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On an Inequality of Pual Turan Concerning Polynomials-II

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Abstract. Let $P(z)$ be a polynomial of degree n and for any complex number α, let $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we obtain inequalities for the polar derivative of a polynomial having all zeros inside a circle. Our results shall generalize and sharpen some well-known results of Turan, Govil, Dewan et al. and others.

Key Words: Polar derivative, polynomials, inequalities, maximum modulus, growth.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then according to the well-known Bernstein's inequality [4] on the derivative of a polynomial, we have

$$\max_{|z|=1}|P'(z)|\le n\max_{|z|=1}|P(z)|. \tag{1.1}$$

The equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P(z)$ having all zeros in $|z|\le 1$, Turan [11] proved that

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and becomes equality for $P(z)=\alpha z^n+\beta$ where $|\alpha|=|\beta|$.

In the literature, there already exists some refinements and generalizations of the inequality (1.2), for example see Aziz and Dawood [3], Govil [5], Dewan and Mir [6], Dewan, Singh and Mir [7], Mir, Dar and Dawood [10] etc.

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Inequality (1.2) was refined by Aziz and Dawood [3] and they proved under the same hypothesis that

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{2}\Big\{\max_{|z|=1}|P(z)|+\min_{|z|=1}|P(z)|\Big\}. \tag{1.3}$$

As an extension of (1.3), it was shown by Govil [5], that if $P(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, then

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{1+k}\Big\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\Big\}. \tag{1.4}$$

For the class of polynomials

$$P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu},\quad 1\leq\mu\leq n,$$

of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, Aziz and Shah [2] proved

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{1+k^\mu}\Big\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=k}|P(z)|\Big\}. \tag{1.5}$$

For $\mu=1$, inequality (1.5) reduces to (1.4).

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to α, then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

Recently Dewan, Singh and Mir [7] besides proving some other results, also proved the following interesting generalization of (1.5).

Theorem 1.1. *If*

$$P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu},\quad 1\leq\mu\leq n,$$

is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, and δ is any complex number with $|\delta|\leq 1$, then for $|z|=1$,

$$|D_\delta P(z)|\leq n\Big(\frac{k^\mu+|\delta|}{1+k^\mu}\Big)\max_{|z|=1}|P(z)|-n\Big(\frac{1-|\delta|}{k^{n-\mu}(1+k^\mu)}\Big)\min_{|z|=k}|P(z)|. \tag{1.6}$$

In this paper, we shall first prove a result which gives certain generalizations of the inequality (1.4) by considering polynomials having all zeros in $|z|\leq k$, $k\leq 1$ with s-fold zeros at $z=0$. We shall also present a refinement of Theorem 1.1. We first prove the following result.