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Fixed Point Theory for 1-Set Weakly Contractive Operators in Banach Spaces

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Abstract. In this work, using an analogue of Sadovskii's fixed point result and several important inequalities we investigate and give new existence theorems for the nonlinear operator equation $F(x)=\mu x$, $(\mu\geq 1)$ for some weakly sequentially continuous, weakly condensing and weakly 1-set weakly contractive operators with different boundary conditions. Correspondingly, we can obtain some applicable fixed point theorems of Leray-Schauder, Altman and Furi-Pera types in the weak topology setting which generalize and improve the corresponding results of [3,15,16].

Key Words: Weakly condensing, weakly sequentially continuous, fixed point theorem, operator equation.

AMS Subject Classifications: 47H10, 47J05, 47J10

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1 Introduction

The Study of fixed point problem for condensing operators and 1-set contractive operators especially for a closed convex subset into itself has been one of the main objects of research in nonlinear functional analysis and was started by [19], Petryshyn [17,18], Nussbaum [14] and Browder [4]. These studies were mainly based on the potential tool of degree theory. Since then, whether a 1-set contractive mapping defined on the closure of bounded open subset of a Banach space has a fixed point, has become an interesting problem. For example, in [12] the author has defined the fixed point index of 1-set-contractive operators, introduced the concept of semi-closed 1-set-contractive operators and obtained some fixed point theorems of such a class of operators. Recently, Some existence theorems for fixed points of semi-closed 1-set-contractive type mappings

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and under several boundary conditions related to degree theory have been considered by some authors (see [13,21,23]). Because the weak topology is the convenient and natural setting to investigate the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned results cannot be easily applied. These equations can be transformed into fixed point problems and nonlinear operator equations involving a broader class of nonlinear operators, in which the operators have the property that the image of any set in a certain sense more weakly compact than the original set itself. The major problem to face is that an infinite dimensional Banach space equipped with its weak topology does not admit open bounded sets. That is, a weakly closed and bounded subset has a weak interior and thus coincides with its weak boundary which yields very difficult the verification of the boundary conditions. To this interest, we introduce the concept of weakly semi-closed operators at the origin (see Definition 2.5), and provide new existence theorems for the nonlinear operator equation $F(x)=\mu x$ $(\mu\geq 1)$, for some class of weakly sequentially continuous, weakly condensing (β-condensing) and 1-set weakly contractive (β-nonexpansive) operators (see Definition 2.1) under several boundary conditions in particular of Furi-Pera type [9] and under weakest assumptions on operators and domains as it is known (here β is the DeBlasi measure of weak noncompactness see [5]). Meanwhile analogues of fixed point theorems of Altman, Leray-Schauder and Furi-Pera types are given in the weak topology setting which generalize and extend relevant and recent ones (see [3,15,16]). In addition, our arguments and methods are elementary in the sense that without any recourse to degree theory or theory of homotopy-extensions.

2 Preliminaries

Let Ω_E be the family of bounded subsets of a Banach space E and let $\mathcal{K}^w$ be the family of weakly compact subsets of E. Also let B_E be the closed unit ball of E. The DeBlasi [5] measure of weak noncompactness is the map $\beta:\Omega_E\to[0,\infty)$ defined by

$$\beta(X)=\inf\big\{t>0:\ \text{there exists } Y\in\mathcal{K}^w \text{ such that } X\subset Y+tB_E\big\},\quad \text{here } X\in\Omega_E.$$

For convenience we recall some properties of β: Let X_1, $X_2\in\Omega_E$. Then

(i) $X_1\subset X_2$ implies $\beta(X_1)\leq\beta(X_2)$.

(ii) $\beta(X_1)=0$ iff $\overline{X_1^w}\in\mathcal{K}^w$, here $\overline{X_1^w}$ is the weak closure of X_1 in E.

(iii) $\beta(X_1)=\beta(\overline{X_1^w})$.

(vi) $\beta(X_1\cup X_2)=\max\{\beta(X_1),\beta(X_2)\}$.

(v) $\beta(\lambda X_1)=\lambda\beta(X_1)$ for all $\lambda>0$.

(vi) $\beta(\mathrm{conv}(X_1))=\beta(X_1)$.

(vii) $\beta(X_1+X_2)\leq\beta(X_1)+\beta(X_2)$.