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Symplectic Euler Method for Nonlinear High Order Schrödinger Equation with a Trapped Term

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Abstract. In this paper, we establish a family of symplectic integrators for a class of high order Schrödinger equations with trapped terms. First, we find its symplectic structure and reduce it to a finite dimensional Hamilton system via spatial discretization. Then we apply the symplectic Euler method to the Hamiltonian system. It is demonstrated that the scheme not only preserves symplectic geometry structure of the original system, but also does not require to resolve coupled nonlinear algebraic equations which is different with the general implicit symplectic schemes. The linear stability of the symplectic Euler scheme and the errors of the numerical solutions are investigated. It shows that the semi-explicit scheme is conditionally stable, first order accurate in time and $2l^{th}$ order accuracy in space. Numerical tests suggest that the symplectic integrators are more effective than non-symplectic ones, such as backward Euler integrators.

AMS subject classifications: 65M06, 65M12, 65Z05, 70H15
Key words: Symplectic Euler integrator, high order Schrödinger equation, stability, trapped term.

1 Introduction

In this paper, we consider a class of high order nonlinear Schrödinger equation with a trapped term (HNSET)

$$iu_t + (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}} + \hbar'(|u|^2)u + \beta g(x)u = 0, \tag{1.1}$$

together with the prescribed initial and periodic boundary conditions

$$u(x,0) = u_0(x), \qquad x \in [0, L], \tag{1.2}$$

$$\frac{\partial^s u(x,t)}{\partial x^s} = \frac{\partial^s u(x+L,t)}{\partial x^s}, \quad t \in [0,T], \quad s = 0,1,2,\cdots,m-1, \tag{1.3}$$

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where $i=\sqrt{-1}$, $m\in\mathbb{N}$, β is a given constant, $\hbar(|u|^2)$ is a bounded real differentiable function of $|u(x,t)|^2\in\mathbb{R}$, and $g(x)$ is a real-valued bounded function with period L, $u_0(x)$ is a given complex-valued function. The trapped term $g(x)$ is to position the solutions near $x=0$. The initial-boundary value problem (1.1)-(1.3) admits at least two conserved quantities:

(1) The charge is invariant

$$\mathcal{Q}(t)=\int_0^L |u(x,t)|^2dx=\int_0^L |u_0(x)|^2dx=\mathcal{Q}(0); \tag{1.4}$$

(2) The energy is conserved

$$\mathcal{E}(t)=\int_0^L \Big(\Big|\frac{\partial^m u(x,t)}{\partial x^m}\Big|^2+\hbar(|u(x,t)|^2)+\beta g(x)|u(x,t)|^2\Big)dx=\mathcal{E}(0). \tag{1.5}$$

Symplectic geometric integrators have been very popular for the Hamiltonian system since the method was proposed by Feng in [1–3]. It has been applied to many practical physics problems, such as quantum mechanics [4]. Eq. (1.1) can be reformulated into Hamiltonian system. In fact, let

$$u(x,t)=p(x,t)+iq(x,t),$$

where $p(x,t)$, $q(x,t)$ are real-valued functions, we have

$$\begin{cases} p_t=-\Big((-1)^m\dfrac{\partial^{2m}q}{\partial x^{2m}}+\hbar'(p^2+q^2)q+\beta g(x)q\Big),\\ q_t=(-1)^m\dfrac{\partial^{2m}p}{\partial x^{2m}}+\hbar'(p^2+q^2)p+\beta g(x)p. \end{cases} \tag{1.6}$$

Suppose

$$z(x,t)=\big[p(x,t),q(x,t)\big]^T.$$

The infinite dimensional Hamiltonian system (1.6) can be written into the standard Hamiltonian system

$$\frac{d}{dt}z=J^{-1}Az, \tag{1.7}$$

where

$$J^{-1}=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix},$$

and

$$A=(-1)^m\begin{bmatrix} \triangle^{2m} & 0\\ 0 & \triangle^{2m}\end{bmatrix}+\Big(\frac{\partial}{\partial|u|^2}\hbar(|u|^2)+\beta g(x)\Big)I_2,$$

with the identity matrix I_2 and the $2m^{th}$ order differential operator $\triangle^{2m}$. The Hamiltonian function is

$$H(p,q)=\frac{1}{2}\int_0^L\Big[\Big(\frac{\partial^m p}{\partial x^m}\Big)^2+\Big(\frac{\partial^m q}{\partial x^m}\Big)^2+\hbar(p^2+q^2)+\beta g(x)(p^2+q^2)\Big]dx.$$