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Closed Range Composition Operators on a General Family of Function Spaces

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Abstract. In this paper, necessary and sufficient conditions for a closed range composition operator C_ϕ on the general family of holomorphic function spaces $F(p,q,s)$ and more generally on α-Besov type spaces $F(p,\alpha p-2,s)$ are given. We give a Carleson measure characterization on $F(p,\alpha p-2,s)$ spaces, then we indicate how Carleson measures can be used to characterize boundedness and compactness of C_ϕ on $F(p,q,s)$ and $F(p,\alpha p-2,s)$ spaces.

Key Words: Composition operators, $F(p,q,s)$ spaces, closed range, Carleson measure, Bloch space, Bergman type space.

AMS Subject Classifications: 47B33, 47B38, 30H25, 30H30

1 Introduction

Let ϕ be a holomorphic self-map of the unit disk $\mathbb{D}$. Associate to ϕ the composition operator C_ϕ is defined by $C_\phi = f\circ\phi$, for any function f that is holomorphic on $\mathbb{D}$.

This is the first setting that composition operators were studied boundedness, compactness, closed range have been studied in this setting. It is natural to study these properties on other function spaces.

In the early 70s, Cima, Thomson and Wogen in [7] were the first to study closed range composition operators, in the context of H^2 on $\mathbb{D}$. Their results are in terms of the boundary behaviour of the symbol ϕ. They asked the question of studying closed range composition operators in terms of properties of ϕ on $\mathbb{D}$ rather than on $\partial\mathbb{D}$. Zorboska in [23] answered the call and studied the problem in H^2 and also in weighted Bergman spaces. Jovovic and MacCluer in [10] studied the problem in weighted Dirichlet spaces; the closed range composition operators on Dirichlet-type spaces $D(\mu)$ (with μ is a positive Borel measure defined on the boundary of the unit disc) were introduced by Chacón in [6]. Also Ghatage, Zheng and Zorboska studied the problem in the Bloch

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space (see [9]), then Akeroyd and Ghatage revisited the problem in the context of the Bergman space (see [3]). Tjani in [16] studied the closed range composition operators on Besov spaces B_p and more generally on α-Besov spaces $B_{p,\alpha}$.

In this paper we study closed range composition operators on α-Besov type spaces $F(p,\alpha p-2,s)$ for $p>2$, and $\alpha,s>0$ with $\alpha p+s>1$. We will define and discuss properties of these spaces in Section 2, also we indicate how Carleson measures can be used to characterize boundedness of C_ϕ on $F(p,\alpha p-2,s)$, and we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,\alpha p-2,s)$.

In Section 3, we concetrate on the spaces $F(p,q,s)$ for $p>2$, $q>-2$ and $s>0$ with $q+s>-1$. Let ϕ be a boundedly valent holomorphic self map of $\mathbb{D}$; we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,q,s)$. We will show that, assuming that C_ϕ is bounded on $F(p,q,s)$, if C_ϕ is closed range on the Bloch space, then it is also closed range on $F(p,q,s)$. Moreover we will show that if ϕ is a boundedly valent holomorphic self map of $\mathbb{D}$ then C_ϕ is closed range on $F(p,q,s)$ if and only if it is closed range on Bloch space.

In Section 4, it has results on $F(p,\alpha p-2,s)$ spaces. Let ϕ be a boundedly valent holomorphic self map of $\mathbb{D}$; again we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,\alpha p-2,s)$. Moreover assuming that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$, if $\phi(\mathbb{D})$ or certain subsets of it contain an outer annulus then C_ϕ is closed range on $F(p,\alpha p-2,s)$.

Let C and K denote a positive and finite constants which may change from one occurrence to the next but will not depend on the functions involved.

Two quantities A_f and B_f, both depending on $f\in\mathcal{H}(\mathbb{D})$, are said to be equivalent, written as $A_f\approx B_f$, if there exists $C>0$ such that $\frac{1}{C}B_f\le A_f\le CB_f$, for every function $f\in\mathcal{H}(\mathbb{D})$. If the quantities A_f and B_f, are equivalent, then in particular we have $A_f<\infty$ if and only if $B_f<\infty$.

2 Prerequisites

This introductory section is dedicated to setting up the notation and introducing the main concepts along with a collection of some fundamental facts required for what is to follow.

- The symbol $\mathbb{D}=\{z:|z|<1\}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ the unit circle.
- The symbol $\mathcal{H}(\mathbb{D})$ denote the family of functions holomorphic on $\mathbb{D}$.
- H^2 the Hilbert space of $\mathcal{H}(\mathbb{D})$ with square summable power series coefficients.
- The symbol A denote two-dimensional Lebesgue measure on $\mathbb{D}$, so that $A(\mathbb{D})\equiv 1$.
- For $a\in\mathbb{D}$ the Möbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z} \quad \text{for } z\in\mathbb{D}.$$