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Analytical Solution for the Lattice Boltzmann Model Beyond Naviers-Stokes

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Abstract. To understand lattice Boltzmann model capability for capturing non-equilibrium effects, the model with first-order expansion of the equilibrium distribution function is analytically investigated. In particular, the velocity profile of Couette flows is exactly obtained for the D2Q9 model, which shows retaining the first order expansion can capture rarefaction effects in the incompressible limit. Meanwhile, it clearly demonstrates that the D2Q9 model is not able to reflect flow characteristics in the Knudsen layer.

AMS subject classifications: 76P05, 82B40

Key words: Lattice Boltzmann method, rarefied gas dynamics, micro flows, Knudsen layer.

1 Introduction

Due to rapid development of micro/nano-technologies and modern material processing techniques such as laser fabrication processing and plasma etching [9, 10, 12], the research interest in rarefied gas dynamics has shifted to low-speed flows under the standard ambient temperature and pressure. For non-equilibrium flows, the linear constitutive relation for stress, which is assumed in the Navier-Stokes equation, is no longer valid. Therefore, kinetic methods or extended hydrodynamic models have to be employed, e.g., the direct simulation Monte Carlo (DSMC) method, and Grad 13 moment model. However, the DSMC simulations is computationally expensive, especially for slow microflows with small Knudsen number. Meanwhile, the direct solution of the Boltzmann equation is still very complex due to the collisional integral. The extended hydrodynamic models are only applicable to the near hydrodynamic regime.

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The lattice Boltzmann (LB) framework can be served as an alternative computationally efficient method for non-equilibrium gas flows. It was originally developed for hydrodynamics and is proved to be a viable numerical tool [1,4,5,17,25]. Compared to the traditional kinetic theory, the LB framework can be efficient since it utilizes a minimal set of velocities in the phase space [5]. Therefore, significant efforts have been devoted to develop or examine the capability of LB models for finite Knudsen number flows, e.g., [1,2,6,11,13,21,22,24,26–30]. It was shown that the LB model with discrete velocity set derived from high-order Gauss Hermite quadratures can provide a computationally efficient way of solving the Boltzmann model equation. It can asymptotically recover the Bhatnagar-Gross-Krook (BGK) equation. With the first order approximation of the equilibrium distribution function, it is equivalent to discrete velocity model (DVM) approach of solving the linearized BGK (LBGK) equation [16]. Therefore, the corresponding LB model can capture non-equilibrium effects.

In this work, we will analytically investigate the capability of LB model for non-equilibrium flows. With the first order expansion, the governing equations for distribution function can be great simplified so that they can be solved directly by using available mathematical techniques [14,19,20]. In particular, the exact velocity profile of Couette flows will be obtained for the so-called D2Q9 model [18].

2 Lattice Boltzmann model

LB models can be constructed by utilizing the Gauss-Hermite quadratures [7,8,15,23,24]. The Boltzmann-BGK equation is

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f + \boldsymbol{g} \cdot \nabla_{\xi} f = -\frac{p}{\mu}\left(f - f^{eq}\right), \tag{2.1}$$

where f denotes the distribution function; $\boldsymbol{\xi}$, the phase velocity; p, the pressure; $\boldsymbol{g}$, the body force; and μ, the gas viscosity. To examine rarefaction effects, it is convenient to use the following non-dimensional variables

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{L}, \qquad \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\sqrt{RT_0}}, \qquad \hat{t} = \frac{\sqrt{RT_0}t}{L}, \tag{2.2a}$$

$$\hat{\boldsymbol{g}} = \frac{L\boldsymbol{g}}{RT_0}, \qquad \hat{\boldsymbol{\xi}} = \frac{\boldsymbol{\xi}}{\sqrt{RT_0}}, \qquad \hat{T} = \frac{T}{T_0}, \tag{2.2b}$$

where $\boldsymbol{u}$ is the macroscopic velocity; R, the gas constant; T, the gas temperature; T_0, the reference temperature; $\boldsymbol{r}$, the spatial position; and L, the characteristic length of the flow system. The symbol *hat*, which denotes dimensionless value, will hereinafter be omitted. The Knudsen number can be defined by using macroscopic properties as

$$Kn = \frac{\mu\sqrt{RT_0}}{pL}. \tag{2.3}$$