d_1}\widehat{B} \tag{4.3}$$

with $\Gamma_{d_i} = X_d J^i Y_d, i = 1, 2, 3$.

To extend $Q_d(\lambda)$ to a class of isospectral WO quadratics, we keep J unchanged and give the transformation $X = P^{-1}X_d$ and $Y = Y_d P^{-T}$ with any nonsingular matrix $P \in \mathbb{R}^{n\times n}$. By Lemma 2.1 again, we can define the quadratic

$$\widetilde{Q}(\lambda) = \lambda^2 A + \lambda B + C, \tag{4.4}$$

where

$$A = \Gamma_1^{-1}, \quad B = -A\Gamma_2 A, \quad C = -A\Gamma_3 A + B\Gamma_1 B \tag{4.5}$$

with $\Gamma_i = XJ^iY (i = 1, 2, 3)$. Since the uniqueness of the Jordan triple of the matrix polynomial can not be guaranteed (see [3, Chap. 1]), the obtained $\widetilde{Q}(\lambda)$ could not cover all WO quadratics isospectral with $Q_d(\lambda)$. Here we still keep coefficient matrices the same with (1.1) to avoid the notational clutter. A direct comparison of (4.3) with (4.5) yields the relations

$$A = P^T P, \quad B = P^T \widehat{B} P, \quad C = P^T \widehat{C} P. \tag{4.6}$$

Remark 4.1. The construction process of $\widetilde{Q}(\lambda)$ guarantees that two extremal solvents $S^{(1)}$ and $S^{(2)}$ of $\widetilde{Q}(S) = AS^2 + BS + C = 0$ have a clear definition and they do exist. Also $\lambda_n = -1$ is semisimple in $S^{(i)}$, $i = 1, 2$. Indeed, it is clear that

$$\widehat{S}^{(1)} = -[I_{m_1} \oplus \Delta_1 \oplus \Delta_2 \oplus I_r] \quad \text{and} \quad \widehat{S}^{(2)} = -[\Sigma_1 \oplus \Sigma_2 \oplus I_{m_2} \oplus I_r]$$

are the extremal solvents of $Q_d(S) = S^2 + \widehat{B}S + \widehat{C} = 0$, and $\lambda_n = -1$ is semisimple in $\widehat{S}^{(i)}$, $i = 1, 2$. By (4.6), the two extremal solvents of $\widetilde{Q}(S) = 0$ have the form

$$S^{(1)} = P^{-1}\widehat{S}^{(1)}P \quad \text{and} \quad S^{(2)} = P^{-1}\widehat{S}^{(2)}P,$$

and $\lambda_n = -1$ is semisimple in $S^{(i)}$, $i = 1, 2$.

The convergence theorem of the CR algorithm for $\widetilde{Q}(\lambda)$ is as follows:

Theorem 4.1. *Let $\widetilde{Q}(\lambda)$ be a weakly overdamped quadratic given by* (4.4)*. Let $S^{(1)}$ and $S^{(2)}$ be the primary and secondary solutions of $\widetilde{Q}(S) = 0$, respectively. Then for any matrix norm $\|\cdot\|$, the CR matrix sequences generated by* (3.1) *satisfy*

$$\limsup_{k\to\infty} \sqrt[k]{\|A_k - \widetilde{B}P^{-1}(0_{m_1+l} \oplus I_{m_2} \oplus 0_r)P\|} \le \frac{1}{2}, \tag{4.7}$$

$$\limsup_{k\to\infty} \sqrt[k]{\|C_k - \widetilde{B}P^{-1}(I_{m_1} \oplus 0_{n-m_1})P\|} \le \frac{1}{2}, \tag{4.8}$$

$$\limsup_{k\to\infty} \sqrt[k]{\|S_k - \widetilde{S}\|} \le \frac{1}{2}, \quad \limsup_{k\to\infty} \sqrt[k]{\|B_k - \widetilde{B}\|} \le \frac{1}{2}, \tag{4.9}$$

where P is given by (4.6), $\widetilde{B} = A(S^{(1)} - S^{(2)}) \ge 0$ is singular and $\widetilde{S} = -{S^{(2)}}^T A$ is nonsingular.

Remark 4.2. Theorem 4.1 shows some different convergence from Theorem 3.1, i.e. the matrix sequence $\{A_k\}$ (or $\{C_k\}$) no longer converges to the zero matrix unless m_2 (or m_1) equals zero. The case $m_1 = m_2 = 0$ implies that the partial multiplicities of λ_n are all 2. In this case, Theorem 4.1 is reduced to Theorem 3.1. So Theorem 4.1 can be seen as a complement of the convergence for the CR algorithm. We will also see from the proof of Theorem 4.1 that the convergence recovers quadratic if the partial multiplicities of λ_n are all 1. This is also reflected from the example in [6, Sec. 4].

5. Proof of Theorem 4.1

This section is dedicated to the proof of Theorem 4.1. The key of the proof is to make out the structure of the eigenspace. We will describe the doubling algorithm in subsection 5.1 and then explore the structure of the eigenspace in subsection 5.2. The proof of Theorem 4.1 will be given in subsection 5.3.

5.1. The doubling algorithm

Different structure-preserving doubling algorithms have been studied in [2–4,7,9,12,20–22]. For general WO quadratics $Q(\lambda)$ in (1.1), the doubling algorithm is as follows [6]

$$\begin{cases} A_0 = A,\ H_0 = B,\ D_0 = 0,\ C_0 = C, \\ A_{k+1} = A_k(D_k - H_k)^{-1}A_k, \\ H_{k+1} = H_k + C_k(D_k - H_k)^{-1}A_k, \\ C_{k+1} = C_k(D_k - H_k)^{-1}C_k, \\ D_{k+1} = D_k - A_k(D_k - H_k)^{-1}C_k. \end{cases} \tag{5.1}$$

It is clear that the CR algorithm (3.1) can be recovered from the doubling algorithm (5.1) by letting

$$B_k = H_k - D_k \quad \text{and} \quad S_k = {H_k}^T. \tag{5.2}$$

It follows from (3.2) that $B_k > 0$ for $k \geq 0$, thus the doubling algorithm (5.1) for $Q(\lambda)$ is well defined. Let

$$M_k = \begin{bmatrix} A_k & 0 \\ -H_k & -I \end{bmatrix} \quad \text{and} \quad L_k = \begin{bmatrix} D_k & I \\ C_k & 0 \end{bmatrix} \tag{5.3}$$

for $k \geq 0$. The doubling algorithm can also be described in block matrices [2,6]

$$M_k^{-1}L_k = (M_0^{-1}L_0)^{2^k} \tag{5.4}$$

for $k \geq 0$.

The next lemma shows that the doubling algorithm, when applied to WO quadratics $\widetilde{Q}(\lambda)$, really acts on the diagonal quadratic $Q_d(\lambda)$.

Lemma 5.1. *Let*

$$\widehat{M}_0 = \begin{bmatrix} I & 0 \\ -\widehat{B} & -I \end{bmatrix} \quad \textit{and} \quad \widehat{L}_0 = \begin{bmatrix} 0 & I_n \\ \widehat{C} & 0 \end{bmatrix}$$

be the initial block matrices associated with $Q_d(\lambda)$. Let

$$M_0 = \begin{bmatrix} A_0 & 0 \\ -H_0 & -I \end{bmatrix} \quad \textit{and} \quad L_0 = \begin{bmatrix} D_0 & I \\ C_0 & 0 \end{bmatrix}$$

be the initial block matrices associated with $\widetilde{Q}(\lambda)$. *When the doubling algorithm* (5.1) *is applied to* $Q_d(\lambda)$ *and* $\widetilde{Q}(\lambda)$, *we have for* $k \geq 0$

$$A_k = P^T \widehat{A}_k P, \quad C_k = P^T \widehat{C}_k P, \quad D_k = P^T \widehat{D}_k P, \quad H_k = P^T \widehat{H}_k P, \tag{5.5}$$

$$M_k = D_{P_1}^T \widehat{M}_k D_{P_2}, \quad L_k = D_{P_1}^T \widehat{L}_k D_{P_2}, \tag{5.6}$$

where M_k *and* L_k *are defined by* (5.3),

$$\widehat{M}_k = \begin{bmatrix} \widehat{A}_k & 0 \\ -\widehat{H}_k & -I \end{bmatrix}, \ \widehat{L}_k = \begin{bmatrix} \widehat{D}_k & I \\ \widehat{C}_k & 0 \end{bmatrix}, \ D_{P_1} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \ \text{and} \ D_{P_2} = \begin{bmatrix} P & 0 \\ 0 & P^{-T} \end{bmatrix}$$

with nonsingular P *in* (4.6).

Proof. We prove the lemma by induction. It is easy to see from (4.6) and (5.1) that (5.5) is true for $k = 0$ with $\widehat{A}_0 = I_n$, $\widehat{C}_0 = \widehat{C}$, $\widehat{D}_0 = 0$ and $\widehat{H}_0 = \widehat{B}$. By the definition of $\widehat{M}_0$, $\widehat{L}_0$, M_0 and L_0, (5.6) holds for $k = 0$.

Suppose that (5.5) and (5.6) are true for $k = i \geq 0$. We are going to show that they are true for $i + 1$ too. By the doubling iterates (5.1), we have

$$\begin{aligned} A_{i+1} &= P^T \widehat{A}_i (\widehat{D}_i - \widehat{H}_i)^{-1} \widehat{A}_i P = P^T \widehat{A}_{i+1} P, \\ H_{i+1} &= P^T \widehat{H}_i P + P^T \widehat{C}_i (\widehat{D}_i - \widehat{H}_i)^{-1} \widehat{A}_i P = P^T \widehat{H}_{i+1} P, \\ C_{i+1} &= P^T \widehat{C}_i (\widehat{D}_i - \widehat{H}_i)^{-1} \widehat{C}_i P = P^T \widehat{C}_{i+1} P, \\ D_{i+1} &= P^T \widehat{D}_i P - P^T \widehat{A}_i (\widehat{D}_i - \widehat{H}_i)^{-1} \widehat{C}_i P = P^T \widehat{D}_{i+1} P \end{aligned}$$

and

$$\begin{aligned} M_{i+1} &= \begin{bmatrix} A_{i+1} & 0 \\ -H_{i+1} & -I_n \end{bmatrix} = \begin{bmatrix} P^T \widehat{A}_{i+1} P & 0 \\ -P^T \widehat{H}_{i+1} P & -P^T P^{-T} \end{bmatrix} = D_{P_1}^T \widehat{M}_{i+1} D_{P_2}, \\ L_{i+1} &= \begin{bmatrix} D_{i+1} & I_n \\ C_{i+1} & 0 \end{bmatrix} = \begin{bmatrix} P^T \widehat{D}_{i+1} P & P^T P^{-T} \\ P^T \widehat{C}_{i+1} P & 0 \end{bmatrix} = D_{P_1}^T \widehat{L}_{i+1} D_{P_2}. \end{aligned}$$

So (5.5) and (5.6) hold for all $i + 1 \geq 0$. The proof is complete. $\square$

By the use of (5.4) and (5.6), we have

$$M_k^{-1} L_k = (M_0^{-1} L_0)^{2^k} = D_{P_2}^{-1} (\widehat{M}_0^{-1} \widehat{L}_0)^{2^k} D_{P_2} = D_{P_2}^{-1} (\widehat{M}_k^{-1} \widehat{L}_k) D_{P_2}, \tag{5.7}$$

i.e., the doubling algorithm on $\widetilde{Q}(\lambda)$ is really on $Q_d(\lambda)$ via a similarity transformation.

To obtain the convergence of the CR algorithm for $\widetilde{Q}(\lambda)$, we need to make out the structure of the eigenspace for $M_0^{-1} L_0$. To this purpose, let

$$J_v = \begin{bmatrix} -\Sigma_1 \oplus -\Sigma_2 \oplus -I_{m_2} \oplus -I_r & 0_{m_1} \oplus 0_l \oplus 0_{m_2} \oplus I_r \\ 0 & -I_{m_1} \oplus -\Delta_1 \oplus -\Delta_2 \oplus -I_r \end{bmatrix}, \tag{5.8}$$

$$J_w = \begin{bmatrix} -\Sigma_1 \oplus -\Sigma_2 \oplus -I_{m_2} \oplus -I_r & 0 \\ 0_{m_1} \oplus 0_l \oplus 0_{m_2} \oplus I_r & -I_{m_1} \oplus -\Delta_1 \oplus -\Delta_2 \oplus -I_r \end{bmatrix} \tag{5.9}$$

and $X_{\hat{v}}$ and $X_{\hat{w}}$ be two $n \times 2n$ matrices by rearranging columns of X_d (see (4.2)) corresponding to J_v and J_w, respectively. Since $\lambda \widehat{M}_0 - \widehat{L}_0$ is a linearization of $Q_d(\lambda)$ and $(X_{\hat{v}}, J_v)$ (or $(X_{\hat{w}}, J_v)$)

is a Jordan pair of $Q_d(\lambda)$, we then have, by [5, Thm 1.20] and Lemma 5.1, two basic equalities

$$\begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}}J_v \end{bmatrix}^{-1} \widehat{M_0}^{-1}\widehat{L}_0 \begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}}J_v \end{bmatrix} = \begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}}J_v \end{bmatrix}^{-1} D_{P_2}(M_0^{-1}L_0)D_{P_2}^{-1} \begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}}J_v \end{bmatrix} = J_v,$$
$$\begin{bmatrix} X_{\hat{w}} \\ X_{\hat{w}}J_w \end{bmatrix}^{-1} \widehat{M_0}^{-1}\widehat{L}_0 \begin{bmatrix} X_{\hat{w}} \\ X_{\hat{w}}J_w \end{bmatrix} = \begin{bmatrix} X_{\hat{w}} \\ X_{\hat{w}}J_w \end{bmatrix}^{-1} D_{P_2}(M_0^{-1}L_0)D_{P_2}^{-1} \begin{bmatrix} X_{\hat{w}} \\ X_{\hat{w}}J_w \end{bmatrix} = J_w.$$

If we denote

$$\widehat{V} = \begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}}J_v \end{bmatrix} \quad \text{and} \quad \widehat{W} = \begin{bmatrix} X_{\hat{w}} \\ X_{\hat{w}}J_w \end{bmatrix}, \tag{5.10}$$

the structure of matrices

$$V = D_{P_2}^{-1}\widehat{V} \quad \text{and} \quad W = D_{P_2}^{-1}\widehat{W} \tag{5.11}$$

(i.e., the structure of the eigenspace for $M_0^{-1}L_0$) can be derived by exploring the structure of matrices $\widehat{V}$ and $\widehat{W}$.

5.2. The structure of matrices $\widehat{V}$ and $\widehat{W}$

We first give the following lemma which reveals the structure of $X_{\hat{v}}$ and $X_{\hat{w}}$. The proof will be shown in Appendix A.

Lemma 5.2. *Let Φ_{m_1}, Φ_l, Ψ_l and Ψ_{m_2} be $m_1 \times m_1$, $l \times l$, $l \times l$ and $m_2 \times m_2$ nonsingular diagonal matrices. Let Φ_r, Φ_{m_2} and Ψ_{m_1} be $r \times r$, $m_2 \times m_2$ and $m_1 \times m_1$ nonsingular matrices. Suppose that $\Phi_{jr}(j = 1,2,3)$, Φ_k and $\Psi_k(k = 0,1)$ are all arbitrary real matrices with the respective dimension $m_1 \times r$, $m_2 \times r$, $r \times r$, $m_1 \times m_1$, $r \times m_2$, $m_2 \times m_1$ and $r \times m_1$. The structure of $X_{\hat{v}}$ and $X_{\hat{w}}$ are as follows:*

$$X_{\hat{v}} = \left[\begin{array}{cccc:cccc} \Phi_{m_1} & & \Phi_0 & & \Psi_{m_1} & & & \Phi_{1r} \\ & \Phi_l & & & & \Psi_l & & \\ & & \Phi_{m_2} & & \Psi_0 & & \Psi_{m_2} & \Phi_{2r} \\ & & \Phi_1 & \Phi_r & \Psi_1 & & & \Phi_{3r} \end{array}\right], \tag{5.12}$$

$$X_{\hat{w}} = \left[\begin{array}{cccc:cccc} \Phi_{m_1} & & \Phi_0 & \Phi_{1r} & \Psi_{m_1} & & & \\ & \Phi_l & & & & \Psi_l & & \\ & & \Phi_{m_2} & \Phi_{2r} & \Psi_0 & & \Psi_{m_2} & \\ & & \Phi_1 & \Phi_{3r} & \Psi_1 & & & \Phi_r \end{array}\right]. \tag{5.13}$$

The next lemma indicates that $X_{\hat{v}}$ and $X_{\hat{w}}$ have more concise structure, i.e. the blocks Φ_0 and Ψ_0 in (5.12) and (5.13) are really zero matrices. It also gives the inverse structure of the nonsingular sub-block .

Lemma 5.3. *Matrices $\widehat{V}$ and $\widehat{W}$ in* (5.10) *have the following block forms:*

$$\widehat{V} = \begin{bmatrix} \widehat{V}_1 & \widehat{V}_3 \\ \widehat{V}_2 & \widehat{V}_4 \end{bmatrix}, \qquad \widehat{W} = \begin{bmatrix} \widehat{W}_1 & \widehat{W}_3 \\ \widehat{W}_2 & \widehat{W}_4 \end{bmatrix},$$

where $\widehat{V}_1$ *and* $\widehat{W}_3$ *are nonsingular* $n \times n$ *matrices with*

$$\widehat{V}_1^{-1} = \begin{bmatrix} \Phi_{m_1}^{-1} & & & \\ & \Phi_l^{-1} & & \\ & & \Phi_{m_2}^{-1} & \\ & & -\Phi_r^{-1}\Phi_1\Phi_{m_2}^{-1} & \Phi_r^{-1} \end{bmatrix}, \tag{5.14}$$

$$\widehat{W}_3^{-1} = \begin{bmatrix} \Psi_{m_1}^{-1} & & & \\ & \Psi_l^{-1} & & \\ & & \Psi_{m_2}^{-1} & \\ -\Psi_r^{-1}\Psi_1\Psi_{m_1}^{-1} & & & \Phi_r^{-1} \end{bmatrix}. \tag{5.15}$$

Proof. Since Φ_{m_1}, Φ_l, Φ_{m_2}, Ψ_{m_1}, Ψ_l, Ψ_{m_2} and Φ_r in (5.12) and (5.13) are all nonsingular block matrices , we can define

$$\widetilde{V}_1 = \begin{bmatrix} \Phi_{m_1}^{-1} & & -\Phi_{m_1}^{-1}\Phi_0\Phi_{m_2}^{-1} & \\ & \Phi_l^{-1} & & \\ & & \Phi_{m_2}^{-1} & \\ & & -\Phi_r^{-1}\Phi_1\Phi_{m_2}^{-1} & \Phi_r^{-1} \end{bmatrix},$$

$$\widetilde{W}_3 = \begin{bmatrix} \Psi_{m_1}^{-1} & & & \\ & \Psi_l^{-1} & & \\ -\Psi_{m_2}^{-1}\Psi_0\Psi_{m_1}^{-1} & & \Psi_{m_2}^{-1} & \\ -\Psi_r^{-1}\Psi_1\Psi_{m_1}^{-1} & & & \Phi_r^{-1} \end{bmatrix}.$$

Since $\widetilde{V}_1\widehat{V}_1 = \widehat{V}_1\widetilde{V}_1 = I_n$ and $\widetilde{W}_3\widehat{W}_3 = \widehat{W}_3\widetilde{W}_3 = I_n$. So $\widehat{V}_1$ and $\widehat{W}_3$ are nonsingular with inverses $\widetilde{V}_1$ and $\widetilde{W}_3$, respectively. It then suffices to show that Φ_0 and Ψ_0 are zero matrices. As mentioned in Remark 4.1, $\widehat{S}^{(1)} = -[I_{m_1} \oplus \Delta_1 \oplus \Delta_2 \oplus I_r]$ and $\widehat{S}^{(2)} = -[\Sigma_1 \oplus \Sigma_2 \oplus I_{m_2} \oplus I_r]$ are the extremal solvents of $Q_d(S) = 0$. By the proof of Theorem 4.6 in [6], we also have

$$\widehat{W}_4\widehat{W}_3^{-1} = \widehat{S}^{(1)}, \qquad \widehat{V}_2\widehat{V}_1^{-1} = \widehat{S}^{(2)}. \tag{5.16}$$

However, a direct computation shows

$$\widehat{V}_2\widetilde{V}_1 = -\begin{bmatrix} \Sigma_1 & & (I_{m_1} - \Sigma_1)\Phi_0\Phi_{m_2}^{-1} & \\ & \Sigma_2 & & \\ & & I_{m_2} & \\ & & & I_r \end{bmatrix},$$

$$\widehat{W}_4\widetilde{W}_3 = -\begin{bmatrix} I_{m_1} & & & \\ & \Delta_1 & & \\ (I_{m_2} - \Delta_2)\Psi_0\Psi_{m_1}^{-1} & & \Delta_2 & \\ & & & I_r \end{bmatrix},$$

where

$$\widehat{V}_2 = -\begin{bmatrix} \Phi_{m_1}\Sigma_1 & & \Phi_0 & \\ & \Phi_l\Sigma_2 & & \\ & & \Phi_{m_2} & \\ & & \Phi_1 & \Phi_r \end{bmatrix}, \quad \widehat{W}_4 = -\begin{bmatrix} \Psi_{m_1} & & & \\ & \Psi_l\Delta_1 & & \\ \Psi_0 & & \Psi_{m_2}\Delta_2 & \\ \Psi_1 & & & \Phi_r \end{bmatrix}.$$

Hence by the nonsingularity of $I_{m_1} - \Sigma_1$ and $I_{m_2} - \Delta_2$, Φ_0 and Ψ_0 are zero matrices. So the inverses of $\widehat{V}_1$ and $\widehat{W}_3$ have the form (5.14) and (5.15). $\square$

Based on Lemma 5.3, we can describe the structure of $\widehat{V}$ and $\widehat{W}$ as follows:

$$
\widehat{V} = \left[\begin{array}{c:c} \widehat{V}_1 & \widehat{V}_3 \\ \hdashline \widehat{V}_2 & \widehat{V}_4 \end{array}\right] \tag{5.17}
$$

$$
= \left[\begin{array}{cccc:cccc}
\Phi_{m_1} & & & & \Psi_{m_1} & & & \Phi_{1r} \\
& \Phi_l & & & & \Psi_l & & \\
& & \Phi_{m_2} & & & & \Psi_{m_2} & \Phi_{2r} \\
& & \Phi_1 & \Phi_r & \Psi_1 & & & \Phi_{3r} \\
\hdashline
-\Phi_{m_1}\Sigma_1 & & & & -\Psi_{m_1} & & & -\Phi_{1r} \\
& -\Phi_l\Sigma_2 & & & & -\Psi_l\Delta_1 & & \\
& & -\Phi_{m_2} & & & & -\Psi_{m_2}\Delta_2 & -\Phi_{2r} \\
& & -\Phi_1 & -\Phi_r & -\Psi_1 & & & \Phi_r - \Phi_{3r}
\end{array}\right],
$$

$$
\widehat{W} = \left[\begin{array}{c:c} \widehat{W}_1 & \widehat{W}_3 \\ \hdashline \widehat{W}_2 & \widehat{W}_4 \end{array}\right] \tag{5.18}
$$

$$
= \left[\begin{array}{cccc:cccc}
\Phi_{m_1} & & & \Phi_{1r} & \Psi_{m_1} & & & \\
& \Phi_l & & & & \Psi_l & & \\
& & \Phi_{m_2} & \Phi_{2r} & & & \Psi_{m_2} & \\
& & \Phi_1 & \Phi_{3r} & \Psi_1 & & & \Phi_r \\
\hdashline
-\Phi_{m_1}\Sigma_1 & & & -\Phi_{1r} & -\Psi_{m_1} & & & \\
& -\Phi_l\Sigma_2 & & & & -\Psi_l\Delta_1 & & \\
& & -\Phi_{m_2} & -\Phi_{2r} & & & -\Psi_{m_2}\Delta_2 & \\
& & -\Phi_1 & \Phi_r - \Phi_{3r} & -\Psi_1 & & & -\Phi_r
\end{array}\right].
$$

Moreover, (5.17) and (5.18) help to yield the following lemma that is useful in the proof of Theorem 4.1.

Lemma 5.4. *Let $\widehat{V}$ and $\widehat{W}$ be partitioned as in* (5.17) *and* (5.18)*, respectively. Let* $\Xi_{(1,k)} = \widehat{W}_3(I_{m_1} \oplus \Delta^{2^k} \oplus I_r)\widehat{W}_3^{-1}$, $\Xi_{(2,k)} = \widehat{V}_1(\Sigma^{-2^k} \oplus I_{m_2} \oplus 0_r)\widehat{V}_1^{-1}$, $\Xi_{(3,k)} = \widehat{V}_1(\Sigma^{-2^k} \oplus I_{m_2} \oplus I_r)\widehat{V}_1^{-1}$ *and* $\Xi_{(4,k)} = \widehat{W}_3(I_{m_1} \oplus \Delta^{2^k} \oplus 0_r)\widehat{W}_3^{-1}$ *with* $\Sigma = \Sigma_1 \oplus \Sigma_2$ *and* $\Delta = \Delta_1 \oplus \Delta_2$*. Then for sufficiently large k, matrices*

$$
I - \Xi_{(1,k)}\Xi_{(2,k)} + (I_n - \Xi_{(1,k)})\widehat{V}_3(0_{n-r} \oplus (-2)^{-k} I_r)\widehat{V}_1^{-1}, \tag{5.19}
$$

$$
I - \Xi_{(3,k)}\Xi_{(4,k)} + (\Xi_{(3,k)} - I_n)\widehat{W}_1(0_{n-r} \oplus (-2)^{-k} I_r)\widehat{W}_3^{-1}, \tag{5.20}
$$

are both nonsingular. Moreover, we have

$$
\lim_{k\to\infty} \Xi_{(2,k)}(I - \Xi_{(1,k)}\Xi_{(2,k)})^{-1} = \widehat{V}_1(0_{m_1+l} \oplus I_{m_2} \oplus 0_r)\widehat{V}_1^{-1} =: \Xi_2, \tag{5.21}
$$

$$
\lim_{k\to\infty} \Xi_{(4,k)}(I - \Xi_{(3,k)}\Xi_{(4,k)})^{-1} = \widehat{W}_3(I_{m_1} \oplus 0_{n-m_1})\widehat{W}_3^{-1} =: \Xi_4. \tag{5.22}
$$

Proof. Note that Σ is a diagonal matrix with nonzero entries greater than 1 while Δ is a diagonal matrix with nonzero entries less than 1. Therefore, matrices $\Xi_{(j,k)}(j = 1,2,3,4)$ have the limits $\Xi_j(j = 1,2,3,4)$ with $\Xi_1 = \widehat{W}_3(I_{m_1} \oplus 0_{l+m_2} \oplus I_r)\widehat{W}_3^{-1}$, $\Xi_2 = \widehat{V}_1(0_{m_1+l} \oplus I_{m_2} \oplus 0_r)\widehat{V}_1^{-1}$, $\Xi_3 = \widehat{V}_1(0_{m_1+l} \oplus I_{m_2} \oplus I_r)\widehat{V}_1^{-1}$ and $\Xi_4 = \widehat{W}_3(I_{m_1} \oplus 0_{n-m_1})\widehat{W}_3^{-1}$. Observing (5.14) and (5.15), the matrix sequences defined by (5.19) and (5.20) converge to nonsingular matrices

$$I-\Xi_1\Xi_2=\begin{bmatrix} I_{m_1} & & & \\ & I_l & & \\ & & I_{m_2} & \\ & & -\Phi_1\Phi_{m_2}^{-1} & I_r \end{bmatrix}, \quad I-\Xi_3\Xi_4=\begin{bmatrix} I_{m_1} & & & \\ & I_l & & \\ & & I_{m_2} & \\ -\Phi_1\Phi_{m_2}^{-1} & & & I_r. \end{bmatrix}.$$

Relations (5.21) and (5.22) can be obtained by (5.14), (5.15), (5.17) and (5.18). □

5.3. The proof of Theorem 4.1

We prove the convergence for the diagonal quadratic $Q_d(\lambda)$ by doubling algorithm, then the results in Theorem 4.1 can be obtained via the relation (5.5). By (5.7), we have $\widehat{M}_k^{-1}\widehat{L}_k=(\widehat{M}_0^{-1}\widehat{L}_0)^{2^k}$. This together with $\widehat{V}^{-1}(\widehat{M}_0^{-1}\widehat{L}_0)\widehat{V}=J_v$ and $\widehat{W}^{-1}(\widehat{M}_0^{-1}\widehat{L}_0)\widehat{W}=J_w$ yield

$$\widehat{L}_k\widehat{V}=\widehat{M}_k\widehat{V}J_v^{2^k}, \qquad \widehat{L}_k\widehat{W}=\widehat{M}_k\widehat{W}J_w^{2^k}. \tag{5.23}$$

Equating (5.23) by 2×2 blocks of $\widehat{M}_k$, $\widehat{L}_k$ in Lemma 5.1 and $\widehat{V}$, $\widehat{W}$ in Lemma 5.3, we obtain

$$\widehat{D}_k\widehat{V}_1+\widehat{V}_2=\widehat{A}_k\widehat{V}_1(\Sigma^{2^k}\oplus I_{m_2}\oplus I_r) \tag{5.24}$$
$$\widehat{C}_k\widehat{V}_1=-(\widehat{H}_k\widehat{V}_1+\widehat{V}_2)(\Sigma^{2^k}\oplus I_{m_2}\oplus I_r) \tag{5.25}$$
$$\widehat{D}_k\widehat{V}_3+\widehat{V}_4=\widehat{A}_k\widehat{V}_1(0_{n-r}\oplus(-2)^kI_r)+\widehat{A}_k\widehat{V}_3(I_{m_1}\oplus\Delta^{2^k}\oplus I_r) \tag{5.26}$$
$$\widehat{C}_k\widehat{V}_3=-(\widehat{H}_k\widehat{V}_1+\widehat{V}_2)(0_{n-r}\oplus(-2)^kI_r)-(\widehat{H}_k\widehat{V}_3+\widehat{V}_4)(I_{m_1}\oplus\Delta^{2^k}\oplus I_r) \tag{5.27}$$

and

$$\widehat{D}_k\widehat{W}_1+\widehat{W}_2=\widehat{A}_k\widehat{W}_1(\Sigma^{2^k}\oplus I_{m_2}\oplus I_r)+\widehat{A}_k\widehat{W}_3(0_{n-r}\oplus(-2)^kI_r) \tag{5.28}$$
$$\widehat{C}_k\widehat{W}_1=-(\widehat{H}_k\widehat{W}_1+\widehat{W}_2)(\Sigma^{2^k}\oplus I_{m_2}\oplus I_r)-(\widehat{H}_k\widehat{W}_3+\widehat{W}_4)(0_{n-r}\oplus(-2)^kI_r) \tag{5.29}$$
$$\widehat{D}_k\widehat{W}_3+\widehat{W}_4=\widehat{A}_k\widehat{W}_3(I_{m_1}\oplus\Delta^{2^k}\oplus I_r) \tag{5.30}$$
$$\widehat{C}_k\widehat{W}_3=-(\widehat{H}_k\widehat{W}_3+\widehat{W}_4)(I_{m_1}\oplus\Delta^{2^k}\oplus I_r), \tag{5.31}$$

where Σ and Δ are both diagonal matrices with $\Sigma=\Sigma_1\oplus\Sigma_2$ and $\Delta=\Delta_1\oplus\Delta_2$.

Post-multiplying (5.24) and (5.26) by $\Sigma^{-2^k}\oplus I_{m_2}\oplus 0_r$ and $0_{n-r}\oplus(-2)^{-k}I_r$ respectively and then adding the resulting equalities yields

$$\begin{aligned}&\widehat{A}_k\widehat{V}_1+\widehat{A}_k\widehat{V}_3(0_{n-r}\oplus(-2)^{-k}I_r)\\ =&\,(\widehat{D}_k\widehat{V}_1+\widehat{V}_2)(\Sigma^{-2^k}\oplus I_{m_2}\oplus 0_r)+(\widehat{D}_k\widehat{V}_3+\widehat{V}_4)(0_{n-r}\oplus(-2)^{-k}I_r).\end{aligned} \tag{5.32}$$

By (5.30), we have

$$\widehat{D}_k=\Big(\widehat{A}_k\widehat{W}_3(I_{m_1}\oplus\Delta^{2^k}\oplus I_r)-\widehat{W}_4\Big)\widehat{W}_3^{-1}. \tag{5.33}$$

Inserting (5.33) into (5.32) and then post-multiplying it by $\widehat{V}_1^{-1}$ gives

$$\begin{aligned}&\widehat{A}_k\Big(I_n-\Xi_{(1,k)}\Xi_{(2,k)}+(I_n-\Xi_{(1,k)})\widehat{V}_3(0_{n-r}\oplus(-2)^{-k}I_r)\widehat{V}_1^{-1}\Big)\\ =&\,(\widehat{V}_2-\widehat{W}_4\widehat{W}_3^{-1}\widehat{V}_1)(\Sigma^{-2^k}\oplus I_{m_2}\oplus 0_r)\widehat{V}_1^{-1}\\ &+(\widehat{V}_4-\widehat{W}_4\widehat{W}_3^{-1}\widehat{V}_3)(0_{n-r}\oplus(-2)^{-k}I_r)\widehat{V}_1^{-1},\end{aligned}$$

where $\Xi_{(1,k)}$ and $\Xi_{(2,k)}$ are defined by Lemma 5.4. By (5.16) and (5.21), we get

$$\begin{aligned}&\widehat{A}_k - (\widehat{V}_2\widehat{V}_1^{-1} - \widehat{W}_4\widehat{W}_3^{-1})\Xi_{(2,k)}(I_n - \Xi_{(1,k)}\Xi_{(2,k)})^{-1}\\ &= \widehat{A}_k - (\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_2 = \mathcal{O}((-2)^{-k}).\end{aligned} \tag{5.34}$$

It then follows from (5.33) that

$$\widehat{D}_k + \widehat{W}_4\widehat{W}_3^{-1} = \widehat{A}_k\Xi_{(1,k)} = \mathcal{O}((-2)^{-k}), \tag{5.35}$$

here we have used the fact $\Xi_2(I_n - \Xi_1\Xi_2)^{-1}\Xi_1 = \Xi_2\Xi_1 = 0$. Post-multiplying (5.29) by $0_{n-r} \oplus (-2)^{-k}I_r$ and then subtracting it from (5.31) gives

$$\begin{aligned}&\widehat{C}_k\widehat{W}_3 - \widehat{C}_k\widehat{W}_1(0_{n-r} \oplus (-2)^{-k}I_r)\\ &= (\widehat{H}_k\widehat{W}_1 + \widehat{W}_2)(0_{n-r} \oplus (-2)^{-k}I_r) - (\widehat{H}_k\widehat{W}_3 + \widehat{W}_4)(I_{m_1} \oplus \Delta^{2^k} \oplus 0_r).\end{aligned} \tag{5.36}$$

From (5.25), we have

$$\widehat{H}_k = -\Big(\widehat{C}_k\widehat{V}_1(\Sigma^{-2^k} \oplus I_{m_2} \oplus I_r) + \hat{V}_2\Big)\widehat{V}_1^{-1}. \tag{5.37}$$

Inserting (5.37) into (5.36) and then post-multiplying the result by $\hat{W}_3^{-1}$, we get

$$\begin{aligned}&\widehat{C}_k\Big(I_n - \Xi_{(3,k)}\Xi_{(4,k)} + (\Xi_{(3,k)} - I_n)\widehat{W}_1(0_{n-r} \oplus (-2)^{-k}I_r)\widehat{W}_3^{-1}\Big)\\ &= (\widehat{W}_2 - \widehat{V}_2\widehat{V}_1^{-1}\widehat{W}_1)(0_{n-r} \oplus (-2)^{-k}I_r)\widehat{W}_3^{-1} + (\widehat{V}_2\widehat{V}_1^{-1} - \widehat{W}_4\widehat{W}_3^{-1})\Xi_{(4,k)},\end{aligned}$$

where $\Xi_{(3,k)}$ and $\Xi_{(4,k)}$ are defined by Lemma 5.4. By (5.16) and (5.22), we have

$$\begin{aligned}&\widehat{C}_k - (\widehat{V}_2\widehat{V}_1^{-1} - \widehat{W}_4\widehat{W}_3^{-1})\Xi_{(4,k)}(I_n - \Xi_{(3,k)}\Xi_{(4,k)})^{-1}\\ &= \widehat{C}_k - (\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_4 = \mathcal{O}((-2)^{-k}).\end{aligned} \tag{5.38}$$

This together with (5.37) implies

$$\widehat{H}_k + \widehat{V}_2\widehat{V}_1^{-1} = \widehat{C}_k\Xi_3 = \mathcal{O}((-2)^{-k}), \tag{5.39}$$

here we have used the fact $\Xi_4(I_n - \Xi_3\Xi_4)^{-1}\Xi_3 = \Xi_4\Xi_3 = 0$. In view of (5.5), we get from (5.34), (5.35), (5.38) and (5.39)

$$A_k - P^T(\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_2 P = \mathcal{O}((-2)^{-k}), \tag{5.40}$$

$$D_k + P^T\widehat{S}^{(1)}P = \mathcal{O}((-2)^{-k}), \tag{5.41}$$

$$C_k - P^T(\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_4 P = \mathcal{O}((-2)^{-k}), \tag{5.42}$$

$$H_k + P^T\widehat{S}^{(2)}P = \mathcal{O}((-2)^{-k}). \tag{5.43}$$

Since $(\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_2 = (\widehat{S}^{(2)} - \widehat{S}^{(1)})(0_{m_1+l} \oplus I_{m_2} \oplus 0_r)$ and $(\widehat{S}^{(2)} - \widehat{S}^{(1)})\Xi_4 = (\widehat{S}^{(2)} - \widehat{S}^{(1)})(I_{m_1} \oplus 0_{n-m_1})$, (5.40) and (5.42) together with $S^{(i)} = P^{-1}\widehat{S}^{(i)}P$ $(i = 1, 2)$ and $A = P^TP$ imply (4.7) and (4.8). Note that $B_k = H_k - D_k$, $S_k = H_k^T$. Following a similar way to the proof of Corollary 4.7 in [6], we obtain (4.9). □

Remark 5.1. When the constant r equals zero, we know from (5.34), (5.35), (5.38) and (5.39) in Theorem 4.1 that the $r \times r$ block matrices disappear. In this case, the convergence rate is quadratic, which is also reflected from the example in [6, Sec. 4]. When the constant r is greater than zero, Theorem 4.1 shows that the iterate sequences $\{B_k\}$ and $\{S_k\}$ have the same limit with that in Theorem 3.1. However, sequences $\{A_k\}$ and $\{C_k\}$ no longer converge to the zero matrix unless m_1 and m_2 are both zero.

Remark 5.2. We prove Theorem 4.1 by using the same tool in [6]. This result can also be derived by directly applying the Kronecker form to $\widetilde{Q}(S) = 0$ and its dual equation (as in [2]). Once the structure of V and W is clear, there is no difference in essence of the two methods. However, our proof can not cover all quadratics isospectral with $Q_d(\lambda)$, even for those in which the two extremal solvents have a semisimple λ_n.

6. Numerical Experiments

In this section, we test the convergence behavior of the CR algorithm in Theorem 4.1. We first provide a way to generate a class of WO quadratics $\widetilde{Q}(\lambda)$ (see also in [13] and [14]). Let

$$X_{\hat{v}} = [X_{m_1}, X_l, X_{m_2}, X_r, X_{m_1}\Theta_1, X_l\Theta_2, X_{m_2}\Theta_3, 0_{n\times r}] \tag{6.1}$$

with

$$\begin{aligned} X_{m_1} &= [(\Sigma_1 - I_{m_1})^{-\frac{1}{2}}, 0_{(n-m_1)\times m_1}]^T, \\ X_l &= [0_{m_1\times l}, (\Sigma_2 - \Theta_2\Delta_2\Theta_2^T)^{-\frac{1}{2}}, 0_{(m_2+r)\times l}]^T, \\ X_{m_2} &= [0_{(m_1+l)\times m_2}, (I_{m_2} - \Delta_2)^{-\frac{1}{2}}, 0_{r\times m_2}]^T, \\ X_r &= [0_{(n-r)\times r}, I_r]^T, \end{aligned}$$

where $\Theta_i, i = 1, 2, 3$ are real orthogonal matrices. Let

$$P_{\hat{v}} = \begin{bmatrix} -I_{m_1} \oplus -I_l \oplus -I_{m_2} \oplus 0_r & 0_{m_1} \oplus 0_l \oplus 0_{m_2} \oplus I_r \\ 0_{m_1} \oplus 0_l \oplus 0_{m_2} \oplus I_r & I_{m_1} \oplus I_l \oplus I_{m_2} \oplus 0_r \end{bmatrix}$$

be a standard involuntary permutation matrix [5, Chap. 10] and $Y_{\hat{v}} = P_{\hat{v}} X_{\hat{v}}^T$. Then we have

$$\begin{bmatrix} X_{\hat{v}} \\ X_{\hat{v}} J_v \end{bmatrix} Y_{\hat{v}} = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}.$$

Such a triple $(X_{\hat{v}}, J_v, Y_{\hat{v}})$ is called a *self-adjoint Jordan triple* [5] for the monic quadratic matrix polynomial. If J_v is stable (i.e. all eigenvalues of J_v are in the open left half of the complex plane), we define the moments

$$\Gamma_i = X_{\hat{v}} J_v^i Y_{\hat{v}}, \quad i = -1, 1, 2$$

from the triple $(X_{\hat{v}}, J_v, Y_{\hat{v}})$. The next lemma describes how to generate weakly overdamped quadratics $\widetilde{Q}(\lambda)$.

Lemma 6.1. *Given a nonsingular real $n \times n$ matrix P and a choice of $X_{\hat{v}}$ by (6.1). If J_v is stable, then the coefficient matrices A, B and C of $\widetilde{Q}(\lambda)$ defined by*

$$A = PP^T, \quad B = -P\Gamma_2 P^T, \quad C = -P\Gamma_{-1}^{-1} P^T \tag{6.2}$$

are all symmetric and positive definite.

Proof. By the nonsingularity of P, it is clear that A is symmetric and positive definite. By the definition of Γ_i, we have

$$\begin{aligned}
\Gamma_2 &= X_{\hat{v}} J_v^2 Y_{\hat{v}} \\
&= X_{m_1}(I - \Sigma_1^2){X_{m_1}}^T + X_l(\Theta_2\Delta_1^2\Theta_2^T - \Sigma_2^2){X_l}^T \\
&\quad + X_{m_2}((\Delta_2^2 - I){X_{m_2}}^T - 2X_r^{(-1)}{X_r^{(-1)}}^T, \\
\Gamma_{-1} &= X_{\hat{v}} J_v^{-1} Y_{\hat{v}} \\
&= X_{m_1}(\Sigma_1^{-1} - I){X_{m_1}}^T + X_l(\Sigma_2^{-1} - \Theta_2{\Delta_1^{-1}}\Theta_2^T){X_l}^T \\
&\quad + X_{m_2}(I - \Delta_2^{-1}){X_{m_2}}^T - X_r^{(-1)}{X_r^{(-1)}}^T.
\end{aligned}$$

Since Σ_j and $\Delta_j (j = 1, 2)$ are diagonal matrices with diagonal elements greater than 1 and less than 1, respectively, we deduce from Weyl's theorem and the fact $\lambda(MN) = \lambda(NM)$ (see [19] for example) that

$$\lambda(\Theta_2\Delta_1^2\Theta_2^T - \Sigma_2^2) \le \lambda(\Delta_1^2) - \lambda_{\min}(\Sigma_2^2) < 0,$$

Thus $\Theta_2\Delta_1^2\Theta_2^T - \Sigma_2^2$ is negative definite, and so is $\Sigma_2^{-1} - \Theta_2{\Delta_1^{-1}}\Theta_2^T$. By the choice of $X_{\hat{v}}$ in (6.1), Γ_2 and Γ_{-1} are both symmetric and negative definite. The required symmetry and definiteness of B and C then follows from (6.2). $\square$

Example 6.1. Take $-2^i(i = \pm1, \pm2, \pm3, \pm4)$ and -1 with $r = m_1 = m_2 = 2$ as eigenvalues. Let $X_{\hat{v}}$ be of the form in (6.1) with $\Theta_i = I_2 (i = 1, 2, 3)$ and P be the real random orthogonal matrix (`gallery('randsvd',8)` in MATLAB). As suggested in [2], it would be reasonable to stop the iteration of the CR when $\|S_k - S_{k-1}\|/\|S_k\| < 10^{-7}$, and take S_k as an approximation to the exact $-S^{(2)}A$. Further iterations may not be able to improve the accuracy significantly in view of that B_k are nearly singular for large k. Table 6.1 shows the average iterations and the maximum, minimum 1-norm of A_k and C_k over 20 quadratics for which the stop criterion is satisfied. The third row in Table 6.1 listed the corresponding results of a version of the CR algorithm to balance $\|A_k\|$ and $\|C_k\|$ (BCR) [8].

Table 6.1 Average iterations and the maximum, minimum 1-norm of A_k and C_k performed by the CR and its balanced version BCR

	averiter	maxnormA	minnormA	maxnormC	minnormC
CR	20.5	30.1370	4.2443	222.5990	79.1581
BCR	20.8	67.1519	22.9727	67.1519	22.9727

We note from Table 6.1 that minimum 1-norm of A_k and C_k does not converge to zero. In fact, they are found between 4.2518 and 4.2443 for A_k and 79.3675 and 79.1581 for C_k from $k = 5$ to the termination. This shows that for WO quadratics $\widetilde{Q}(\lambda)$ in (4.4), the convergence behavior of the CR is largely dictated by the results in Theorem 4.1.

7. Concluding Remarks

We have established the convergence of the CR algorithm for a class of weakly overdamped quadratics. This convergence theory does not require the assumption that the partial multiplicities of λ_n are all equal to 2. Also, the matrices sequences $\{A_k\}$ and $\{C_k\}$ generated by the CR algorithm, unlike Guo, Higham and Tisseur's convergence theorem in [6], no longer converge to the zero matrix if the partial multiplicities of λ_n include 1. Therefore, our derived result can be seen as a complement to the convergence of the CR algorithm. However, the behavior of the CR algorithm for general weakly overdamped quadratics without any assumption on the partial multiplicity of λ_n still deserves more research.

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Appendix

Proof of Lemma 5.2

We can utilize Definition 2.1 to find Jordan chains of the diagonal quadratic $Q_d(\lambda)$ corresponding to each eigenvalue. Let $\lambda_0 = -\sigma_t$ for $m_1 + l \geq t \geq 1$. In view of (2.1) for $k = 0$, we have $Q_d(-\sigma_t)\phi^{(-\sigma_t)} = 0$. Solving the system for $\phi^{(-\sigma_t)}$, we have

$$\phi^{(-\sigma_t)} = [\overbrace{0, \cdots, 0}^{t-1}, \phi^{(-\sigma_t)}_{(0,t)}, 0, \cdots, 0]^T, \tag{A1}$$

where $\phi^{(-\sigma_t)}_{(0,t)}$ can be an arbitrary nonzero real scalar. A further computation by (2.1) for $k = 1$ shows that $\phi^{(-\sigma_t)} = 0$. Hence the length of the Jordan chain corresponding to $-\sigma_t(m_1 + l \geq t \geq 1)$ is 1. Similarly, we can obtain the Jordan chain

$$\phi^{(-\delta_s)} = [\overbrace{0, \cdots, 0}^{m_1+s-1}, \phi^{(-\delta_s)}_{(0,s)}, 0, \cdots, 0]^T \tag{A2}$$

with the length 1 corresponding to $-\delta_s(m_2 + l \geq s \geq 1)$, where $\phi^{(-\delta_s)}_{(0,s)}$ can be an arbitrary nonzero real scalar. Finally we find the Jordan chains of $Q_d(\lambda)$ corresponding to -1. From the equation $Q_d(-1)\phi^{(-1)}_0 = 0$, we get

$$\begin{aligned}\phi^{(-1)}_0 = [&\phi^{(-1)}_{(0,1)}, \cdots, \phi^{(-1)}_{(0,m_1)}, \overbrace{0, \cdots, 0}^{l}, \phi^{(-1)}_{(0,m_1+l+1)}, \cdots, \phi^{(-1)}_{(0,m_1+l+m_2)},\\ &\phi^{(-1)}_{(0,m_1+l+m_2+1)}, \cdots, \phi^{(-1)}_{(0,n)}]^T\end{aligned} \tag{A3}$$

with arbitrary real constants $\phi^{(-1)}_{(0,j)}$ not all equal zero for $j = 1, \cdots, m_1, m_1 + l + 1, \cdots, n$. Note that $\phi^{(-1)}_0$ in (A3) indeed includes $m_1 + m_2 + r$ linearly independent vectors (see also in (4.2)) corresponding to -1. Let $k = 1$ in (2.1) and solve the system $Q'_d(-1)\phi^{(-1)}_0 + Q_d(-1)\phi^{(-1)}_1 = 0$ for $\phi^{(-1)}_0$ (based on (A3)) and $\phi^{(-1)}_1$. We obtain

$$\phi^{(-1)}_0 = [\overbrace{0, \cdots, 0}^{m_1+l+m_2}, \phi^{(-1)}_{(0,m_1+l+m_2+1)}, \cdots, \phi^{(-1)}_{(0,n)}]^T, \tag{A4}$$

and

$$\begin{aligned}\phi^{(-1)}_1 = [&\phi^{(-1)}_{(1,1)}, \cdots, \phi^{(-1)}_{(1,m_1)}, \overbrace{0, \cdots, 0}^{l}, \phi^{(-1)}_{(1,m_1+l+1)}, \cdots, \phi^{(-1)}_{(1,m_1+l+m_2)},\\ &\phi^{(-1)}_{(1,m_1+l+m_2+1)}, \cdots, \phi^{(-1)}_{(1,n)}]^T\end{aligned} \tag{A5}$$

with arbitrary real constants $\phi_{(1,j)}^{(-1)}$ for $j = 1, \cdots, m_1, m_1 + l + 1, \cdots, n$. An analogous computations to

$$\frac{1}{2} Q_d''(-1)\phi_0^{(-1)} + Q_d'(-1)\phi_1^{(-1)} + Q_d(-1)\phi_2^{(-1)} = 0$$

gives $\phi_0^{(-1)} = 0$. Thus the Jordan chain of the eigenvalue -1 terminated at the length 2.

Arranging the vectors (A1)$-$(A5) according to J_v, we obtain the structure of $X_{\hat{v}}$ as in (5.12), where Φ_{m_1}, Φ_l, Ψ_l and Ψ_{m_2} are $m_1 \times m_1$, $l \times l$, $l \times l$ and $m_2 \times m_2$ nonsingular diagonal matrices, respectively, with nonzero entries corresponding to (A1) and (A2). Note that $\phi_0^{(-1)}$ in (A4) includes r linearly independent eigenvectors corresponding to -1. Thus Φ_r is an $r \times r$ nonsingular matrix. $\Phi_{jr}(j = 1, 2, 3)$, Φ_k and $\Psi_k (k = 0, 1)$ are all arbitrary real matrices with the respective dimension $m_1 \times r$, $m_2 \times r$, $r \times r$, $m_1 \times m_1$, $r \times m_2$, $m_2 \times m_1$ and $r \times m_1$. Without loss of the generality, we can assume that Φ_{m_2} and Ψ_{m_1} are $m_2 \times m_2$ and $m_1 \times m_1$ nonsingular matrices. In fact, an appropriate permutation to vectors formed by (A3) always gives such nonsingular Φ_{m_2} and Ψ_{m_1}, since we have mentioned that (A3) includes $m_1 + m_2 + r$ linearly independent vectors corresponding to -1. Similarly, rearranging the vectors (A1)$-$(A5) according to J_w can yield (5.13). $\square$

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