Analysis in Theory and Applications
Anal. Theory Appl., Vol. **29**, No. 3 (2013), pp. 255-266

DOI: 10.4208/ata.2013.v29.n3.5

Common Fixed Points for a Countable Family of Quasi-Contractive Mappings on a Cone Metric Space with the Convex Structure

Yongjie Piao* and Hua Nan

Department of mathematics, College of Science, Yanbian University, Yanji 133002, China

Received 9 January 2013; Accepted (in revised version) 19 April 2013

Available online 30 September 2013

Abstract. In this paper, we consider a countable family of surjective mappings $\{T_n\}_{n\in\mathbb{N}}$ satisfying certain quasi-contractive conditions. We also construct a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ by the quasi-contractive conditions of $\{T_n\}_{n\in\mathbb{N}}$ and the boundary condition of a given complete and closed subset of a cone metric space X with convex structure, and then prove that the unique limit x^* of $\{x_n\}_{n\in\mathbb{N}}$ is the unique common fixed point of $\{T_n\}_{n\in\mathbb{N}}$. Finally, we will give more generalized common fixed point theorem for mappings $\{T_{i,j}\}_{i,j\in\mathbb{N}}$. The main theorems in this paper generalize and improve many known common fixed point theorems for a finite or countable family of mappings with quasi-contractive conditions.

Key Words: Common fixed point, the convex property, cone metric space.

AMS Subject Classifications: 47H05, 47H10

1 Introduction

Huang and Zhang [1] recently have introduced the concept of cone metric spaces, where the set of real number is replaced by an ordered Banach space, and they have established some fixed point theorems for a contractive type mappings in a normal cone metric space. Subsequently, some other authors [2–7] have generalized the results of Huang and Zhang [1] and have studied the existence of common fixed points of a finite self mappings satisfying a contractive type condition in the framework of normal or non-normal cone metric spaces. In [8], the authors discussed some common fixed point problems for a pair of non-self mappings defined on a nonempty closed subset of a non-normal cone metric space, the results improved and generalized many similar common fixed point theorems for a pair of self mappings on a cone metric space. On the other hand, the author recently

*Corresponding author. *Email addresses:* `pyj6216@hotmail.com` (Y. J. Piao), `nanhua@ybu.edu.cn` (H. Nan)

have discussed and obtained some unique existence theorems of common fixed points for a countable family of contractive or quasi-contractive mappings on 2-metric spaces or metrically convex metric spaces respectively, see [9–12].

In this paper, we will give some common fixed point theorems for a countable family of surjective quasi-contractive mappings defined on a non-normal cone metric space with the convex structure. Our main results improve and generalize many known common fixed point theorems.

Let E be a real Banach space. A subset P_0 of E is called a cone if and only if:

(i) P_0 is closed, nonempty, and $P_0 \neq \{0\}$;

(ii) $a,b \in \mathbb{R}$, $a,b \geq 0$ and $x,y \in P_0$ implies $ax+by \in P_0$;

(iii) $P_0 \cap (-P_0) = \{0\}$.

Given a cone $P_0 \subset E$, we define a partial ordering $\leq$ on E with respective to P_0 by $x \leq y$ if and only if $y-x \in P_0$. We will write $x<y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in \mathrm{int}P_0$ (interior of P_0).

The cone P_0 is called normal if there is a number $L>0$ such that for all $x,y \in E$,

$$0 \leq x \leq y \ \text{ implies } \ \|x\| \leq L\|y\|.$$

The least positive number satisfying the above is then called the normal constant of P_0.

Let X be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies

(d1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y)=0$ if and only if $x=y$;

(d2) $d(x,y)=d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \leq d(x,z)+d(z,y)$, for all $x,z,y \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space. The concept of a cone metric space is more general than a metric space.

Let (X,d) be a cone metric space. We say that $\{x_n\}$ is

(e) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $n,m>N$, $d(x_m,x_n) \ll c$;

(f) convergent sequence if for every $c \in E$ with $0 \ll c$, there is an N such that for all $n>N$ such that $d(x_n,x) \ll c$ for some $x \in E$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X .

A cone metric space X is said to have the convex structure if for each nonempty closed subset C of X and each $x \in C$ and $y \notin C$, there exists a point $z \in \partial C$ such that

$$d(x,z)+d(z,y)=d(x,y).$$

A metric space is said to be metrically convex [13, 14], if for any $x,y \in X$ with $x \neq y$, there exists a point $z \in X$ such that $d(x,z)+d(z,y)=d(x,y)$.