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Construction of Multivariate Tight Framelet Packets Associated with Dilation Matrix

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Abstract. In this paper, we present a method for constructing multivariate tight framelet packets associated with an arbitrary dilation matrix using unitary extension principles. We also prove how to construct various tight frames for $L^2(\mathbb{R}^d)$ by replacing some mother framelets.

Key Words: Wavelet, tight frame, framelet packet, matrix dilation, extension principle, Fourier transform.

AMS Subject Classifications: 42C40, 42C15, 65T60

1 Introduction

A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Tight wavelet frames provide representations of signals and images in applications, where redundancy of the representation is preferred and the perfect reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets, is kept (see [6]). The main tools for construction and characterization of wavelet frames are the several extension principles, the unitary extension principle (UEP) and oblique extension principle (OEP) as well as their generalized versions, the mixed unitary extension principle (MUEP) and the mixed oblique extension principle (MOEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). These essential methods were firstly introduced by Ron and Shen in [11] and in the fundamental work of Daubechies et al. [4] for scalar refinable functions. The resulting tight wavelet frames are

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based on a multiresolution analysis, and the generators are often called framelets. The advantages of MRA-based tight wavelet frames and their promising features in applications have attracted a great deal of interest and effort in recent years to extensively study them. To mention only a few references on tight wavelet frames, the reader is referred to [4–9] and many references therein.

However, wavelet frames provide poor frequency localization in many applications as they are not suitable for signals whose domain frequency channels are focused only on the middle frequency region. Therefore, in order to make more kinds of signals suited for analyzing by wavelet frames, it is necessary to extend the concept of wavelet frames to a library of wavelet frames, called framelet packets or wavelet frame packets. The original idea of framelet packets was introduced by Coifman et al. in [3] to provide more efficient decomposition of signals containing both transient and stationary components. Chui and Li [2] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate orthogonal wavelets such that they may be used in a wider field. Other notable generalizations are the wavelet packets and framelet packets related to the Walsh polynomials [12, 15], wave packets and tight framelet packets on local fields of positive characteristic [13, 14], the vector-valued wavelet packets [1], the vector-valued multivariate wavelet frame packets [17] and the tight framelet packets on $\mathbb{R}^d$ for dilation factor 2 [10].

Recently, Shah and Debnath [16] have introduced a general construction scheme for a class of stationary M-band tight framelet packets in $L^2(\mathbb{R})$ via extension principles. In this paper, we generalize the concept of univariate framelet packets to the case of multivariate tight framelet packets associated with an expansive dilation matrix using unitary extension principles and our approach is quite different as described in [16].

This paper is organized as follows. In Section 2 we review some basic facts about tight wavelet frames associated with dilation matrix using extension principles. In Section 3, we prove our main results regarding the construction of multivariate tight framelet packets in $L^2(\mathbb{R}^d)$.

2 Notations and preliminaries

Throughout this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be all real and complex numbers, respectively. $\mathbb{N}$ and $\mathbb{N}_0$ denote all natural and non-negative integers, respectively. $\mathbb{Z}^d$ and $\mathbb{R}^d$ denote the set of all d-tuples integers and d-tuples of reals, respectively. Assume that we have an expansive dilation matrix A, i.e., all its eigenvalues lie outside the unit circle, the matrix A is known as the dilation matrix. For a $d\times d$ real matrix A, we denote by A^* the transpose of A and B be a $d\times d$ non-singular matrix.

For a $d\times d$ expansive matrix A, define the dilation operator D and the shift operator