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Renormalized Solutions for Nonlinear Parabolic Systems with Three Unbounded Nonlinearities in Weighted Sobolev Spaces

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Abstract. We prove an existence result without assumptions on the growth of some nonlinear terms, and the existence of a renormalized solution. In this work, we study the existence of renormalized solutions for a class of nonlinear parabolic systems with three unbounded nonlinearities, in the form

$$\begin{cases} \dfrac{\partial b_1(x,u_1)}{\partial t} - div\big(a(x,t,u_1,Du_1)\big) + div\big(\Phi_1(u_1)\big) + f_1(x,u_1,u_2) = 0 & \text{in } Q, \\ \dfrac{\partial b_2(x,u_2)}{\partial t} - div\big(a(x,t,u_2,Du_2)\big) + div\big(\Phi_2(u_2)\big) + f_2(x,u_1,u_2) = 0 & \text{in } Q, \end{cases}$$

in the framework of weighted Sobolev spaces, where $b(x,u)$ is unbounded function on u, the Carathéodory function a_i satisfying the coercivity condition, the general growth condition and only the large monotonicity, the function ϕ_i is assumed to be continuous on $\mathbb{R}$ and not belong to $(L^1_{loc}(Q))^N$.

Key Words: Nonlinear parabolic system, existence, truncation, weighted Sobolev space, renormalized solution.

AMS Subject Classifications: 35K45, 35K61, 35K65

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2<p<\infty$, $Q=\Omega\times[0,T]$ and $w=\{w_i(x): 0\le i\le N\}$ be a vector of weight functions (i.e., every component

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$w_i(x)$ is a measurable almost everywhere strictly positive function on Ω), satisfying some integrability conditions (see Section 2).We prove the existence of a renormalized solution for the nonlinear parabolic systems

$$\frac{\partial b_i(x,u_i)}{\partial t} - div\left(a(x,t,u_i,Du_i)\right) + div\left(\Phi_i(u_i)\right) + f_i(x,u_1,u_2) = 0 \quad \text{in } Q, \tag{1.1a}$$

$$u_i = 0 \quad \text{on } \Gamma = (0,T)\times\partial\Omega, \tag{1.1b}$$

$$b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) \quad \text{in } \Omega, \tag{1.1c}$$

in the framework of weighted Sobolev spaces, where $i=1,2$. $b_i:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and The data $b_i(x,u_{i0})$ lie in $L^1(\Omega)$. The functions ϕ is just assumed to be continnous of $\mathbb{R}$ with values in $\mathbb{R}^N$. The operator $div(a(x,t,u,Du))$ is a Leray-Lions operator which is coercive, and which grows like $|Du|^{p-1}$ with respect to $|Du|$, but which is not restricted by any growth condition with respect to u and only the large monotonicity (see Assumption (H4)) and $b(x,u)$ is unbounded function on u. such that for every $x\in\Omega$, $b_i(x,\cdot)$ is a strictly increasing C^1-function with $b_i(x,0)=0$.

Moreover, we suppose that for $i=1,2$, $f_i:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (see Assumption (H5)).

The main difficulty when dealing with problem (1.1a)-(1.1c) is due to the fact that the functions $a(x,t,u_i,Du_i)$, $\Phi_i(u_i)$ and $f_i(x,u_1,u_2)$ are not in $(L^1_{loc}(Q))^N$ in general, since the growth of $a(x,u_i,Du_i)$, $\Phi_i(u_i)$ and $f_i(x,u_1,u_2)$ are not controlled with respect to u_i, so that proving existence of a weak solution (i.e., in the distribution meaning) seems to be an arduous task. To overcome this difficulty, we use in this paper the framework of renormalized solutions due to Lions and DiPerna [28] for the study of Boltzmann equations (see also Lions [29] for a few applications to fluid mechanics models). This notion was then adapted to the elliptic version of (1.1a)-(1.1c) in Boccardo, Diaz, Giachetti, Murat [16], in Lions and Murat [30] .

The particular case where $b_i(x,u_i)=b_i(u_i)$, $i=1,2$ has been studied in Redwane [5] and for the parabolic version in classical Sobolev spaces of (1.1a)-(1.1c), existence and uniqueness results are already proved in [8] (see also [33] and [31]) in the case where $f_i(x,u_1,u_2)$ is replaced by $f+div(g)$ where $f\in L^1(Q)$ and $g\in L^{p'}(Q)^N$.

In the case where $a(x,t,s,\xi)$ is independent of s, $\Phi_i=0$ and $g=0$, existence and uniqueness are established in [6]; in [7] and in the case where $a(x,t,s,\xi)$ is independent of s and linear with respect to ξ, existence and uniqueness are established in [10].

This article is organized as follows: in Section 3, we specify the notation and give the definition of a renormalized solution of (1.1a)-(1.1c). Then, in Section 6, we establish the existence of such a solution (see Theorem 6.1).

2 Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2<p<\infty$ and $w=\{w_i(x), 0\le i\le N\}$ be a vector of weight functions; i.e., every component $w_i(x)$ is a measurable