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Classical Fourier Analysis over Homogeneous Spaces of Compact Groups

Arash Ghaani Farashahi*

Numerical Harmonic Analysis Group (NuHAG), Faculty Mathematics, University of Vienna, Vienna, Austria.

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Abstract. This paper introduces a unified operator theory approach to the abstract Fourier analysis over homogeneous spaces of compact groups. Let G be a compact group and H be a closed subgroup of G. Let G/H be the left coset space of H in G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then, we present a generalized abstract framework of Fourier analysis for the Hilbert function space $L^2(G/H,\mu)$.

Key Words: Compact group, homogeneous space, dual space, Fourier transform, Plancherel (trace) formula, Peter-Weyl Theorem.

AMS Subject Classifications: 20G05, 43A85, 43A32, 43A40, 43A90

1 Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or precisely left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis [2–4, 12], theoretical and particle physics [1, 9–11, 13]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics and scientific computing (computational engineering), see [6, 7, 13–18] and references therein.

Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. The left coset space G/H is considered as a compact homogeneous space, which G acts on it via the left action. This paper which contains 5 sections, is organized as follows. Section 2 is devoted to fix notations and preliminaries including a brief summary on Hilbert-Schmidt operators,

*Corresponding author. *Email addresses:* `arash.ghaani.farashahi@univie.ac.at`, `ghaanifarashahi@hotmail.com` (A. Ghaani Farashahi)

non-Abelian Fourier analysis over compact groups, and classical results on abstract harmonic analysis over locally compact homogeneous spaces. We present some abstract harmonic analysis aspects of the Hilbert function space $L^2(G/H,\mu)$, in Section 3. Then we define the abstract notion of dual space $\widehat{G/H}$ for the homogeneous space G/H and we will show that this definition is precisely the standard dual space for the compact quotient group G/H, when H is a closed normal subgroup of G. We then introduce the definition of abstract operator-valued Fourier transform over the Banach function space $L^1(G/H,\mu)$ and also generalized version of the abstract Plancherel (trace) formula for the Hilbert function space $L^2(G/H,\mu)$. The paper closes by a presentation of Peter-Weyl Theorem for the Hilbert function space $L^2(G/H,\mu)$.

2 Preliminaries and notations

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T\in\mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k\|Te_k\|^2<\infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$ and for $T\in\mathrm{HS}(\mathcal{H})$ the Hilbert-Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2=\sum_k\|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$ and if $\mathcal{H}$ is finite-dimensional we have $\mathrm{HS}(\mathcal{H})=\mathcal{B}(\mathcal{H})$. An operator $T\in\mathcal{B}(\mathcal{H})$ is trace-class, whenever $\|T\|_{\mathrm{tr}}=\mathrm{tr}[|T|]<\infty$, if $\mathrm{tr}[T]=\sum_k\langle Te_k,e_k\rangle$ and $|T|=(TT^*)^{1/2}$ [20].

Let G be a compact group with the probability Haar measure dx. Then each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations, see [1,10]. The set of of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, since each character of an Abelian group is a one dimensional representation of G. If π is any unitary representation of G, for $\zeta,\xi\in\mathcal{H}_\pi$ the functions $\pi_{\zeta,\xi}(x)=\langle\pi(x)\zeta,\xi\rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The notation $\mathcal{E}_\pi$ is used for the linear span of the matrix elements of π and the notation $\mathcal{E}$ is used for the linear span of $\bigcup_{[\pi]\in\widehat{G}}\mathcal{E}_\pi$. Then Peter-Weyl Theorem [1,10] guarantees that if G is a compact group, $\mathcal{E}$ is uniformly dense in $\mathcal{C}(G)$, $L^2(G)=\bigoplus_{[\pi]\in\widehat{G}}\mathcal{E}_\pi$, and $\{d_\pi^{-1/2}\pi_{ij}:i,j=1,\cdots,d_\pi,[\pi]\in\widehat{G}\}$ is an orthonormal basis for $L^2(G)$. For $f\in L^1(G)$ and $[\pi]\in\widehat{G}$, the Fourier transform of f at π is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\widehat{f}(\pi)=\int_G f(x)\pi(x)^*dx. \tag{2.1}$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x))\in\mathbb{C}^{d_\pi\times d_\pi}$. Then $\widehat{f}(\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$ is the matrix with entries given by $\widehat{f}(\pi)_{ij}=d_\pi^{-1}c_{ji}^\pi(f)$ which satisfies

$$\sum_{i,j=1}^{d_\pi}c_{ij}^\pi(f)\pi_{ij}(x)=d_\pi\sum_{i,j=1}^{d_\pi}\widehat{f}(\pi)_{ji}\pi_{ij}(x)=d_\pi\mathrm{tr}[\widehat{f}(\pi)\pi(x)],$$