Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 2 (2015), pp. 138-153

DOI: 10.4208/ata.2015.v31.n2.4

Riesz Transforms Associated with Schrödinger Operators Acting on Weighted Hardy Spaces

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Received 23 June 2014; Accepted (in revised version) 25 March 2015

Abstract. Let $L=-\Delta+V$ be a Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n\geq 1$, where $V\not\equiv 0$ is a nonnegative locally integrable function on $\mathbb{R}^n$. In this article, we will introduce weighted Hardy spaces $H^p_L(w)$ associated with L by means of the square function and then study their atomic decomposition theory. We will also show that the Riesz transform $\nabla L^{-1/2}$ associated with L is bounded from our new space $H^p_L(w)$ to the classical weighted Hardy space $H^p(w)$ when $n/(n+1)<p<1$ and $w\in A_1\cap RH_{(2/p)'}$.

Key Words: Weighted Hardy space, Riesz transform, Schrödinger operator, atomic decomposition, A_p weight.

AMS Subject Classifications: 42B20, 42B30, 35J10

1 Introduction

Let $n\geq 1$ and V be a nonnegative locally integrable function defined on $\mathbb{R}^n$, not identically zero. We define the form $\mathcal{Q}$ by

$$\mathcal{Q}(u,v)=\int_{\mathbb{R}^n}\nabla u\cdot\nabla v dx+\int_{\mathbb{R}^n}Vuvdx$$

with domain $\mathcal{D}(\mathcal{Q})=\mathcal{V}\times\mathcal{V}$ where

$$\mathcal{V}=\Big\{u\in L^2(\mathbb{R}^n):\frac{\partial u}{\partial x_k}\in L^2(\mathbb{R}^n)\ \text{ for }\ k=1,\cdots,n,\ \text{ and }\ \sqrt{V}u\in L^2(\mathbb{R}^n)\Big\}.$$

It is well known that this symmetric form is closed. Note also that it was shown by Simon [27] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$ (the space of C^∞ functions with compact supports). In other words, $C_0^\infty(\mathbb{R}^n)$ is a core of the form $\mathcal{Q}$.

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Let us denote by L the self-adjoint operator associated with $\mathcal{Q}$. The domain of L is given by

$$\mathcal{D}(L)=\Big\{u\in\mathcal{V}:\exists v\in L^2(\mathbb{R}^n) \text{ such that } \mathcal{Q}(u,\varphi)=\int_{\mathbb{R}^n} v\cdot\bar{\varphi}dx,\ \ \forall\varphi\in\mathcal{V}\Big\}.$$

Formally, we write $L=-\Delta+V$ as a Schrödinger operator with the potential V. Let $\{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $p_t(x,y)$ be its kernel. Since V is a nonnegative function on $\mathbb{R}^n$, then the Feynman–Kac formula implies that

$$0\le p_t(x,y)\le\frac{1}{(4\pi t)^{n/2}}\exp\Big\{-\frac{|x-y|^2}{4t}\Big\} \tag{1.1}$$

for all $t>0$ and all $x,y\in\mathbb{R}^n$ (see [24]).

The operator $\nabla L^{-1/2}$ is called the Riesz transform associated with L, which is defined by

$$\nabla L^{-1/2}(f)(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\nabla e^{-tL}(f)(x)\frac{dt}{\sqrt{t}}. \tag{1.2}$$

This operator is bounded on $L^2(\mathbb{R}^n)$ (see [17]). Moreover, it was proved in [1,5] that by using the molecular decomposition of functions in the Hardy space $H^1_L(\mathbb{R}^n)$ associated with L, the operator $\nabla L^{-1/2}$ is bounded from $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, and hence, by interpolation, is bounded on $L^p(\mathbb{R}^n)$ for all $1<p\le 2$. Now we assume that $V\in RH_q$ (Reverse Hölder class). In [25], Shen showed that $\nabla L^{-1/2}$ is a Calderón-Zygmund operator if $q\ge n$. On the other hand, when $n/2\le q<n$, then $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p\le p_0$, where $1/p_0=1/q-1/n$, and the above range of p is optimal (see also [25]). For more information about the Hardy spaces $H^p_L(\mathbb{R}^n)$ associated with Schrödinger operators when $0<p\le 1$, we refer the readers to [6–10,29,31] and the references therein.

In the weighted case, in [28], Song and Yan introduced the weighted Hardy spaces $H^1_L(w)$ associated with L in terms of the square function and established their atomic decomposition theory. Furthermore, they also showed that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for $1<p<2$, and bounded from $H^1_L(w)$ to the classical weighted Hardy space $H^1(w)$.

As a continuation of [28], the main purpose of this paper is to define the weighted Hardy spaces $H^p_L(w)$ associated with L for $0<p<1$ and then study their atomic characterizations. Moreover, we will also prove that $\nabla L^{-1/2}$ is bounded from $H^p_L(w)$ to the classical weighted Hardy space $H^p(w)$ for $n/(n+1)<p<1$. Our main result is stated as follows.

Theorem 1.1. *Suppose that $L=-\Delta+V$. Let $n/(n+1)<p<1$ and $w\in A_1\cap RH_{(2/p)'}$. Then the operator $\nabla L^{-1/2}$ is bounded from $H^p_L(w)$ to the classical weighted Hardy space $H^p(w)$.*

It is worth pointing out that when $L=-\Delta$ is the Laplace operator on $\mathbb{R}^n$, then the space $H^p_L(w)$ coincides with the classical weighted Hardy space $H^p(w)$. Therefore, in this special case, we can derive that the classical Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded